Analysing the mathematical experience: Posing the 'What is mathematics?' question

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Have your students ever wondered what mathematics is, and exactly what it is that a mathematician does? In this paper different schools of thought are discussed and compared to encourage lively classroom discussion and interest in mathematics for high achieving Form 12 students and first (or higher) year university students enrolled in a mathematics degree program. (The topic also fits well under the rationale for Queensland Senior Mathematics B Syllabus, Queensland Studies Authority, 2008.) In particular the work and views of two mathematicians, Kurt Gödel (1931) and Ian Stewart (1996), mathematician and professor Reuben Hersh (1998) and university lecturer, researcher and writer Robyn Arianrhod (2003) are used to illustrate different views of mathematics. Two documentaries are suggested for viewing by students: *Dangerous Knowledge*, relating the work and place of Gödel in the history and foundations of mathematics (Malone & Tanner, 2008), and *How Kevin Bacon Cured Cancer* (Jacques, 2008) which illustrates how mathematicians and scientists work together developing and applying mathematics.

The philosophy of mathematics

Does the mathematician 'create' an elegant theorem, or does he or she 'discover' it? Is mathematics 'mind independent' or 'mind dependent'? (VCAA, 2008). Mathematical investigations have always been connected with a critical analysis of their foundations, according to the accepted knowledge of the time. The situation at the beginning of the twentieth century was as follows, there were three main schools of thought: the logistic or realist, the formalist, and the intuitionist, see Table 1.

Of course the summary in Table 1 is a simplification, even mathematicians and philosophers may alter their views and may indeed belong to more than one camp throughout their lives, but it is useful to describe the situation at the beginning of the twentieth century. As for Gödel, Stewart and Hersh it seems reasonable to refer to the opinion of the people who actually do mathematics—mathematicians—what, in their opinion, mathematics is.

Table 1. Traditional schools of thought in the philosophy of mathematics at the beginning of the twentieth century.

	Logistic/realist	Formalist	Intuitionist
Abstract entities (e.g., numbers)	Numbers etc. exist in and of themselves.	Numbers etc. and their manipulations are 'games with marks on paper' with no real interpretation.	Numbers etc. are 'creatures of the mind'.
Mind…	independent		dependent
Definition	Mathematics is not created, mathematicians discover and describe it. Number laws can be reduced to logic alone.	Mathematics is simply a formalised manipulation of symbols according to carefully prescribed rules.	Mathematics is autonomous and self sufficient with no need of support by extended logic or rigorous formalisation.
Statements	Statements must be true or false (law of excluded middle).	Statements are neither true nor false.	Statements can be true, false or neither.

At the beginning of the twentieth century, the mathematician David Hilbert challenged mathematicians to prove that the axioms (or assumptions) of arithmetic are consistent—that a finite number of logical steps based on them cannot lead to contradictory results. A decade later the *Principia Mathematica* was published (Russell & Whitehead, 1910–1913) in which the authors attempted to prove that all mathematics is based on logic, that all pure mathematics can be derived from a small number of fundamental logical principles. But they failed to prove the consistency of arithmetic.

Kurt Gödel: Realist/Platonist

Kurt Gödel responded to Hilbert's challenge. However, his incompleteness theorems show that there are an endless number of true arithmetical statements which cannot be formally deduced from any given set of axioms by a fixed, or predetermined, set of rules of inference (Nagel & Newman, 2001). He showed that any proof that a 'formal system' is free from contradictions necessitates methods beyond those provided by the system itself. A formal system S, with a formal language L, is an *idealised model of mathematical reasoning*. It is described as *complete* if each sentence A of L either A, or its negation –A, is provable. It is said to be *incomplete*, if for some sentence A, both A and not –A are unprovable. It is described as consistent if there is no sentence A, such that both A and –A are provable in S. For more detail, see Padula (2011). He demonstrated that mathematical statements can be 'undecidable', that is, undemonstrable or unprovable within the system.

In essence, he combined an ingenious numbering system, or code, which mapped or mirrored number-theoretical statements onto their meta-mathematical translations. Meta-mathematics is an aspect of mathematical logic; it is not concerned with the symbolism and operations of arithmetic primarily, but with the interpretation of these signs and rules (Boyer, 1968). It is the study of mathematics itself by mathematical methods. The formula $x = x$ belongs to mathematics because it is built up entirely of mathematical signs, but the statement, "'*x*' is a variable," belongs to meta-mathematics because it characterises a certain mathematical sign, '*x*', as belonging to a specific class of signs, the class of variables (Nagel & Newman, 2001).

Gödel realised that a statement of number theory could be about a statement of number theory (possibly even itself), if only numbers could stand for statements. His code numbers are made to stand for symbols and sequences of symbols. Each statement of number theory, a sequence of symbols, acquires a Gödel number by which it can be referred to and in this way statements of number theory can be understood on two different levels: as statements of number theory, and also as meta-mathematical statements about number theory (Hofstadter, 1999).

Up to a point, Gödel uses an ancient paradox from philosophy called the Liar's paradox: *This statement is false* analogically, to argue that a statement from number theory can be true but not provable within the system, so that *Principia Mathematica* (PM) and related formal systems are 'incomplete' (i.e., it is just not possible to deduce all arithmetical truths from the axioms and rules of these systems). He introduces a lemma or argument which says: "This statement is unprovable," (or, more precisely: "not demonstrable using the rules of PM", Nagel & Newman, 2001), but which he then shows to be true in a formal arithmetic (Peano Arithmetic, laid down by Giuseppe Peano in the 1890s). But, if it is true it must be false and if it is false it must be true and we have a contradiction—since it cannot be both provable and unprovable. Therefore PM and related formal systems are inconsistent—not free of contradiction.

Gödel's paper showed that the axiomatic method had certain inherent limitations and he proved that it is impossible to establish the internal logical consistency of a very large class of deductive systems, number theory (formal arithmetic) being one, unless you adopt principles of reasoning so complex that their internal consistency is as open to doubt as that of the systems themselves. On the other hand his paper introduced into the study of the foundations (of mathematics) a new technique of analysis. This technique suggested new problems for logical and mathematical investigation and it provoked an investigation, still happening, of widely held philosophies of mathematics, and of philosophies of knowledge in general (Nagel & Newman, 2001). Gödel also showed the "in principle inexhaustibility" of pure mathematics, in the sense of the never ending need for new axioms or postulates (Feferman, 2006a).

Gödel's philosophy

Gödel was a mathematical Platonist. He believed that mathematics is discovered, not created. In other words he believed that mathematical concepts had an independent existence. It is well to note here that Gödel actually attributed his success not so much to mathematical invention as to attention to philosophical distinctions (Hersh, 1998).

Gödel himself wrote, describing classes and concepts, fundamental aspects of set theory:

> Classes and concepts may … be conceived as real objects … existing independently of our definitions and constructions. It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. (Gödel, 1944, p. 137)

Again, when he talks about set theory, Gödel states:

> Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception. … This, too, may represent an aspect of objective reality. (Hersh, 1998, p. 10)

Furthermore, when Gödel explained his achievements for an article by Hao Wang, he pointed out: "How indeed could one think of expressing metamathematics in the mathematical systems themselves, if the latter are considered to consist of meaningless symbols which acquire some substitute of meaning only *through* meta-mathematics" (Feferman, 1988, p. 107).

Gödel was not alone in his views; according to Goldstein (2005) mathematician after mathematician has testified, like G. H. Hardy (1940) and Paul Erdös (Barabási, 2002), to their Platonist conviction that they are discovering, rather than creating, mathematical truths. Hofstadter (1999) claims that many famous mathematicians are basically Platonist—even formalists:

> The formalist philosophy claims that mathematicians only deal with abstract symbols, and that they couldn't care less whether these symbols have any applications to or connections with reality. But that is quite a distorted picture. Nowhere is this clearer than in metamathematics. If the theory of numbers is itself used as an aid in gaining factual knowledge about formal systems, then mathematicians are tacitly showing that they believe these ethereal things called "natural numbers" are actually part of reality – not just figments of the imagination. (Hofstadter, 1999, p. 458).

Hersh (1998) holds a similar opinion. He states that Platonism was and is believed by nearly all mathematicians; like an underground religion it is: "observed in private, rarely mentioned in public" (p. 162).

Philosphical consequences

Feferman (2006b), a scholar, academic and Gödel's editor (Gödel, 1931/1986), does not think that the incompleteness theorems support mathematical Platonism. However, he concedes that Gödel himself did support Platonism from his university-undergraduate days (Feferman, 1988), although he did not become firm in his philosophical views until 1940 (Feferman, 2006a).

Feferman (2006a) believes the incompleteness theorems do not have direct philosophical consequences but that they raise questions of great philosophical interest. These questions are:

> If mathematics is to be founded on systems of axioms, on what basis are these systems to be chosen? (Whatever system is chosen, one will need further axioms to arrive at previously unprovable truths.)
> Which axioms and why? (Various answers have been proposed but none has gained universal acceptance.)
> If mathematics is about a non-physical Platonic reality, how can mathematicians gain knowledge of it? (Feferman, p. 9)

G. H. Hardy (1940) would perhaps have some suggestions here regarding "taking notes of our observations". Perhaps by looking for patterns and relationships in the world and nature and writing down these observations in a shorthand way, with mathematical notation?

However, "if it is a human creation, as formalists assert, what 'confers on it its exceptional certainty, which distinguishes it from most other areas of human thought'? … From both (realist and formalist) points of view, how come mathematics is so useful in describing the physical world?" (Feferman, 2006a, p. 9).

These are all valid and fascinating questions (even though Gödel and Feferman, 2006a, agree that *99.9 percent of mathematics follows from a small settled part of the axiomatic theory of sets*). Let us try and answer the last one, possibly the last two, and maybe some of the previous ones in the process.

A modern mathematician's view—Ian Stewart

Professor Hilton (1991) states that:

> the great areas of mathematics: algebra, real analysis, complex analysis, number theory, combinatorics, probability theory, statistics, topology, geometry, and so on—have undoubtedly arisen from our experience of the world around us, in order to systematize that experience, to give it order and coherence, and thereby to enable us to predict and perhaps control future events. However, … progress is often made with no reference to the real world, but in response to what might be called the mathematician's apprehension of the natural dynamic of mathematics itself. (Hilton, 1991, p. xxi)

According to mathematician and author Ian Stewart (1996), mathematicians neither discover nor invent mathematics (it is a bit of both and neither word is adequate), but what they do is complex, context dependent, and a mixture of invention and discovery. The world of mathematical ideas is a collection of more or less identical individual sub-consciousnesses, made similar by their common social context, and mathematics is distributed throughout the minds of the world's mathematicians, each with his or her own mathematics inside his or her head. (This is not the same as the rather vague idea of a 'world soul', Stewart adds.)

As a mathematician himself Stewart (1996) writes that it feels like discovery when you are carrying out mathematical research in a previously defined area because there is no choice about what the answer is; but when you are trying to formalise an elusive idea or find a new method, it feels more like invention. Once you have made a few assumptions or axioms, then everything that follows is predetermined by those axioms; but, Stewart continues, this summing up excludes the most crucial features: significance, simplicity, elegance, and how compelling the argument is.

Also, Stewart continues, mathematics is so powerful because it is an abstraction, an abstraction that came out of reality (2 + 2 = 4, whether it is sheep, cows, wolves, warts or witches). Since the abstraction came out of reality, it is no surprise it applies to reality.

However, Stewart continues, mathematics has an internal structure of *logical deduction* that allows it to grow in unexpected ways. New ideas can be generated internally too.

Mathematics, claims Stewart, is the art of drawing necessary conclusions, independently of interpretations: two plus two just has to be four! (Unless you are counting clouds or something equally amorphous. As Hofstadter (1999, p. 457) relates, you have to match the kind of mathematics to the patterns you are trying to find: "Mathematics only tells you answers to questions in the real world after you have taken the one vital step of choosing which kind of mathematics to apply.")

Our minds search for pattern because human minds evolved in the real world and they *learnt to detect patterns in order to survive* (Stewart, 1996). If none of the patterns detected by these minds bore any relation to the world they would not have helped their owners survive and so would have died out. He is sure that there are definitely some mathematical things in the Universe, the most obvious being the mind of the mathematician.

A physicist's view: Wigner

However, Eugene Wigner, physicist, in his 1960 paper is surprised that mathematics is applicable to reality. He claims that mathematics is the science of skilful operations with concepts and rules invented just for this purpose with the principal emphasis on the *invention of concepts*. Mathematics would soon run out of interesting theorems if these had to be formulated in terms of the concepts which already appear in the axioms. Further, whereas the concepts of elementary mathematics, particularly elementary geometry, were formulated to describe entities suggested by the actual world, the same does not seem to be true of the more advanced concepts, in particular the concepts which play an important role in physics. He argues that: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve" (Wigner, 1960, p. 11).

Mathematics as language: Arianrhod

Arianrhod (2003) agrees that mathematics is language. She states that the uncanny predictive power of physical theories like James Clerk Maxwell's on electromagnetism lies in the *difference* between mathematics and ordinary language. The differences being of: precision in its measurement of quantity; its form: the symbolism that enables you to see at a glance, patterns and generalities, similarities and differences; the *economy of thought* arising from that symbolism; and its *linguistic structure* which seems to reflect hidden, often unimaginable physical structures. She gives the example of Einstein's theory of relativity: the language, in the form of the equation $E = mc^2$ came first; only later did experimental physicists discover that it described something real.

The humanist view: Hersh

Hersh (1998), mathematics professor and humanist, argues that mathematics is a social-historical-cultural phenomenon without need of Platonism or formalism or intuitionism, and that mathematical 'objects' are also *processes*—they change. The rules of mathematics, like those of language, are historically determined by the workings of society that evolve from the inner workings and interactions of social groups, the physical and biological environment of earth, and simultaneously by the biological properties, especially the nervous systems, of individual humans. Those biological properties and nervous systems have permitted us to survive on earth, so of course they somehow reflect the physical and biological properties of this planet.

In illustration of how these processes, and consequently our view of mathematical objects change, Hersh describes how our perception of 2 has altered over the centuries. For Pythagoras, God was 1, 2 was the 'female principle'. Today 2 is no longer only a counting or natural number it is: an integer, a rational number, an element in a dense ordered set and a point in the complex plane.

Hersh reports that some mathematicians say that if history started all over, mathematics would evolve in much the same way, in much the same order. The opportunities and questions arise from what we know, and decide what advances are made.

Working together with scientists, physicists

An interesting case in point is James Clerk Maxwell's mathematical expression of Michael Faraday's electro-magnetic fields. Michael Faraday thought nature contained more complex patterns than the 'Newtonian' forces (of gravity) and fortunately for the theory of electromagnetism, so did mathematics. As Maxwell showed, a relatively new branch of calculus, and some new mathematical objects called 'vectors', turned out to be perfect for describing Faraday's field idea, and for summarising the new science of electromagnetism. But even this complex new language had been derived from the simple patterns of arithmetic, in a historical and intellectual process (Arianrhod, 2003; see also the Appendix).

However, sometimes mathematicians develop mathematics at the same time without working with scientists or other mathematicians.

Synchronicity

Hersh (1998) recalls that Igor Shafarevich, an algebraic geometer, said that mathematicians do not make mathematics, *they are instruments for mathematics to make itself.* Although this theory sounds strange it is supported, Hersh remarks, by many examples of repeated or simultaneous discovery.

Certainly, it is possible that two mathematicians could arrive at the same solution to a mathematical problem at (approximately) the same time in different countries without communicating with each other—in the same way that scientists working in different places can make a scientific breakthrough, or create new technologies (such as the telephone) simultaneously. The mathematician Gottfried Leibniz devised the calculus independently and around the same time as Sir Isaac Newton (Hawking, 2005). In a similar way logarithms were devised by John Napier, a Scotsman, in 1614, and by Joost Burgi, a Swiss, in 1620. Napier's approach was algebraic and Burgi's, geometric (Marcus, 2008; Pont, 2007). Hersh (1998) gives many more examples of mathematical synchronicity.

Conclusion

Today some mathematicians still call themselves formalists or constructivists (constructivists only accept mathematics that is obtained from the natural numbers by a finite construction). In philosophical circles one hears more often of Platonists versus fictionalists. Fictionalists reject Platonism: they can be formalists, constructivists, or something else (Hersh, 1998).

These days mathematical philosophers are not easy to classify. They study questions and results from different, partially contradictory standpoints as they try not to overstep the perimeters of widely accepted knowledge. The study of the foundations of mathematics has however contributed in a practical way with, for example, the theory of algorithms which has led to computer languages such as ALGOL and FORTRAN (and many others), and the theory of formal systems and computation.

As Davis and Hersh (1981) ask, perhaps we need to rethink our aims for mathematical philosophy.

> Do we really have to choose between a formalism that is falsified by our everyday experience, and a Platonism that postulates a mythical fairyland where the uncountable and the inaccessible lie waiting to be observed by the mathematician whom God blessed with a good enough intuition? It is reasonable to propose a different task for mathematical philosophy, not to seek indubitable truth, but to give an account of mathematical knowledge as it really is—fallible, corrigible, tentative, and evolving, as is every other kind of human knowledge. Instead of continuing to look in vain for foundations, or feeling disoriented and illegitimate for lack of foundations, we have tried to look at what mathematics really is, and account for it as a part of human knowledge in general. We have … to reflect honestly on what we do when we use, teach, invent, or discover mathematics. (Davis & Hersh, 1981, p. 406)

Davis and Hersh's comment has implications for teaching mathematical philosophy.

Implications for teaching

Firstly, teachers reflecting on what they do when they teach mathematics need to realise that how they themselves define mathematics *is* important. Do they have a view of mathematics that is sufficiently broad, of a mathematics that is evolving, and a firm conviction that mathematics, including mathematical philosophy, is inherently interesting?

Secondly, upon reflection, teachers will realise that mathematics education begins and proceeds in language (with the learning of basic concepts by pre-school-age children (Padula & Stacey, 1990) and the mathematics children learn in primary school); it advances or stumbles because of language, and its outcomes are often assessed in language (Durkin & Shire, 1991)—and, it *is* language, a highly abstract, symbolic, economical and ubiquitous language (Patel, 2008). That part of philosophy adapted by Gödel in the incompleteness theorems is expressed in a linguistic device, a paradox. Moreover, as Arianhrod (2003) remarks, we define ourselves as a species mainly by our ability both to *create abstract languages* and to *appreciate patterns*, so mathematics is the *quintessential expression of a defining aspect of what it means to be human*.

Thirdly, along with the knowledge that mathematics is language, teachers need to (if they do not already) realise the importance of communication skills in mathematics teaching and learning. Not only must the teacher be able to communicate and illuminate the topic, but also students themselves must learn to communicate what they know—and not just at examination time. As Sillence (2008) comments: the mathematics graduate, still excited by the power, beauty and utility of mathematics, no matter how brilliantly they have solved the problem in industry, must be able to communicate both the relationship between the mathematically-posed problem and the industrially-posed parent problem and how the solution materially affects the price of the resultant product or process. Students will surely be willing to try to improve their ability to communicate mathematical ideas when they realise this. Discussing mathematical philosophy is a good way to learn how to communicate mathematical ideas.

As for teaching mathematical philosophy to upper-high-school students, Sriraman (2003, 2004) found that even 13–14 year old students could discuss mathematical philosophy at an elementary level. (They were evenly divided between the Platonist, or realist camp—mathematics is discovered, mind independent—and the Formalist view—mathematics is invented, mind dependent.) Some of our more thoughtful students wonder about these questions; it is up to teachers to kick-start the discussion and make them want to engage more passionately in the study of both pure and applied mathematics.

The two documentaries previously mentioned are an excellent way to start the journey. The first (Malone & Tanner, 2008) will help them to understand Gödel's place in the study of the foundations of mathematics and the history of human thought, and the second (Barabási, 2002; Jacques, 2008; also see Appendix) will assist them to gain an understanding of the way mathematicians and scientists, working collaboratively and learning from each other's data and published papers, formulated an important part of modern-day mathematics. A viewing of these documentaries at the beginning (or end) of a course will illustrate the development of mathematical theory and practice and hopefully inspire many mathematicians (and badly needed teachers of mathematics) of the future.

Appendix

In further illustration of Hersh's (1998) view that mathematics evolves historically, that the opportunities and questions arise from what we know and decide what advances are made, consider the development of graph theory, the shared knowledge supporting networks science.

Graph theory, the science of networks

In 1736 Leonhard Euler considered the problem of the Bridges of Konigsberg. Was it possible to cross all seven bridges over the Pregel River only once between four land areas? With four nodes (areas of land) and seven links (bridges) he proved it was not. This was the beginning of graph theory.

Then in 1960 Paul Erdös and Alfréd Rényi concluded that random networks showed the probability (P) that a vertex (point or node) has k links (edges) and follows a Poisson distribution (the probability of a number of events occurring in a fixed period of time), thus:

$$P(k) = \frac{e^{-\lambda}\lambda^{k}}{k!}$$

where

$$\lambda = N \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

(Barabási, Albert, & Yeong, 1999)

Erdös and Rényi's paper was followed by the work of Steven Strogatz and Australian Duncan Watts (Strogatz & Watts, 1998) who studied the synchronicity of the sound made by snowy tree crickets, followed by a study of the network of Hollywood movie stars. They were inspired by the idea of 'six degrees of separation': if a person is one step away from each person they know and two steps away from each person who is known by one of the people they know, then everyone is an average of six 'steps' away from each person on earth (Guare, 1990). They devised a 'small-world' model where N vertices form a one-dimensional lattice, each vertex being connected to its nearest and next-nearest neighbours. With probability p each vertex is reconnected to a vertex chosen randomly (Strogatz & Watts, 1998).

Meanwhile, Hungarian physicist Albert-Laszlo Barabási was studying the World Wide Web and he and graduate student Réka Albert discovered 'hubs' —nodes that had more connections than others—after reading Strogatz and Watts' (1998) paper and borrowing their data on the Hollywood network. These hubs, formed through *growth* and *preferential attachment* were governed by a *power law,* not the well-known bell curve of distribution. (In a random network the peak of the distribution implies that the majority of the nodes have the same number of links, and nodes deviating from the average are extremely rare. In a power-law distribution we see a continuous hierarchy of nodes, spanning from the rare hubs to the numerous tiny nodes.) Barabási devised an equation $P(k) \sim k^{-\gamma}$ where $P(k)$ stands for the probability that a node in the network is connected to k other nodes and where the parameter γ is the degree exponent of the resulting graph (Barabási & Albert, 1999).

Applications were found by Alessandro Vespignani, an expert on diffusion, and geneticist, Mark Vidal, who was studying cancer. Vespignani realised that highly sexually active people were *hubs* in the *small world* of sexual relations, with important implications for the spread of diseases such as HIV/AIDS. Vidal devised a map, or network, of diseases and genes, an important advance in the research into treatment of diseases such as breast cancer and individualised health treatment (Jacques, 2008).

Networks science has implications for understanding networks in economics (highly relevant today, in light of the recent economic downturn), health research, epidemiology, sociology, military strategy and saving endangered species. The story of how Andrew Wiles solved Fermat's last theorem (Singh, 2005) by building on the work of other mathematicians is yet another example of mathematics growing and evolving historically.

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