On periodicity of trigonometric functions and connections with elementary number theoretic ideas

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The notion of periodicity stands for regular recurrence of phenomena in a particular order in nature or in the actions of man, machine, etc. Many examples can be given from daily life featuring periodicity: day and night, the weekdays, the months of the year, the circulation of blood in our body, the function of the heart, the operation of a clock, the natural circulation of water, crop rotation, and tree crop rotation. In astronomy there are many periodical phenomena: the revolution of planets around the sun, the Solar Cycle and the Lunar Cycle, the cycle of intercalation in a period of 19 years: "Every 19 years of which 7 are intercalary and 12 are regular, is called a Period" (Maimonides, Sanctification of the Month 6). Mathematically the meaning of periodicity is that some value recurs with a constant frequency.

Students learn about the periodicity of the trigonometric functions—sine, cosine, tangent and cotangent, after they study the subject of extension of the trigonometric circle and the graphical representation of these functions. However, students encounter the notion of the period even before their studies in trigonometry. Here are some examples:

1. The digit of units in the value of the expression 2^n (n is a natural number) recurs with a periodicity of 4:
   $\underline{2}, \underline{4}, \underline{8}, 1\underline{6}, 3\underline{2}, 6\underline{4}, 12\underline{8}, 25\underline{6}, \ldots$
2. When converting the common fraction $\frac{83}{333}$ to a decimal fraction, one obtains the periodical decimal fraction $0.\underline{249}249249\ldots$, which has a periodicity of 3.
3. The expression $(-a)^n$ where $a > 0$, and n is a natural number gives a positive or negative value with a periodicity of 2.

In subsequent studies in mathematics, students encounter additional phenomena featuring periodicity.

4. The high-order derivatives of the function $y = \sin x$ show a periodicity of 4 in the value of the derivatives of different orders:
   $y = \sin x,\ y' = \cos x,\ y'' = -\sin x,\ y^{(3)} = -\cos x,\ y^{(4)} = \sin x,\ y^{(5)} = \cos x,\ \ldots$

5. The value of the expression i^n (i is the imaginary number $\sqrt{-1}$) repeats with a periodicity of 4.
$i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ \ldots$

In this article we consider the properties of periodical functions, and how to find the period of the sum, the difference and the product functions of trigonometric functions. We shall discuss the issue of the smallest period, and find a connection between the period of a product of trigonometric functions and the period of their sum. A surprising connection between different mathematics topics is presented in the process of determining the period of the sum or product of trigonometric functions.

Trigonometry is an integral part of the current curricula and the proposed new Senior Secondary *Australian Curriculum: Mathematics*, as specified by the Australian Curriculum, Assessment and Reporting Authority (ACARA, 2010). Even though mathematics education in Australia varies considerably between various States due to the different education systems in place in each State, the trigonometric functions and subsequently solving trigonometric equations including the periodicity of those functions are included in the various levels of mathematics learned in Years 11–12. For example, reviewing the mathematics syllabuses presented by the Board of Studies New South Wales (2012), one can see that the topic of Trigonometry is included repeatedly in all the three levels: General Mathematics, Mathematics and Mathematics Extension 1, and Mathematics Extension 2 (previously called 2 Unit, 3 Unit, and 4 Unit).

Moore (2010), Pimpalak et al. (2011), Weber (2005) and others report research studies revealing that both students and teachers have difficulty understanding and using trigonometric functions, as well as that they hold weak understanding of ideas foundational for learning trigonometry. There is no doubt that periodicity of the trigonometric functions is a key idea and should be a central part of teaching and learning the topic of trigonometry. The following material is intended to strengthen and enrich teachers' knowledge of the concept of periodicity of trigonometric functions and equip them with tools and ideas to be presented within their teaching, especially within the advanced mathematics courses.

The period of trigonometric functions

Why is it important to know the periodicity of a trigonometric function?

In the solution of trigonometric equations or inequalities, we need to investigate and represent graphically functions with trigonometric parts, or to calculate the area between the graphs of trigonometric functions. Knowledge of the periodicity of the function simplifies this task greatly. To this end we present methods for determining the period of different trigonometric functions. First let us define what a periodical function is:

Definition 1

A function $f(x)$ is called a periodical function if there exists a constant number T, $(T \neq 0)$, so that for every x in the domain of the function there holds: $f(x+T) = f(x)$, provided that x + T belongs to the domain of the function.

Note: A periodical function may have points of entire ranges in which it is not defined. In this case, the definition of periodicity only applies to the points in which the function is defined.

Clearly, if T satisfies the equality $f(x+T) = f(x)$, then any nT, $n \in Z$ (integer numbers) also satisfies it, because for example $(n = 2)$:

$$f(x+2T) = f[(x+T)+T] = f(x+T) = f(x).$$

Definition 2

The smallest period (if exists) of function is the smallest positive number T that satisfies the equality $f(x+T) = f(x)$.

Occasionally, the smallest period is called 'the period' or the 'prime period' of the function. A function that is not periodic is called aperiodic.

Note: If the function is constant (i.e., of the form $f(x) = c$) then it is periodical (because for any positive number T, we have: $f(x+T) = f(x) = c$), but it does not have a smallest period. It is interesting to simulate this case with asking students to indicate the smallest positive number, or the nearest number to zero that is not zero. Finding the period of a function, and especially finding the smallest period, is not a simple task. In the present article we shall discuss some common special cases.

Li et al. (2009) present several strict mathematical definitions, 20 basic properties, and some examples of periodic functions, especially trigonometric functions. The six basic trigonometric functions ($\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\mathrm{cosec}(x)$) are most commonly used periodic functions. By inspecting the graphs of the basic trigonometric functions, one can see that the period of $\sin(x)$ and $\cos(x)$ is $2\pi k$, and the period of the tangent and cotangent functions is πk, where k is a whole number, and the smallest period is obtained for $k = 1$.

First we give some examples, followed by the theorem and its proof.

Example 1

What is the period of the function $f(x) = \sin(2x + \alpha)$, where α is a constant?

Solution

$f(x+T) = f(x)$

$\sin[2(x+T)+\alpha] = \sin(2x+\alpha)$

$\sin[2(x+T)+\alpha] - \sin(2x+\alpha) = 0$

$2\cos\,(2x+\alpha+T)\sin T = 0$

The equality holds for any x only if $\sin T = 0$, meaning that $T = \pi k$, k is a whole number. The smallest period is obtained for $k = 1$ and then it is equal to π.

Example 2

What is the period of $f(x) = \cos\frac{x}{2}$?

Solution

$f(x+T) = f(x)$

$\cos\frac{x+T}{2} = \cos\frac{x}{2}$

$\cos(\frac{x}{2} + \frac{T}{2}) = \cos\frac{x}{2}$

As we know, the smallest period of the cosine function is 2π, thus:

$$\frac{T}{2} = 2\pi \qquad T = 4\pi$$

(See Figure 1 to link to movie 1 and interact with the interactive work sheet).

Now we formulate several theorems for finding the period of a function constructed using operations between basic functions.

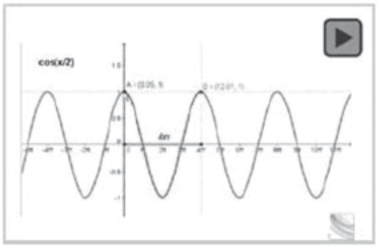

Figure 1.
Movie 1:
http://youtu.be/rGUMxsczcHc
Interactive worksheet:
http://highmath.haifa.ac.il/stupel/1.html

Theorem 1

If $f(x)$ is a periodical trigonometric function with a smallest period of T, then the function $a \cdot f(mx+n) + b$ (where a, b, m and n are constant numbers, $a \neq 0$, $m \neq 0$) is also periodical, with a smallest period of $\frac{T}{|m|}$.

Proof

We denote by T_1 the smallest period of the function, $a \cdot f(mx+n) + b$ meaning that for any x in the domain of the function there holds:

$$a \cdot f[m(x+T_1) + n] + b = a \cdot f(mx+n) + b, \text{ or:}$$
$$f[mx + n + mT_1] = f(mx+n)$$

From the statement, we know that the smallest positive number satisfying the last equality is T, therefore:

$$mT_1 = T, \;\; T_1 = \frac{T}{|m|}$$

The effect of the parameters in a trigonometric function on the size of the period and other properties is clearly seen in movie 2 (see Figure 2) and can be manipulated by entering the interactive work sheet. The movie uses the sine function as an illustration. Similar operations can be performed on any other trigonometric function.

Following is the calculation for finding the smallest period by using Theorem 1:

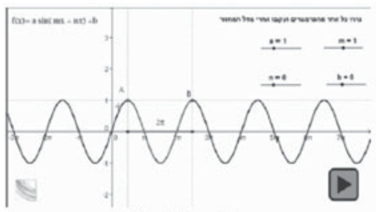

Figure 2.
Movie 2:
http://youtu.be/EboR0hcWuPY
Interactive work sheet:
http://highmath.haifa.ac.il/stupel/3.html

Example 3

$f(x) = \sin(2x)$

$T_{f(x)} = \frac{T_{\sin x}}{2} = \frac{2\pi}{2} = \pi$

Example 4

$f(x) = 2 + 3\sin x$

$T_{f(x)} = \frac{T_{\sin x}}{1} = 2\pi$

Example 5

$f(x) = \cos(4x + \frac{\pi}{3})$

$T_{f(x)} = \frac{T_{\cos x}}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

Determining the period of a sum, a difference and a product of trigonometric functions

In the case of a function composed of operations between other functions, a single function that is suitable for implementing Theorem 1 must be obtained primarily, if possible. Following are some examples based on the utilisation of basic knowledge of trigonometric identities.

Example 6

$f(x) = \sin(x) + \cos(x)$

$f(x) = \sin(x) + \sin(\frac{\pi}{2} - x) = 2\sin\frac{\pi}{4}\cos(x - \frac{\pi}{4}) = \sqrt{2}\cos(x - \frac{\pi}{4})$

$T_{f(x)} = T_{\cos x} = 2\pi$

This result is in agreement with the property: if T is a period of F and a period of G, then T is a period of $F + G$.

Example 7

$f(x) = \tan^2 x$

$f(x) = \frac{1}{\cos^2 x} - 1 = \frac{2}{1 + \cos 2x} - 1$

By application of the property: if T is a period of F, then T is a period of F^{-1}, we obtain

$$T_{f(x)} = \frac{2\pi}{2} = \pi$$

This final result is in agreement with the property: if T is a period of F, then T is a period of F^2.

Example 8

$f(x) = \cot x - \tan x$

$$f(x) = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\frac{1}{2}\sin 2x} = 2\cot 2x$$

$$T_{f(x)} = \frac{\pi}{2}$$

Obviously, in this case, we received a smaller period ($\pi/2$) than the one (π) is indicated by the property: if T is a period of F and a period of G, then T is a period of $F–G$.

It should be emphasised that functions of the above types not always can be transformed into one simple form that fits Theorem 1. For example, the sum of two functions such that one function with a period π and the second function with a period 3, need not be periodic. When it cannot, we might not find the smallest period, or any period at all.

Theorem 2

Let $f(x)$ be a periodical function with a period of T_1, and let $g(x)$ be a periodical function with a period T_2. If

$$\frac{T_1}{T_2} = \frac{m}{n}$$

where m and n are whole numbers and the fraction $\frac{m}{n}$ is reduced to lowest terms, then the function $f(x) \pm g(x)$ is periodical, and has the period $T = nT_1 = mT_2$.

Before we prove Theorem 2, let present the definition of the least common multiple of two integers.

Definition 3

The least common multiple of two non-zero integers a and b, denoted by LCM (a, b), is the positive integer m satisfying: (1) a/m and b/m; (2) if a/c and b/c with $c > 0$, then $m \leq c$ (Burton, 1985).

Thus, it seems that a period T of a sum (or difference) of two functions with periods of T_1 and T_2 is connected to the least common multiple of T_1 and T_2. Nevertheless, T_1 and T_2 are not always integers, hence we cannot use this term at this point, unless we define it for the proper numbers other than integers. We will deal with that in the next section of this article.

Proof

T is a multiple of the periods of the function $f(x)$ and the function $g(x)$, and therefore it is a period of both functions, therefore it is a period of the sum.

Note: The period found from this theorem is not necessarily the smallest period. Example 8 demonstrated this and the next example is also showing that for two forms of the same function, we might obtain two different results for the period of the given function. However, it is not clear yet in general which procedure will give the smallest period.

Example 9

$f(x) = \sin^2 x + \cos^2 x$

Since $$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

then the smallest period of each part is π and hence the period of $f(x)$ is π. However, $f(x) = 1$, and a constant function does not have a smallest period, even though π is also a period of the constant function.

We now present additional examples for the use of Theorem 2:

Example 10

$$f(x) = 4\sin\frac{x}{2} + \cos\frac{x}{3}$$

$$T_{\sin\frac{x}{2}} = \frac{2\pi}{\frac{1}{2}} = 4\pi, \qquad T_{\cos\frac{x}{3}} = \frac{2\pi}{\frac{1}{3}} = 6\pi$$

$$\frac{T_1}{T_2} = \frac{4\pi}{6\pi} = \frac{2}{3} \qquad T = 3T_1 = 2T_2 = 12\pi$$

Example 11

$$f(x) = \sin\frac{\pi}{4}x + \cos\frac{\pi}{3}x + 2\tan\frac{\pi}{12}x$$

$$T_1 = \frac{2\pi}{\frac{1}{4}} = 8\pi, \qquad T_2 = \frac{2\pi}{\frac{1}{3}} = 6\pi, \qquad T_3 = \frac{\pi}{\frac{1}{12}} = 12\pi$$

$$\frac{T_1}{T_2} = \frac{8\pi}{6\pi} = \frac{4}{3} \qquad T^* = 3T_1 = 4T_2 = 24\pi$$

$$\frac{T^*}{T_3} = \frac{24\pi}{12\pi} = \frac{2}{1} \qquad T = T^* = 2T_3 = 24\pi$$

T* is the period of the first two terms of $f(x)$ and $Tf(x) = T$.

Arguments similar to those that lead to Theorem 2, lead to the conclusion that the product of the two periodical functions $f(x) \cdot g(x)$ is also a periodical function. In this case as well, we use the same procedure as exemplified above. In addition, it is clear that Theorem 2 can be extended to a sum of more than two functions by at least finding the period of first pair and proceeding to the next components of the sum step by step.

The connection between periodicity of trigonometric functions and elementary number theoretic ideas

Why do all textbooks (see, for example, Weber (2005)) recommend replacing a product of two trigonometric functions by a sum of two trigonometric functions before calculating the period? To answer this question we shall prove below that expressing a product of trigonometric functions as a sum of trigonometric functions may, in some cases, yield a period smaller than the least common multiple of the periods of the functions in the product (in the other cases the product remains the same—and in no case shall it be larger).

It is the author's belief that finding connections and combining methods from different branches of mathematics, serve to illustrate the connections between the various areas of mathematics, reveals the beauty and aesthetics of mathematics (Sinclair, 2006; 2011), and illustrates how mathematics is a conjunction of intertwined fields. In addition, presenting the material from different points of view and including employment of dynamic geometry software (DGS), enhance the student's mathematical understanding. Making connections between different fields of mathematics is also one of the standards of mathematics curriculum in many countries, including America (NCTM, 2000) and Australia. For example, the *Australian Curriculum: Mathematics* Rationale states, "Mathematics ensures that the links between the various components of mathematics, as well as the relationship between mathematics and other disciplines, are made clear. Mathematics is composed of multiple but interrelated and interdependent concepts and systems which students apply beyond the mathematics classroom" (ACARA, 2011).

Since we will make connections between periodicity of trigonometric functions and the concepts of greatest common divisor and the least common multiple that are defined only for integers, we need to provide extended definition of the terms "divisor", "multiple", "greatest common divisor" and the "least common multiple" of rational numbers. Subsequently, we shall present the properties of those new definitions of the greatest common divisor and the least common multiple to explain the claims related to the periodicity.

Definition 4

Let a and b be rational numbers with b not zero. We say that b divides a if there is an integer n such that $a = nb$. In this case, we also say that a is a multiple of b.

Definition 5

Let a and b be rational numbers. A rational number d is said to be the greatest common divisor of a and b if d is a divisor of both a and b and if e is any other common divisor of a and b, then $d \geq e$.

Definition 6

Let a and b be two nonzero rational numbers. A rational positive number m is said to be the least common multiple of a and b if m is a multiple of both a and b and if c is any other common multiple of a and b, then $m \leq c$.

Since we will use the term "relatively prime", the following definition is also given.

Definition 7

Two integers a and b, not both of which are zero, are said to be relatively prime whenever the greatest common divisor of a and b is equal to 1 (Burton, 1985).

Note: If the two numbers are equal 1 each, then they are relatively prime numbers.

Following the definitions 4–6 above, we denote by (a, b) the greatest common divisor (GCD) of the rational numbers a and b, and by $[a, b]$ the least common multiple (LCM) of the nonzero rational numbers a and b.

Example 12

$(247, 52) = 13$ $(136, 40) = 8$

$[5, 7]= 35$ $[6, 9] = 18$

$(\frac{1}{2}, \frac{3}{4}) = \frac{1}{4}$ $[\frac{1}{2}, \frac{3}{4}] = \frac{3}{2}$

When it is not easy to find the LCM of two fractions, it is possible to apply the formula presented by Singh (2011) for finding the LCM of two reduced fractions ($p, q, e, f \in Z$ and $e, f \neq 0$:

$$\text{LCM}(\frac{p}{q}, \frac{e}{f}) = \frac{\text{LCM of the numerators}}{\text{GCD of the denominators}} = \frac{[p, e]}{(q, f)}$$

Example 13

$[\frac{5}{6}, \frac{2}{3}] = \frac{[5,2]}{(6,3)} = \frac{10}{3}$, therefore $\frac{10}{3} : \frac{5}{6} = 4$ and $\frac{10}{3} : \frac{2}{3} = 5$.

This example also demonstrates property number 2 ahead.

At this point, we would like to present our formula for finding the GCD of two reduced fractions ($p, q, e, f \in Z$ and $e, f \neq 0$:

$$\text{GCF}(\frac{p}{q}, \frac{e}{f}) = \frac{\text{GCD of the numerators}}{\text{LCM of the denominators}} = \frac{(p, e)}{[q, f]}$$

Example 14

$(\frac{5}{6}, \frac{2}{3}) = \frac{(5,2)}{[6,3]} = \frac{1}{6}$, therefore $\frac{5}{6} : \frac{1}{6} = 5$ and $\frac{2}{3} : \frac{1}{6} = 4$.

This example also demonstrates property number 1 ahead.

The above definitions for the rational numbers retain properties of (a, b) and $[a, b]$ that are related to integers. Here, we make use of the following properties:

1. A common divisor d of the rational numbers a and b is a GCD of a and b, if and only if $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime whole numbers.
2. A common multiple m of the rational numbers a and b is a LCM if and only if $\frac{m}{a}$ and $\frac{m}{b}$ are relatively prime whole numbers.
3. For any pair of numbers a and b there holds $[a, b] \cdot (a, b) = a \cdot b$.

Another issue is, if we need LCM of two irrational numbers, as in the case of two periods of trigonometric functions. In this case, we adapt the rule presented by Singh (2011) above for finding the LCM of rational numbers. According to Singh (2011, p. 369): "This rule also works for irrational numbers of similar type like $2\sqrt{2}/3$, $3\sqrt{2}/3$ etc. or $\pi/2$, $3\pi/2$ etc. However, it is not possible to find LCM of irrational numbers of different kind like $2\sqrt{2}$ and π. Similarly, there is no LCM for combination of rational and irrational numbers."

Example 15:

Finding the LCM of $\frac{\pi}{3}$ and $\frac{3\pi}{2}$.

$$[\frac{\pi}{3},\frac{3\pi}{2}]=\frac{\text{LCM of numerators}}{\text{GCF of denominators}}=\frac{[\pi,3\pi]}{(3,2)}=\frac{3\pi}{1}=3\pi$$

Claim 1

The GCD of two rational numbers a and b is a common divisor of their sum and difference: $(a+b,\ a-b) \geq (a,\ b)$.

Proof

Given that $a=(a,\ b)\cdot n_1$ and $b=(a,\ b)\cdot n_2$ (where n_1 and n_2 are relatively prime whole numbers according to property (1) above), then:

$$a+\mathrm{b}=(n_1+n_2)\cdot(a,\ b)$$
$$a-\mathrm{b}=(n_1-n_2)\cdot(a,\ b)$$

which means that $(a,\ b)$ is a common divisor of $a+b$ and $a-b$, hence by the definition of GCD we obtain $(a+b,\ a-b)\geq(a,\ b)$.

Example 16

$a=35,\ b=15$.

Hence, $(a,\ b)=(35,\ 15)=5$, $a+b=50$, $a-b=20$, $(a+b,\ a-b)=10$ and in this case indeed $(a+b,\ a-b)\geq(a,\ b)$.

Note: the result that any common divisor of a and b is also a common divisor of their sum and difference is not necessarily true for the converse statement. For example: if a and b are both odd numbers, then the sum of them $a+b$ and the difference $a-b$ are both even. Hence a common even divisor of the sum and the difference cannot be a common divisor of a and b.

After laying the foundations, we apply them to the subject of periodicity. We assume that the given are functions: $F(x)=\sin(\alpha x+\gamma)$, $G(x)=\cos(\beta x+\delta)$, and α, β are rational numbers.

From Theorem 1 the smallest positive period of $F(x)$ is $\frac{2\pi}{\alpha}$, and the smallest positive period of $G(x)$ is $\frac{2\pi}{\beta}$.

The least common multiple of the periods of these functions is

$$[\frac{2\pi}{\alpha},\frac{2\pi}{\beta}]=\frac{2\pi}{(\alpha,\beta)}$$

This number is the period of $F(x)\pm G(x)$, and also of $F(x)\cdot G(x)$. After presenting the product of the trigonometric functions $F(x)\cdot G(x)$ as a sum of trigonometric functions, we obtain:

$$F(x)\cdot G(x)=\sin(\alpha x+\gamma)\cdot\cos(\beta x+\delta)$$
$$=\frac{1}{2}\left[\sin[(\alpha+\beta)x+\gamma+\delta]+\sin[(\alpha-\beta)x+\gamma-\delta]\right]$$

The period of the obtained sum of functions is:

$$[\frac{2\pi}{\alpha+\beta},\frac{2\pi}{\alpha-\beta}]=\frac{2\pi}{(\alpha+\beta,\alpha-\beta)}$$

From Claim 1 $(a+b, a-b) \geq (a, b)$ therefore

$$\frac{2\pi}{(\alpha+\beta,\alpha-\beta)} \leq \frac{2\pi}{(\alpha,\beta)}$$

Thus, upon replacing the product of functions by a sum of functions, we obtained a period smaller than or equal to the period of the product. Note that this is the reason why we replace the product by a sum when looking for the smallest period of functions composed of operations between functions.

Following are several additional examples for finding the period of a trigonometric function based on the presented theory and guidelines. The solution of the examples is based on knowledge of trigonometrical identities, and on skills in algebraic technique.

Example 17

$f(x) = \cos 3x \cdot \cos x$

From Theorem 2, $T_1 = \frac{2\pi}{3}$, $T_2 = 2\pi$, and $\frac{T_1}{T_2} = \frac{1}{3}$, and therefore $Tf(x) = 3T_1 = T_2 = 2\pi$.

However, when going over to the sum:

$$f(x) = \frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x$$

Thus, $T_1 = \frac{2\pi}{4} = \frac{\pi}{2}$, $T_2 = \frac{2\pi}{2} = \pi$, and $\frac{T_1}{T_2} = \frac{1}{2}$, and therefore $T_{f(x)} = \pi$.

We obtained a period that is smaller than the period obtained using the product.

Example 18

$f(x) = \sin 6x \cdot \cos 4x$

From Theorem 2, $T_1 = \frac{2\pi}{6} = \frac{\pi}{3}$, $T_2 = \frac{2\pi}{4} = \frac{\pi}{2}$, and $\frac{T_1}{T_2} = \frac{2}{3}$, therefore $T_{f(x)} = \pi$.

When going over to the sum: $f(x) = \frac{1}{2}\sin 10x + \frac{1}{2}\sin 2x$, we have:

$T_1 = \frac{2\pi}{10} = \frac{\pi}{5}$, $T_2 = \frac{2\pi}{2} = \pi$ and $\frac{T_1}{T_2} = \frac{1}{5}$, therefore $T_{f(x)} = \pi$,

and in this case we have the same period.

Example 19

$$f(x) = \frac{\sin x}{1+\cos x}$$

Based on the trigonometric identity $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$,

we have $f(x) = \tan\frac{x}{2}$, and therefore $T = T_{\tan\frac{x}{2}} = \frac{\pi}{\frac{1}{2}} = 2\pi$ is the smallest period.

Example 20

$$f(x) = \frac{1-\cos x}{1+\cos x}$$

Based on the identity $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$, we have $f(x) = \tan^2\frac{x}{2}$, or

$$f(x) = \frac{1}{\cos^2\frac{x}{2}} - 1$$

Using the trigonometric identity $\cos 2x = 2\cos^2 x - 1$, we obtain $f(x) = \frac{2}{1+\cos x} - 1$ and therefore $T = T_{\cos x} = 2\pi$ is the smallest period.

Example 21

$$f(x) = \tan\frac{3}{4}x + \cot\frac{2}{3}x$$

$$\left.\begin{aligned} T_1 &= \frac{\pi}{\frac{3}{4}} = \frac{4\pi}{3} \\ T_2 &= \frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2} \end{aligned}\right\} \quad \frac{T_1}{T_2} = \frac{4\frac{\pi}{3}}{3\frac{\pi}{2}} = \frac{8}{9}$$

$Tf(x) = 9T_1 = 8T_2 = 12\pi$.

Example 22

$$f(x) = \cos\frac{x}{4} + \cos\frac{x}{5} + \cos\frac{x}{8}$$

The least common multiple of 8π, 10π and 16π is 80π, therefore $T = 80\pi$.

Example 23

In this example we find the period of the function $f(x) = \tan^2 x$.
Based on the definition of periodicity, $f(x+T) = f(x)$, we obtain:
$\tan^2(x+T) = \tan^2 x \Rightarrow \tan^2(x+T) - \tan^2 x = 0$, or:

$$[\tan(x+T)+\tan x]\,[\tan(x+T)-\tan x] = 0$$

$$[\frac{\sin(x+T)}{\cos(x+T)} + \frac{\sin x}{\cos x}]\,[\frac{\sin(x+T)}{\cos(x+T)} - \frac{\sin x}{\cos x}] = 0$$

$$[\frac{\sin(x+T)\cos x + \sin x\cos(x+T)}{\cos(x+T)\cos x}]\,[\frac{\sin(x+T)\cos x - \sin x\cos(x+T)}{\cos(x+T)\cos x}] = 0$$

or, otherwise:

$$\frac{\sin(2x+T)}{\cos(x+T)\cos x}\cdot\frac{\sin T}{\cos(x+T)\cos x} = 0$$

The equality holds for any x only when $\sin T = 0$, or $T = \pi k$. The smallest period of the function is π.

Example 24

We prove that the function $f(x) = x\sin x$ is aperiodical function.

Indirect proof

We assume that $f(x)$ is in fact periodic, with a period of T (so that $T > 0$). From the definition of periodicity we obtain:

$(x+T)\sin(x+T) = x\sin x$

Therefore at the point $x = 0$ there must hold $\sin T = 0$, implying that $T = \pi k$, where k is a positive whole number.

At the point $x = \frac{\pi}{2}$, from the definition of periodicity $f(\frac{\pi}{2}+T) = f(\frac{\pi}{2})$, we obtain:

$$(\frac{\pi}{2}+T)\sin(\frac{\pi}{2}+T) = \frac{\pi}{2}\sin\frac{\pi}{2}$$

Substituting $T = \pi k$, we obtain: $(\frac{\pi}{2}+\pi k)\sin(\frac{\pi}{2}+\pi k) = \frac{\pi}{2}\sin\frac{\pi}{2}$

Since $\sin(\frac{\pi}{2}+\pi k) = \pm 1$, we obtain the equation:

$$(\frac{\pi}{2}+\pi k)\,(\pm 1) = \frac{\pi}{2}$$

There is no positive whole number k, for which the equality holds, therefore, the conclusion is that the function is not periodic.

Concluding remarks

We believe that the material presented in this manuscript is of value to teachers who want to build lessons around the topics of mathematical periodicity and proof in mathematics, especially while teaching the trigonometric functions within the advanced mathematical courses offered to students in senior high school mathematics classes. Senior students who will encounter this material on periodicity (usually high ability students) hopefully will find interest in both, the wide range of situations in which periodicity occurs in mathematics and the surprising connections to basic number theory ideas. In addition, we would like to emphasize the incorporation and the role of the technology represented here by the software of *GeoGebra* to provide the students with dynamic visual tools that enable them to investigate the effect of changing parameters and components of the functions on their shape and periodicity. We strongly recommend that the mathematics teachers take advantage of those tools and incorporate them within their teaching to the benefit of their students.

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