Pythagoras’ garden, revisited

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Mack and Czernezkyj (2010) have given an interesting account of primitive Pythagorean triples (PPTs) from a geometrical perspective. We wish here to enlarge on the role of the equicircles (incircle and three excircles), and show there is yet another family tree in Pythagoras’ garden.

Where they begin with four equicircles, we begin with four tangent circles, attached to the corners of a rectangle based on the right triangle. Reflecting these circles in a certain line results in a congruent tangent cluster, having the same six points of tangency, orthogonal to the first four, and ultimately revealed as (jostled) equicircles!

We then develop three celebrated families of triples by elementary means, and tinker with the sequencing rules until the classic Pythagorean family tree magically appears. Using a favourite set of four parameters to identify and tag triples, we find more circle secrets. A second and totally new family tree (invented by Price) is here debuted. We find it equally interesting and valuable, but so far lacking an adequate geometric interpretation.

Getting started

We begin with a few notational differences. Where the previous paper has a sequence (a, b, c) with $a < b$ we use a matrix-like notation [a b c] using spaces rather than commas. This is a triple of positive integers describing legs (a, b) and hypotenuse (c) of a right triangle ΔABC. Moreover, a, c are odd, b is even, and the greatest common divisor of the sides is one. Thus where they have (20, 21, 29), we have [21 20 29]. We state without proof that every right triangle with rational sides is similar to one and only one PPT.

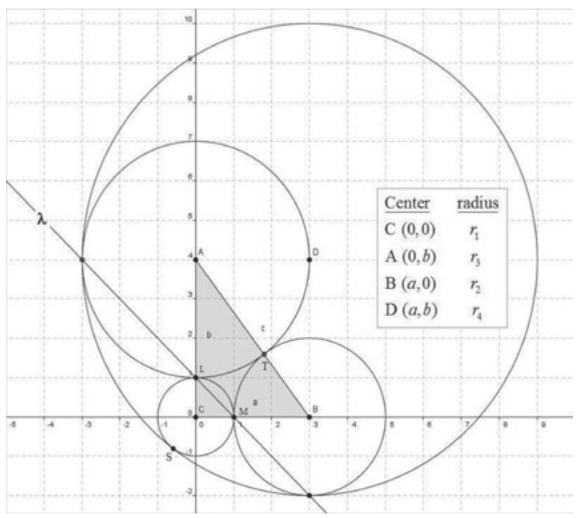

Figure 1

The example in Figure 1 is the venerable [3 4 5] triangle for definiteness, and we have established a coordinate system with *C* as origin, and rays *CB*, *CA* as positive *x*-axis and positive *y*-axis. Our figure shows three circles of radii [1, 2, 3] centred on *C*, *B*, *A* and a fourth circle centred on *D* with radius 6, where *ADBC* makes a rectangle. These are four mutually tangent circles.

Generalisation is quite easy. Just put

$$a = r_1 + r_2,\ b = r_1 + r_3,\ c = r_1 + r_3 \tag{1.1}$$

The radii are positive integers because of the triangle inequality, and since a, c are odd and b is even (solve for the r_1 to see this). Define $r_4 = r_1 + r_2 + r_3$ so that (1.1) is equivalent to

$$a = r_4 - r_3,\ b = r_4 - r_2,\ c = r_4 - r_1 \tag{1.2}$$

Mutual tangency of four circles as in Figure 1 is now almost immediate. It is simply a matter of applying the six equations (1.1), (1.2) to the six line segments determined by *A*, *B*, *C*, *D*.

The Pythagorean identity $a^2 + b^2 = c^2$ becomes $r_1(r_1 + r_2 + r_3) = r_2 \cdot r_3$ or even more simply $r_1 \cdot r_4 = r_2 \cdot r_3$. Two more easily verified forms of this fecund identity are (the first goes back to Dickson (1894); see Gerstein (2005) for history and applications):

$$2r_1^{\,2} = (r_3 - r_1)(r_3 - r_2), \quad (a + b - c)^2 = 2(c - a)(c - b) \tag{1.3}$$

We add as a useful and easily proved fact, that in the sequence r_1, r_2, r_3, r_4 the values alternate odd/even or even/odd. Also r_1 is smallest and r_4 is largest.

What does this have to do with the equicircles of Mack and Czernezkyj? Everything. We reconsider Figure 1. These six points of tangency support a second set of four tangent circles, a general fact which Coxeter (1989) ascribes to a schoolteacher named Beecroft. Though easily proved, we can take a nice shortcut because of the right triangle and get even more.

The descending 45° line λ in Figure 1 contains the diagonals of three successive squares (sides *AL*, *LC*, *MB*), and so must pass through four tangency points (a *harmonic set*). It happens that the other two points of tangency *S*, *T* define a segment for which λ is the perpendicular bisector! This is an exercise in elementary analytic geometry. Here are some hints to make the work less tedious. The line equations are $x + y = r_1$ for λ and

$$\frac{x}{a} + \frac{y}{b} = 1$$

for the hypotenuse line *AB*.

The coordinates of *S*, *T* are easily found as weighted averages (or use similar triangles). This is because *T* divides *AB* in the ratio $r_3 : r_2$, and *S* divides *CD* in the negative ratio $(-r_1) : r_4$.

$$S = \frac{\left[r_4 C - r_1 D\right]}{c} = \left(\frac{-ar_1}{c},\ \frac{-br_1}{c}\right), \quad T = \frac{[r_3 B + r_2 A]}{c} = \left(\frac{ar_3}{c},\ \frac{br_2}{c}\right)$$

The slope of *ST* computes to +1, and the midpoint lies on $x + y = r_1$.

We now may conclude that reflection in λ exchanges *S* with *T* and fixes the other four tangency points. The four circles, on reflection, give a congruent set of four circles, in fact the Beecroft dual configuration. Combining the eight circles makes a beautiful diagram that we call the "Beecroft butterfly" (Figure 2).

Moreover, there are four congruent right triangles in the rectangle $ABCD$, and each one is matched with a different circle in the dual system, showing it to be an excircle or incircle. These four cases are displayed as two pairs in Figure 3.

Our conclusion requires confirmation of the required tangencies for the dual circles. Most are easily seen. But consider the common tangent to two circles at S and again at T in Figure 1. The lines CD, AB have slopes b/a, $-b/a$. The two common tangents are perpendicular to these, and have slopes $-a/b$, a/b, which on reflection in λ are changed to reciprocals, passing through T, S.

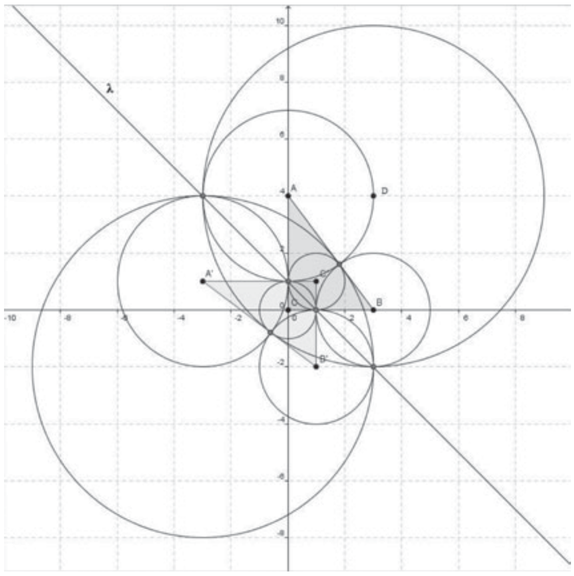

Figure 2. Beecroft 'butterfly'.

If we were to display the four equicircles for a single one of the triangles, tangency is sacrificed. See Bernhart and Price (2005) if desired, to see how reflections in the line λ followed in some cases by reflections in the vertical and/or horizontal midlines of the rectangle $ABCD$ move the original four circles into position as the four equicircles of our triangle ABC.

Some readers may be curious to know how the four tangent circles come to satisfy a celebrated equation found by Descartes and rediscovered by Beecroft and Soddy. To answer this query we can rescale the diagram and the circles as follows.

Dividing all lengths by the value $r_1 \cdot r_4 = r_2 \cdot r_3$ converts each radius r_i into $\frac{1}{r_j}$ where $i + j = 5$. The *curvatures* are the inverses of these new radii. We write $(-r_1, r_2, r_3, r_4)$ for the curvatures, adding a minus sign because the large circle is concave to the others. With this convention, the Cartesian condition for four tangent circles with (signed) curvatures (x, y, z, w) is

$$2(x^2 + y^2 + z^2 + w^2) = (x + y + z + w)^2$$

As a challenge, substitute curvatures $(-r_1, r_2, r_3, r_4)$ in this equation and verify it using our favourite identity $(r_1 \cdot r_4 = r_2 \cdot r_3)$, and of course, $r_4 = r_1 + r_2 + r_3$.

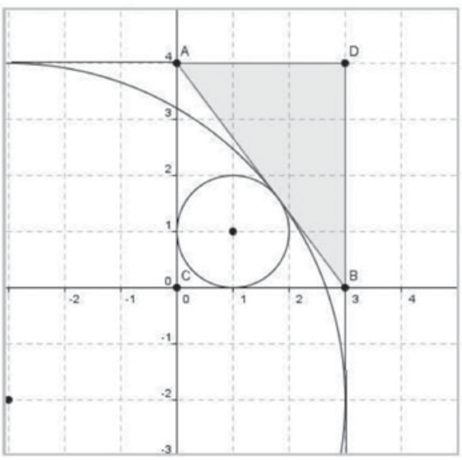

Figure 3 (a)

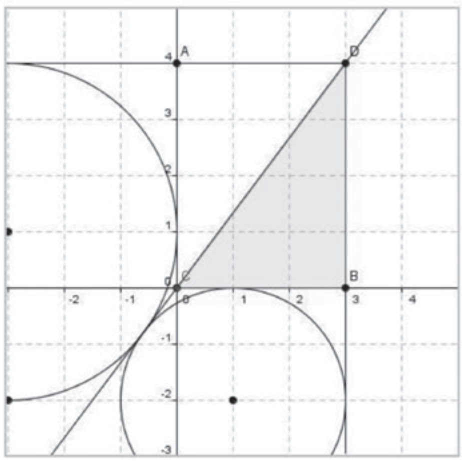

Figure 3 (b)

So the same four circles, moved around, can be a set of tangent circles, a second system of tangent circles (Figure 2), or the set of equicircles for ΔABC.

In a similar application, we can show that the Hero(n) formula for the area G of the triangle becomes $G^2 = r_1 \cdot r_2 \cdot r_3 \cdot r_4$ or $G = r_1 \cdot r_4 = r_2 \cdot r_3$. This is because the semiperimeter s and the other Heron factors are $s = r_4$, $s - c = r_1$, $s - b = r_2$, $s - a = r_3$, which are but simple deductions from equations (1.1), (1.2).

Three families of Pythagorean triples

Mack and Czernezkyj carry out a geometric construction of the *Pythagorean Tree* using each excircle. In three cases they have another PPT, and the sides (a', b', c') of the new triple are simple linear combinations of the sides (a, b, c) of the old. We will now sketch a way to do this, starting with simple questions.

In our notation a, c are odd and b is even, and triple [3 4 5] consists of consecutive integers. No other PPT has consecutive integers, but:

F1. What about $b + 1 = c$?

F2. What about $a + 2 = c$?

F3. What about $a = b + 1$ or $a + 1 = b$?

Suppose that $c - b = 1$. Then $a^2 = c^2 - b^2 = (c + b)(c - b) = c + b$. We find b, c by splitting the square of any odd number into "almost equal halves". Thus $a = 9$ has square $81 = 40 + 41$, and we have produced triple [9 40 41] which satisfies F1.

Similarly suppose that $c - a = 2$. Accepting that consecutive odd numbers like a, c add to a multiple of four (easy number theory) we get $b^2 = c^2 - a^2 = (c + a)(c - a) = 2(c + a)$ is divisible by eight, implying that even leg b is divisible by four. We claim that if x is the integer between a, c then $x = \left(\frac{1}{2}b\right)^2$. So take any multiple of four, such as $b = 12$. The square of $\frac{1}{2}b = 6$ lies between integers a, c so we have produced triple [35 12 37] that satisfies F2.

The two infinite families just described are ascribed to Pythagoras and Plato, respectively. The family which answers to F3 grows more rapidly, and requires some skill to 'bag'. It was first securely bagged by Fermat, so we call it the Fermat family. Our approach came to us in connection with some facts concerning simple continued fractions and Pell equations, but we will take a less arduous path.

Suppose that $b - a = 1$ or $a - b = 1$. Write $\{a, b\} = \{x, x + 1\}$ to cover both cases. A bit of algebra and the Pythagorean identity shows that $(a + b)^2$ and $2c^2$ differ by one, i.e., $(a + b)^2 = (2x + 1)^2 = A^2$ and $B^2 = c^2$ lead to (#1) $A^2 - 2B^2 = -1$. Trial and error with small numbers finds a few solutions to (#1), and also solutions to (#2) $A2 - 2B^2 = +1$ form a series of "fractions" A/B.

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \ldots$$

We will conveniently ignore the fact that these fractions have something to do with the square root of two, because that might be a public school bonus topic. Instead, we just scrutinise the numbers, until we serendipitously notice that the sum $(A + B)$ of a numerator and denominator equals the next denominator, and that each numerator A is the sum of its denominator B and the previous denominator, say B'. Differently put, we have stitched together short Fibonacci-style sequences.

$$1, 1, 2, 3;\ \ 3, 2, 5, 7;\ \ 7, 5, 12, 17;\ \ 17, 12, 29, 41;\ \ldots$$

In each 'quartet', the last two numbers are exchanged to begin the next quartet, and all quartets have the Fibonacci form $(a, b, a + b, a + 2b)$ or equivalently $(b - a, a, b, a + b)$. Query/challenge: do two adjacent values in the quartets ever have a common factor >1?

Our first concern is whether this 'rule' is just an accident, or a persistent pattern. Given any fraction $\frac{x}{y}$, we can use the alleged rule to go forward to $\frac{x+2y}{x+y}$, and, if $x < 2y$, we can go backward to $\frac{-x+2y}{x-y}$. Now use elementary algebra to show that if $\frac{x}{y}$ is a solution of (#1) (or of (#2)), then both the left hand and right hand fractions $\frac{-x+2y}{x-y}$ and $\frac{x+2y}{x+y}$ are solutions of (#2) (or of (#1)). That is, solutions of the two cases *alternate*. Naturally, this is why we kept the two cases in tandem. But we knew the answer before opening the envelope to read the question.

Given any solution to either case, we can back up with the rule until arriving at a fraction $\frac{x}{y}$ which fails the condition $x < 2y$. But then it is not hard to argue that y is small enough that we must have one of the *ad hoc* small solutions already compiled by trial and error. This concludes the argument that our sequence, extended by the rule, necessarily finds *all* the answers to both cases (#1), (#2).

Formula, formula

For those who crave formulas, we can supply $[(2n+1)\ 2n(n+1)\ (2n(n+1)+1)]$ for the Pythagoras family, and $[4n\ (4n^2 - 1)\ (4n^2 + 1)]$ for the Plato family. Both are easily found to square with our descriptions. We can also define two sequences (P_i) and (Q_i) having a generating pattern displayed as follows: $(\ldots, x, y, x + 2y, \ldots)$. For the former we begin with $(0, 1, \ldots)$ and for the latter we begin with $(1, 1, \ldots)$. There exist exact formulas for both, too involved to present here.

Where P_n is odd, split Q_n into "halves" $[\frac{1}{2}Q_n]$, and $[\frac{1}{2}Q_n + 1]$. Here [.] is greatest integer function (or round down). Thus $\frac{41}{29}$ gives 41 = 20 + 21, yielding the Fermat family member[21 20 29]. Likewise, $\frac{239}{169}$ gives 239 = 119 + 120 and thus another member, [119 120 169]. Since there are exact functions of n for P_n and Q_n, the same could be said of Fermat family triples.

Recursion is boss

What we intend to do is play a bit with recursive patterns until something astounding happens.

A simple geometric diagram (Figure 4) convinces us that the step from 5^2 to 7^2 is just 5 + 5 + 7 + 7.

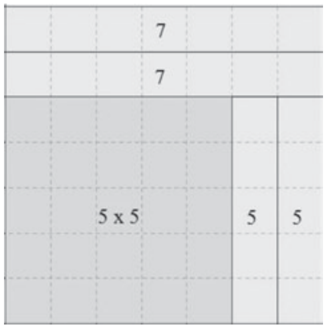

Figure 4

More generally, we advance from $[a\ b\ c]$ in the Pythagoras family to the next member $[a'\ b'\ c']$ as follows:

(Pythagorus)

$$a \to a' = a + 2,\ b \to b' = b + (a + a') = b + 2a + 2,\ c \to c' = \mathrm{c} + (a + a') = c + 2a + 2$$

Note that increase $2(a + a')$ is divided between b and c.

Anyone quite satisfied with a formula may look at this askance. We claim that this supports a simple algorithm for converting one family member to the next. This is a recursive emphasis that will pay enormous dividends, given a further tweak.

A geometric diagram similar to the last evaluates the step from 8^2 to 12^2. It gives the difference as

$$(8 + 8 + 8 + 8) + (12 + 12 + 12 + 12) = 4(8 + 12).$$

Thus recursively, when a, c each increase by $b + b'$, $2(c + a)$ then increases by $4(b + b')$ as with $b = 8$ and $b' = 12$.

(Plato)

$$b \to b' = b + 4,\ a \to a' = a + (b + b') = a + 2b + 4,\ c \to c' = c + (b + b') = c + 2b + 4$$

Finally, a Fermat member $[a\ b\ c]$ is represented in the fraction sequence as $\frac{a+b}{c}$, or since we have assumed that $a + b = 2x + 1$, this fraction equals $\frac{2x+1}{c}$.

Generating two further steps by our simple rule produces

$$\frac{a+b}{c} \to \frac{a+b+2c}{a+b+c} \to \frac{3a+3b+4c}{2a+2b+3c} = \frac{6x+3+4c}{2a+2b+3c}.$$

We want to see this last fraction as $\frac{a'+b'}{c'}$, where $[a'\ b'\ c']$ is the very next Fermat triple. The almost equal halves of the numerator are $3x + 1 + 2c$ and $3x + 2 + 2c$. Certainly one of these is the same as $a + 2b + 2c$ and the other is $2a + b + 2c$ and since a, a' are odd and b, b' are even, the former must be a' and the latter must be b'. Thus

(Fermat)

$$a \to a' = a + 2b + 2c,\ b \to b' = 2a + b + 2c,\ c \to c' = 2a + 2b + 3c$$

Checking the difference of the legs a', b' we find $(a' - b') = (b - a)$. In other words, the difference between the legs merely changes sign.

We have a nice little linear transformation/substitution with the following pattern of coefficients,

Fermat	a	b	c
$a' =$	1	2	2
$b' =$	2	1	2
$c' =$	2	2	3

put for convenience in a matrix-like table.

There are coefficients but no constants, unlike what we obtained for the Pythagoras and Fermat families. Therefore something wonderful happens. Let $[a\ b\ c]$ be *any* PPT, not necessarily one of the Fermat family. We test $[a'\ b'\ c']$ by putting the above expressions into the Pythagorean equation: $(a + 2b + 2c)^2 + (2a + b + 2c)^2 = (2a + 2b + 3c)^2$. This can be simplified, and behold: it turns into the given equation $a^2 + b^2 = c^2$. So applying this transformation to an arbitrary PPT obtains another (that the triple is primitive is a small lemma). Moreover, the difference between the legs changes sign only.

No special favours for Fermat

Why can we not get the same generality for the other recursions? Only the constants are stopping us, so we get rid of them!

In [Pythagoras] above, the constant 2 appears. But we know that $c - b = 1$ and thus $2(c - b) = 2$. Replace each constant two by $2(c - b)$, then proceed as before.

In [Plato] above, the constant 4 appears. Since we have $c - a = 2$, replace each four with $2(c - a)$, and proceed.

When all the shouting is over, each of those linear transformations is homogeneous (constant-free). Namely, just a linear combination of a, b, c. Again the coefficients can be put in a table. For the Pythagoras and Plato families we have

Plato	a	b	c
$a' =$	-1	2	2
$b' =$	-2	1	2
$c' =$	-2	2	3

Pythagoras	a	b	c
$a' =$	1	-2	2
$b' =$	2	-1	2
$c' =$	2	-2	3

Surprise: this is just the previous table, except that the first column (the second column) has changed sign. The three similar transformations apply to any primitive triple $[a\ b\ c]$ to create three immediate successors, or 'children'. This generates an infinite ternary tree.

The three children can be arranged in several orders, but we prefer to put the Pythagoras-like child on the right $c' - b' = c - b$, the Plato-like child on the left $c' - a' = c - a$, and the Fermat-like child squarely in the middle $b' - a' = a - b$.

The classic tree (Figure 5) is here presented like a family tree, growing downward, with triple [3 4 5] at the top, and the three families are infinite paths on the left edge (Plato), the right edge (Pythagoras) and straight down (Fermat).

[3 4 5]

↙ ↓ ↘

Plato [15 8 17] | Fermat [21 20 29] | Pythagoras [5 12 13]

↙ ↓ ↘ | ↙ ↓ ↘ | ↙ ↓ ↘

[35 12 37] [65 72 97] [33 56 65] [77 36 85] [119 120 169] [39 80 89] [45 28 53] [55 48 73] [7 24 25]

↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘ ↙ ↓ ↘

Figure 5. Classic tree.

For space reasons we will skip the lemma that every PPT different than [3 4 5] has a unique ‘parent’ with a smaller hypotenuse for which it is one of the three children. Given that, each PPT occurs once and once only on the infinite family tree.

Our next job is to exhibit a four-parameter system that may be used to construct the usual tree, or, can be used to define an entirely new tree.

Four very special parameters

In the right triangle *ABC* with opposing sides *a*, *b*, *c* there are two acute angles at *A*, *B*. Now a simple geometric diagram can be used to compute the tangents of the half angles α, β. These are found to be

$$\frac{b}{a+c}=\frac{c-a}{b} \quad \text{and} \quad \frac{a}{b+c}=\frac{c-b}{a}$$

The four parameters we need are found by reducing the half-angle tangents to simplest fractions, to obtain a pair of numerators and denominators.

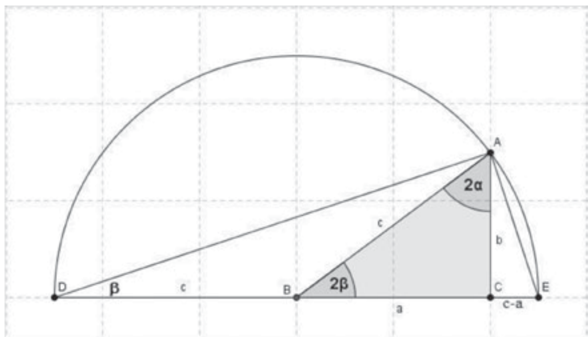

Figure 6

For example, from [21 20 29] we have

$$\frac{29-21}{20}=\frac{2}{5}$$

and

$$\frac{29-20}{21}=\frac{3}{7}$$

These four numbers happen to form a short Fibonacci sequence (3, 2, 5, 7) like the quartets we had in a previous section. We now put this aside, and develop our parameters carefully from the circle radii.

Equation $r_1(r_1+r_2+r_3)=r_2\cdot r_3$ is the gift that keeps on giving. We divide by $r_2\cdot r_3$ and get

$$\left(\frac{r_1}{r_2}\right)\left(\frac{r_1}{r_3}\right)+\left(\frac{r_1}{r_2}\right)+\left(\frac{r_1}{r_3}\right)=1$$

which has form $xy+x+y=1$ for rational numbers x, y. Let the fractions after reduction to lowest terms be $x=\frac{q}{p}$ and $y=\frac{q'}{p'}$ noting (since $r_1\cdot r_4=r_2\cdot r_3$) that

$$x=\frac{q}{p}=\frac{r_1}{r_2}=\frac{r_3}{r_4}$$

and that $$y=\frac{q'}{p'}=\frac{r_1}{r_3}=\frac{r_2}{r_4}$$

Thus, since the radii alternate in parity, $\frac{q'}{p'}$ has both parts odd, whereas $x=\frac{q}{p}$ has one part even. But $xy + x + y = 1$ produces both

$$\frac{p-q}{p+q}=\frac{q'}{p'} \quad \text{and} \quad \frac{p'-q'}{p'+q'}=\frac{2q}{2p}=\frac{q}{p}$$

Equating numerators and denominators we have a sequence of four curious parameters:

$$(q', q, p, p') = (p - q, q, p, p + q)$$

This four-term sequence obeys the Fibonacci rule: $q' + q = p$, $q + p = p'$. We omit the easy verification that all four are relatively prime in pairs, and also that any two determine all four. Moreover, since reduced

$$\frac{q}{p}=\frac{r_1}{r_2}=\frac{r_3}{r_4}$$

one must have positive integers k, m such that

$$r_1 = kq,\ r_2 = kp,\ r_3 = mq,\ r_4 = mp.$$

Any common factor of k, m must divide all r_i and thus a, b, c as well. We conclude that k, m are relatively prime. But then

$$\frac{r_1}{r_3}=\frac{k}{m}=\frac{q'}{p'}$$

forces $k = q'$, $m = p'$. In sum:

$$r_1 = qq',\ r_2 = pq',\ r_3 = qp',\ r_4 = pp' \tag{1.4}$$

In fact, one can choose any positive whole numbers q', q such that q' is odd and they have no common factors. The radii follow by (1.4) and the triple by (1.1). For example:

$$(q', q) = (3, 4) \rightarrow (p, p') = (7, 11) \rightarrow (r_1, r_2, r_3, r_4) = (12, 21, 44, 77)$$
$$\rightarrow [a\ b\ c] = [33\ 56\ 65]$$

Put $G(x, y) = [(x^2 - y^2), 2xy, (x^2 + y^2)]$. Then the PPT is equally $G(p, q)$ or $\frac{1}{2}G(p', q')$. These are just the two standard parametric solutions for obtaining triples. We can say yet more about the fractions $\frac{q}{p}$ and $\frac{q'}{p'}$. Figure 5 shows the right triangle with two acute angles 2α and 2β. The half-angle tangent were found to be

$$\tan\beta=\frac{b}{(c+a)}=\frac{(c-a)}{b} \quad \text{and} \quad \tan\alpha=\frac{a}{(c+b)}=\frac{(c-b)}{a}.$$

Recall that $a = r_1 + r_2$, $b = r_1 + r_3$, $c = r_2 + r_3$. The half-angle tangents are then

$$\frac{r_3-r_1}{r_3+r_1},\ \frac{r_2-r_1}{r_2+r_1}$$

We omit the verification via $r_1(r_1 + r_2 + r_3) = r_2 \cdot r_3$ that once again, we have $\frac{q}{p}$ and $\frac{q'}{p'}$.

That is, $\frac{r_1}{r_2}, \frac{r_1}{r_3}$.

Generation of the three children of a given triple in terms of the four new parameters is singularly simple as we now show.

Family trees using parameters

We state without proof the easy result that if triple [a b c] has associated with it the parameters (q', q, p, p') the three direct descendants of this triple (using the linear transformations given above) have parameter sequences of the form

$$(q', p, \bullet, \bullet)\ (p', q, \bullet, \bullet)\ (p', p, \bullet, \bullet)$$

In other words, q is changed to p, or q' is changed to p', or both. The last two elements in the sequence (•) have to be recomputed by the Fibonacci rule. A mnemonic for this is "flip one or both of the fractions q/p, q'/p', and save only the numerators".

This is clearer with an example. Parameter sequence (3, 2, 5, 7) goes with triple [21 20 20]. We change 3 to 7 or 2 to 5 or both at once. This gives (7, 2, …), (3, 5, …) (7, 5, …) which are completed by Fibonacci addition to

$$\begin{array}{ccc} (7, 2, 9, 11) & (3, 5, 8, 13) & (7, 5, 12, 17) \\ [77\ 36\ 85] & [39\ 80\ 89] & [119\ 120\ 169] \end{array}$$

These three triples obtained from the parameters are the three children of [21 20 29]. One may check the unchanged differences: 21 – 20 = 120 – 119, 29 – 20 = 89 – 80, 29 – 21 = 85 – 77.

Recall that $r_1 = qq'$ is the inradius. In the three children this has changed to pq', qp', pp'. So each exradius r_2, r_3, r_4 is 'promoted' in turn to be the new inradius. This implies that no matter how we pick a path down the tree, we are always increasing the inradius.

It is a bit uncertain who first realized that PPTs had a beautiful family tree, but a 1934 paper in Swedish could be the earliest (Berggren, 1934).

One of us (Price, 2008) played with the sequence and found a totally new tree. Start by dropping parameters q, p. We are left with (q', •, •, p'). Move the last parameter p' left in the sequence in three ways: (q', •, p', •), (p', q', •, •), (q', p', •, •). Re-compute the missing two parameters. It turns out that this is a valid and different way to produce three direct descendants!

To explain the following diagram, we need to say that the small box (we call it a Fibonacci Box) displays the generating parameters (half-angle tangents) in the left and right columns. The parameter sequence is coiled: start in the upper right corner and read counterclockwise. One may notice that for the

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\ [3\ \ 4\ \ 5]$$

$$\swarrow \qquad \downarrow \qquad \searrow$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\ [5\ \ 12\ \ 13] \qquad \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\ [15\ \ 8\ \ 17] \qquad \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}\ [7\ \ 24\ \ 25]$$

$$\swarrow\ \downarrow\ \searrow \qquad \swarrow\ \downarrow\ \searrow \qquad \swarrow\ \downarrow\ \searrow$$

$$\begin{array}{ccccccccc} \begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix} & \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} & \begin{bmatrix} 5 & 1 \\ 6 & 11 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} & \begin{bmatrix} 3 & 5 \\ 8 & 11 \end{bmatrix} & \begin{bmatrix} 5 & 3 \\ 8 & 13 \end{bmatrix} & \begin{bmatrix} 6 & 1 \\ 7 & 13 \end{bmatrix} & \begin{bmatrix} 1 & 7 \\ 8 & 9 \end{bmatrix} & \begin{bmatrix} 7 & 1 \\ 8 & 15 \end{bmatrix} \\ [9\ 40\ 41] & [35\ 12\ 37] & [11\ 60\ 61] & [21\ 20\ 29] & [55\ 48\ 73] & [39\ 80\ 89] & [13\ 84\ 85] & [63\ 16\ 65] & [15\ 112\ 113] \\ \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow & \swarrow\downarrow\searrow \end{array}$$

Figure 7. The new tree (three generations).

curious sequence of fractions (Fermat family) each quartet can be rotated 90° to be a Fibonacci Box for a family member. This extracts the Fermat triples in a different way from the sequence $\{Q_n / P_n\}$. Incidentally, anyone wishing to learn more about the "Pell" sequences $\{P_n\}$ and $\{Q_n\}$ may consult the *Fibonacci Quarterly*, which regularly discusses them.

We have become accustomed to using the Fibonacci Box in preference to the 'line' version (q', q, p, p').

$$\begin{bmatrix} q & q' \\ p & p' \end{bmatrix}$$
$$[a \quad b \quad c]$$

It serves as an excellent mnemonic, and a summary of the intimate interplay of circles, triples, half-angle tangents, and parameters.

In summary:

- The two columns contain the two common parameter pairs used by many authors to generate triples, using the function G(•, •) given earlier.
- The two column ratios give the half-angle tangents.
- The column products are $b/2$, a (~ legs of the right triangle).
- The two row products and two diagonal products are the four radii for equicircles and equally for tangent circles (1.4).
- Hypotenuse c is the sum of the diagonal products or the difference of the row products (1.1), (1.2).

Simple manipulations of the box information finds the three children, for either the classic family tree, or the new tree.

These transformations are easy to reverse if one wants to go up either tree.

Given a sequence (q', q, p, p') for a triple, we have calculated sequences for the children via the new tree, as indicated here (see how one parameter doubles).

$$(q', 2q, p', \bullet);\ (q', p', 2p, \bullet);\ (p', q', 2p, \bullet)$$

Some 'geometric' conclusions may be drawn. In the first case, $b \to 4r_3$, $r_1 \to 2r_1$, $r_2 \to a$. In the second case, $r_1 \to r_4$, $b \to 4r_2$, $r_3 \to 2r_4$. (Notice the pronounced similarity.) In the last case it is still very similar: $r_1 \to a$, $b \to 4r_4$, $r_3 \to 2r_3$. We have no idea if this can be integrated into a larger geometric picture.

There are also three matrices for the new tree. Here is a comparison of them with the matrices of the Classic tree. The labels A, B, C denote *left child*, *middle child*, and *right child* according to our preferred ordering of descendents.

$$\begin{array}{ccc} A & B & C \\ \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \end{array} \quad \text{CLASSIC}$$

$$\begin{array}{ccc} A & B & C \\ \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \end{array} \quad \text{NEW}$$

There are more interesting linkages between equicircles and tangent circles that go beyond the scope of this article. Interested persons may consult our *arXiv* paper.

Here are some challenges for the reader concerning the new tree, to conclude our visit to the "Garden of Pythagoras".

1. Show that the three children of any triple of the Pythagoras class include two more of the same class(A, C) and one of the Plato class (B).
2. Infer the locations of all Pythagoras and Plato triples (from a birds-eye view).
3. Find a triple which has identical positions on both trees (other than [3 4 5]of course).
4. The Classic tree has many geometric interpretations. Try and find one for the new tree.
5. Suppose that [a b c] is *not* Pythagorean, i.e., $a^2 + b^2 - c^2$ is *not* zero. Then that nonzero value is preserved by a classic matrix, and multiplied by four using a new tree matrix.

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