# On Vieta's formulas and the determination of a set of positive integers by their sum and product

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In Years 9 and 10 of secondary schooling students are typically introduced to quadratic expressions and functions and related modelling, algebra, and graphing. This includes work on the expansion and factorisation of quadratic expressions (typically with integer values of coefficients), graphing quadratic functions, finding the roots of quadratic equations and relating these to horizontal axis intercepts of corresponding graphs. For example, given the quadratic expression $x^2 - 5x + 6 = 0$, students would attempt to factorise by seeking two integers $m$ and $n$ such that $m + n = -5$ and $m \times n = 6$, and thus express $x^2 - 5x + 6 = 0$ in the form $(x + m)\cdot(x + n)$. In this case $m = -2$ and $n = -3$ or $m = -3$ and $n = -2$, hence $x^2 - 5x + 6 = 0 = (x - 2)(x - 3)$. The roots of the equation $x^2 - 5x + 6 = 0$ are $x_1 = 2$ and $x_2 = 3$ and they correspond to the horizontal axis intercepts (2, 0) and (3, 0) of $q(x) = x^2 - 5x + 6 = 0$ as shown in Figure 1.

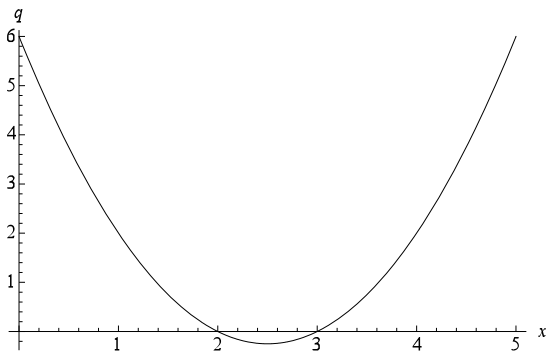


*Figure 1. Part of the graph of $q(x) = x^2 - 5x + 6$ showing horizontal axis intercepts.*

This is subsequently extended to consideration of quadratic expressions of the form $ax^2 + bx + c$ and related functions and equations, typically where $a$, $b$ and $c$ are integers and $a$ is non-zero. This includes application of the quadratic formula for identifying the roots of $ax^2 + bx + c$ such that

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

where $$x_1 + x_2 = \frac{-b}{a} \text{ and } x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Classroom experience suggests that students find Vieta's formulas

$$x_1 + x_2 = \frac{-b}{a} \text{ and } x_1 \cdot x_2 = \frac{c}{a}$$

named after the French mathematician Franciscus Vieta (in French, François Viète; 1540–1603), to be a very appealing way of finding the roots $x_1$, $x_2$ of a quadratic equation $ax^2 + bx + c$ with real coefficients $a \neq 0$, $b$ and $c$ and non-negative discriminant. That is particularly true when the roots $x_1$ and $x_2$ are positive integers.

In the senior secondary years, when students extend their study to polynomial functions, they come across the *rational root theorem* (see www.wikipedia.org). For example, the rational solutions of the equation

$$x^3 - 10x^2 + 31x - 30 = 0$$

must be among ±1, ±2, ±3, ±4, ±5, ±6, ±10, ±15, ±30 which are the integer factors of the constant term –30. Figure 2 shows part of the graph of $q(x) = x^3 - 10x^2 + 31x - 30$ with the horizontal axial intercepts identified.

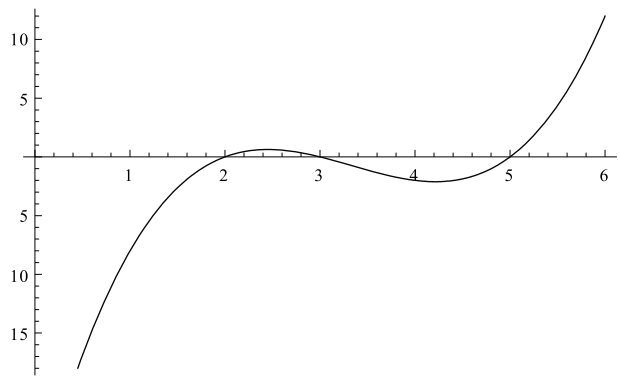


*Figure 2. Part of the graph of $q(x) = x^3 - 10x^2 + 31x - 30$ showing horizontal axis intercepts.*

Letting $x = 1$ in $q(x)$, we can see that $q(1) = -8$. Thus 1 is not a root of the equation. We can also see that by inspecting the graph of $q(x)$. We can therefore shorten the list of the remaining values. Thus by substituting $x = t + 1$ we obtain the polynomial in $t$,

$$q(t) = t^3 - 7t^2 + 14t - 8$$

with constant term –8. Therefore the acceptable values of $t$ are 1, –1, 2, –2, 4, –4, 8 and –8. Using the formula $x = t + 1$, we see that the corresponding values of $x$ are 2, 0, 3, –1, 5, –3, 9, and –7. Values that do not occur on both lists are excluded and the list of the possible roots shrinks to –1, 2, 3, –3, 5.

For a general polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$, with real or complex coefficients and $a \neq 0$, it is well known that the immediate generalisations

$$x_1 + x_2 + \ldots + x_n = \frac{-a_{n-1}}{a_n} \quad \text{and} \quad x_1 \cdot x_2 \ldots x_n = \frac{(-1)^n a_0}{a_n}$$

of Vieta's formulas for a quadratic equation are not sufficient for determining the roots $x_1, x_2, \ldots, x_n$ of the polynomial equation. A good account of Vieta's formulas for general polynomial equations can be found, for example, in Wikipedia (see www.wikipedia.org) and Mathworld (see mathworld.wolfram.com/VietesFormulas.html). Nevertheless, the roots $x_1, x_2, \ldots, x_n$ of $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ are contained in the solutions of the system of equations

$$x_1 + x_2 + \ldots + x_n = \frac{-a_{n-1}}{a_n} \quad \text{and} \quad x_1 \cdot x_2 \ldots x_n = \frac{(-1)^n a_0}{a_n}$$

It is therefore of interest to see if, in analogy with the quadratic case, a simple algorithmic method can be found that will allow us to find the integer solutions $x_1, x_2, \ldots, x_n$ of

$$x_1 + x_2 + \ldots + x_n = \frac{-a_{n-1}}{a_n} \quad \text{and} \quad x_1 \cdot x_2 \ldots x_n = \frac{(-1)^n a_0}{a_n}$$

## Formulation of the problem

The problem described in the previous section can be stated as follows:

Given positive integers $S$, $P$ and $n$, find positive integers $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{n} x_i = S \quad \text{and} \quad \prod_{i=1}^{n} x_i = P$$

Here S and P play the role of $\frac{-a_{n-1}}{a_n}$ and $\frac{(-1)^n a_0}{a_n}$ respectively.

For example, the polynomial equation

$$x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96 = 0$$

has $n = 6$, $S = 17$ and $P = 96$. Its graph is shown in Figure 3.

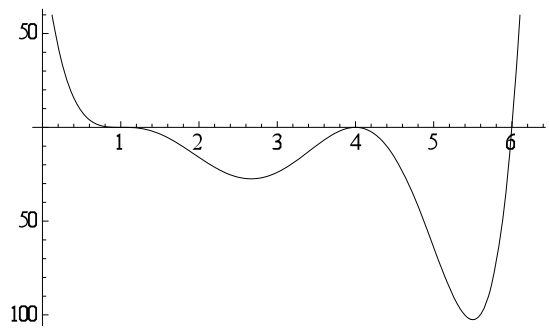


*Figure 3. Part of the graph of $q(x) = x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96$ showing horizontal axis intercepts.*

## The algorithm

Without loss of generality, in order to avoid solutions $x_1, x_2, \ldots, x_n$ that are permutations of one another, we assume that $x_i \geq x_{i+1}$ for all $i \in \{1,2,\ldots,x_{n-1}\}$. Since $x_1 = \max_{1 \leq i \leq n} x_i$, that is, $x_1$ is the bigger one of $x_1, x_2, \ldots, x_n$, we have that $S \leq nx_1$ which implies that

$$x_1 \geq c\left(\frac{S}{n}\right)$$

where $c\left(\frac{S}{n}\right)$ denotes the smallest integer greater than or equal to $\frac{S}{n}$.

Moreover,

$$S = x_1 + \sum_{i=2}^{n} x_i \geq x_1 + \sum_{i=2}^{n} 1 = x_1 + (n-1) \Rightarrow S - (n-1) \geq x_1 \;\Rightarrow\; S - n + 1 \geq x_1$$

Thus
$$S - n + 1 \geq x_1 \geq c\left(\frac{S}{n}\right) \tag{1}$$

Clearly, $x_1$ is a factor of $P$. For such an $x_1$ we can calculate $x_2$ by letting

$$S_1 = S - x_1,\; P_1 = \frac{P}{x_1} \text{ and } n_1 = n - 1.$$

As above,
$$S_1 - n_1 + 1 \geq x_2 \geq c\left(\frac{S_1}{n_1}\right) \tag{2}$$

$x_2$ is a factor of $P_1$. Proceeding in this way, we can locate $x_3, x_4, \ldots, x_n$. This procedure allows us to find all possible values of $x_1, x_2, \ldots, x_n$.

## An example

We illustrate the algorithm described in the previous section with the polynomial presented earlier, namely,

$$x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96 = 0.$$

Since $S = 17$, $P = 96$, and $n = 6$, to find $x_1$, we first notice that $x_1$ is a factor of 96, that is, $x_1$ is one of 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, or 96.
Let $x_1 = 1$, then

$$S_1 = S - x_1 = 17 - 1 = 16,\; P_1 = \frac{P}{x_1} = \frac{96}{1} = 96 \text{ and } n_1 = n - 1 = 5$$

If $x_1 = 1$ then $x_2$ is a factor of 96, that is, $x_2$ is one of 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, or 96 but since we had assumed that $x_i \geq x_{i+1}$ the only permissible value for $x_2$ is 1. Let $x_1 = 1$, then according to (2) $x_2$ must be

$$(S_1 - n_1 + 1 = 16 - 5 + 1 =)12 \geq x_2 \geq \left(c\left(\frac{S_1}{n_1}\right) = c\left(\frac{16}{5}\right) = \right)4$$

Which means that this value for $x_2$ is not accepted and we stop the further search.

Now let $x_1 = 2$ then

$$S_1 = S - x_1 = 17 - 2 = 15,\ P_1 = \frac{P}{x_1} = \frac{96}{2} = 48,\ n_1 = n - 1 = 6 - 1 = 5$$

If $x_1 = 2$ then $x_2$ is a factor of 48, that is, $x_2$ is one of 1, 2, 3, 4, 6, 8, 12, 16, 24, or 48 but since we had assumed that $x_i \geq x_{i+1}$ the permissible values for $x_2$ are 1 or 2. Let $x_1 = 1$ then according to (2) $x_2$ must be

$$(S_1 - n_1 + 1 = 15 - 5 + 1 =)\ \ 11 \geq x_2 \geq 3\ \left(= c\left(\frac{S_1}{n_1}\right) = c\left(\frac{15}{5}\right)\right)$$

which means that the values for $x_2$ are not accepted and we stop the further search.

Let $x_1 = 3$ then

$$S_1 = S - x_1 = 17 - 3 = 14,\ P_1 = \frac{P}{x_1} = \frac{96}{3} = 32,\ n_1 = n - 1 = 6 - 1 = 5$$

If $x_1 = 3$ then $x_2$ is a factor of 32, that is, $x_2$ is one of 1, 2, 4, 8, 16, or 32 but since we had assumed that $x_i \geq x_{i+1}$ the permissible values for $x_2$ are 1 or 2. Let $x_1 = 1$ then according to (2) $x_2$ must be

$$(S_1 - n_1 + 1 = 14 - 5 + 1 =)\ \ 10 \geq x_2 \geq 3\ \left(= c\left(\frac{S_1}{n_1}\right) = c\left(\frac{14}{5}\right)\right)$$

which means that the values for $x_2$ are not accepted and we stop the further search.

Let $x_1 = 4$, then working similarly we reject this value as well.

Let $x_1 = 6$ then

$$S_1 = S - x_1 = 17 - 6 = 11,\ P_1 = \frac{P}{x_1} = \frac{96}{6} = 16,\ n_1 = n - 1 = 6 - 1 = 5$$

If $x_1 = 6$ then $x_2$ is a factor of 16, that is, $x_2$ is one of 1, 2, 4, 8, or 16 but since we had assumed that $x_i \geq x_{i+1}$ the permissible values for $x_2$ are 1, 2, or 4. Let $x_1 = 6$ then according to (2) $x_2$ we reject the values 1 and 2, the value 4 for $x_2$ is accepted since

$$(S_1 - n_1 + 1 = 11 - 5 + 1 =)\ \ 7 \geq x_2 \geq 3\ \left(= c\left(\frac{S_1}{n_1}\right) = c\left(\frac{11}{5}\right)\right)$$

If $x_1 = 6$ and $x_2 = 4$

$$S_2 = S_1 - x_2 = 11 - 4 = 7,\ P_2 = \frac{P_1}{x_2} = \frac{16}{4} = 4, \text{ and } n_2 = n_1 - 1 = 5 - 1 = 4$$

Then $x_3$ must be a factor of 4, that is, $x_3$ is one of 1, 2, or 4. Making use of (1) we exclude 1 and 2 and we accept the value of 4.

If $x_1 = 6$, $x_2 = 4$ and $x_3 = 4$ then

$$S_3 = S_2 - x_3 = 7 - 4 = 3,\ P_3 = \frac{P_2}{x_3} = \frac{4}{4} = 1,\ n_3 = n_2 - 1 = 3$$

then $x_4$ must be a factor of $P_3$, that is, $x_4 = 1$ which is acceptable since

$$(S_3 - n_3 + 1 = 3 - 3 + 1 =) \quad 1 \geq x_4 \geq 1 \quad \left( = c\left(\frac{S_3}{n_3}\right) = c\left(\frac{3}{3}\right)\right)$$

If $x_1 = 6$, $x_2 = 4$, $x_3 = 4$ and $x_4 = 1$, then

$$S_4 = S_3 - x_4 = 3 - 1 = 2,\ P_4 = \frac{P_3}{x_4} = \frac{1}{1} = 1,\ n_4 = n_3 - 1 = 2$$

then $x_5$ must be a factor of $P_4$, that is, $x_5 = 1$ which is acceptable since

$$(S_4 - n_4 + 1 = 2 - 2 + 1 =) \quad 1 \geq x_5 \geq 1 \quad \left( = c\left(\frac{S_4}{n_4}\right) = c\left(\frac{2}{2}\right)\right)$$

If $x_1 = 6$, $x_2 = 4$, $x_3 = 4$, $x_4 = 1$ and $x_5 = 1$, then

$$S_5 = S_4 - x_5 = 2 - 1 = 1,\ P_5 = \frac{P_4}{x_5} = \frac{1}{1} = 1,\ n_5 = n_4 - 1 = 2 - 1 = 1$$

then $x_6$ must be a factor of $P_5$, that is, $x_6 = 1$ which is acceptable since

$$(S_5 - n_5 + 1 = 1 - 1 + 1 =) \quad 1 \geq x_6 \geq 1 \quad \left( = c\left(\frac{S_5}{n_5}\right) = c\left(\frac{1}{1}\right)\right)$$

Thus the solution 6-tuple is

$$x_1 = 6,\ x_2 = 4,\ x_3 = 4,\ x_4 = 1,\ x_5 = 1,\ x_6 = 1$$

and, as we said in (2), is solution of the equation

$$x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96 = 0$$

Applying the algorithm for the next values of $x_1$ we find that we have another solution for $x_1 = 8$. For $x_1 = 6$ and $x_1 = 8$ all possible 6-tuples are listed in Tables 1 and 2. In each case the solution 6-tuple is shown in bold. For $x_1 = 1, 3, 4, 12, 16, 24, 32, 48$ and $96$ there is no solution 6-tuple.
The solution 6-tuple

$$x_1 = 8,\ x_2 = 3,\ x_3 = 2,\ x_4 = 2,\ x_5 = 1,\ x_6 = 1$$

corresponds to equation

$$x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96 = 0.$$

Figure 4 shows the relevant portion of the graph of $q(x) = x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96 = 0$.

*Table 1. $x_1 = 6$.*

| | | | | | | $P$ | $S$ |
|---|---|---|---|---|---|---|---|
| 6 | 3 | 2 | 2 | 2 | 2 | 288 | 17 |
| 6 | 3 | 3 | 2 | 2 | 1 | 216 | 17 |
| 6 | 3 | 3 | 3 | 1 | 1 | 162 | 17 |
| 6 | 4 | 2 | 2 | 2 | 1 | 192 | 17 |
| 6 | 4 | 3 | 2 | 1 | 1 | 144 | 17 |
| 6 | 4 | 4 | 1 | 1 | 1 | 96 | 17 |
| 6 | 5 | 2 | 2 | 1 | 1 | 120 | 17 |
| 6 | 5 | 3 | 1 | 1 | 1 | 90 | 17 |
| 6 | 6 | 2 | 1 | 1 | 1 | 72 | 17 |

*Table 2. $x_1 = 8$.*

| | | | | | | $P$ | $S$ |
|---|---|---|---|---|---|---|---|
| 8 | 2 | 2 | 2 | 2 | 1 | 128 | 17 |
| 8 | 3 | 2 | 2 | 1 | 1 | 96 | 17 |
| 8 | 3 | 3 | 1 | 1 | 1 | 72 | 17 |
| 8 | 4 | 2 | 1 | 1 | 1 | 64 | 17 |
| 8 | 5 | 1 | 1 | 1 | 1 | 40 | 17 |

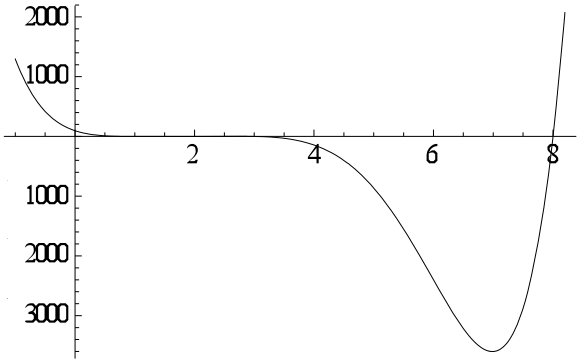


*Figure 4. Part of the graph of $q(x) = x^6 - 17x^5 + 109x^4 - 331x^3 + 494x^2 - 352x + 96 = 0$ showing horizontal axis intercepts*

## Remarks for particular values of n, S and P

1. If $n = 1$, then $x_1 = S = P$. If $n = 2$, then $S = x_1 + x_2$ and $P = x_1 \cdot x_2$ where $x_1$, $x_2$ are the roots of $x^2 - Sx + P = 0$. If $n = S$, then $P = 1$ and $x_1 = x_2 = \ldots = x_n = 1$. If $S$ is a whole multiple of $k$ then the problem has a solution only if

$$S - n + 1 \le P \le \underbrace{k \cdot k \cdot \ldots \cdot k}_{n \text{ factors}} \quad \text{where } k = \frac{S}{n}$$

2. If $S$ is not a whole multiple of $k$, then the problem has a solution only if

$$S - n + 1 \le P \le \underbrace{(k+1)\cdots(k+1)}_{n-x \text{ factors}} \underbrace{k \cdots k}_{x \text{ factors}}$$

where $k$ and $x$ can be found from

$$k \le \frac{S}{n} \le k+1 \text{ and } S + x = n(k+1)$$

For example, suppose that $S = 34$ and $n = 7$; then

$$k = 4 \le \frac{34}{7} \le k+1 = 5 \text{ and } 34 + x = 7(4+1) \text{ or } x = 1$$

Thus the maximum value of $P$ for which the problem is solvable is $P_{\max} = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 4$. If $S = 33$ and $n = 7$, then

$$k = 4 \le \frac{33}{7} \le k+1 = 5 \text{ and } 33 + x = 7(4+1) \text{ or } x = 2$$

Thus $P_{\max} = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 4 \cdot 4$.

3. If $P$ is a prime number, then in order to have a solution, $S$ must be equal to $P + n - 1$. Let $P = 13$ and $n = 4$. Since 13 is prime, the only way to have this number as a product of four numbers is $13 \cdot 1 \cdot 1 \cdot 1$ which gives a sum of $P + (n-1) \cdot 1 = 16$.
4. For specific values of $P$ and $n$, the maximum value of $S$ is $P + n - 1$. For example, let $P = 210$ and $n = 4$, then all possible 4-tuples are listed in Table 3.

*Table 3. P = 210, n = 4.*

| | | | | S |
|---|---|---|---|---|
| 210 | 1 | 1 | 1 | 213 |
| 105 | 2 | 1 | 1 | 109 |
| 70 | 3 | 1 | 1 | 75 |
| 42 | 5 | 1 | 1 | 49 |
| 35 | 6 | 1 | 1 | 43 |
| 35 | 3 | 2 | 1 | 41 |
| 30 | 7 | 1 | 1 | 39 |
| 21 | 10 | 1 | 1 | 33 |
| 21 | 5 | 2 | 1 | 29 |
| 15 | 14 | 1 | 1 | 31 |
| 15 | 7 | 2 | 1 | 25 |
| 14 | 5 | 3 | 1 | 23 |
| 10 | 7 | 3 | 1 | 21 |
| 7 | 6 | 5 | 1 | 19 |
| 7 | 5 | 3 | 2 | 17 |

As we see, the maximum value of $S$ is $P + n - 1 = 210 + 4 - 1 = 213$.