# Enhancing conceptual understanding of trigonometry using Earth geometry and the great circle

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Trigonometry is an integral part of the draft for the Senior Secondary Australian National Curriculum for Mathematics, as it is a topic in Unit 2 of both Specialist Mathematics and Mathematics Methods, and a reviewing topic in Unit 1, Topic 3: Measurement and Geometry of General Mathematics (ACARA, 2010). However, learning trigonometric ideas is difficult for students and the causes of the difficulties seem to be multifaceted and interrelated. First, trigonometric functions are perhaps students' first encounter with operations that cannot be evaluated algebraically, the kinds of operations about which they have trouble reasoning (Weber, 2005). Those introduced to the subject via triangle trigonometry, which is more comprehensible than circle trigonometry at the early stage of learning (Kendal & Stacey 1997), have (i) to relate triangular pictures to numerical relationships, (ii) to cope with trigonometric ratios, and (iii) to manipulate the symbols involved in such relationships (Blackett & Tall, 1991). To facilitate the memorisation of these ratios, students are often taught the mnemonic SOHCAHTOA with the detrimental effect that they stop trying to make sense of the work because they have a simple rule to follow (Cavanagh, 2008). Teaching triangle trigonometry before circle trigonometry also leads to students' understanding of trigonometric functions as taking right triangles, not angle measures, as their arguments (Thompson, 2008). In fact, students may never develop a coherent concept of angle measure. To a certain extent, the same can be said about teachers (Thompson, Carlson, & Silverman, 2007). As such, both teachers and students have trouble transferring to circle trigonometry (Bressoud, 2010; Thompson, 2008), which is the foundation for more advanced topics in science and mathematics, as evidenced by the fact that they have to rely on yet another mnemonic, "All Students Take Calculus," to determine the signs of trigonometric functions in different quadrants (Brown, 2005).

Some of the studies cited above also suggested remedies for the problems of their concern. Blackett & Tall (1991) employed a computer program that draws the desired right triangles to facilitate students' exploration of the relationship between numerical and geometric data. Quinlan (2004) and

Cavanagh (2008) engaged students with hands-on activities involving tangent ratios before formally introducing trigonometric functions. As echoed by Bressoud (2010), Weber (2005) introduced circle trigonometry before triangle trigonometry. He asked students to estimate the results of trigonometric functions, and posed questions that required them to reason about these functions. Thompson, Carlson, & Silverman (2007) used probing questions to help teachers realise the necessity of measuring an angle by its subtending arc length, had them use a string having the same length as the radius to measure angles and estimate sine and cosine, and used another set of questions to help them see that this angle measure leads to a coherent system of meanings in trigonometry.

In order to alleviate the aforementioned problems, we propose in this article an alternative instruction that centres around Earth geometry, a topic occupying a major portion of Unit 4, Topic 3: Time and Place 2 of Essential Mathematics in the ACARA (2010) draft. Being included in Essential Mathematics confirms its importance and usefulness in making sense of the world. While reading the article, keep in mind that it is not intended as standalone instruction in trigonometry, as it is based on our experience of supplementing the main lesson, and that it can be adopted partially, depending on the curriculum. As pointed out by the late Ralph P. Boas, Jr. (1912–1992): "there is not time for enough practice on each new topic; and even if there were, it would be insufferably dull … he gets this practice in less distasteful form by studying more advanced mathematics" (Boas, 1957). At the very least, our instruction provides students with necessary practice based on real-life situations and prepares them for more advanced courses.

In the next section, we present the visual aids to help students visualise the geometry of the Earth, followed by the instruction concerning angle measure, which is the foundation for the instruction in the remaining sections. We conclude our instruction with a method of finding the shortest distance between two points on the surface of the Earth (the great-circle distance).

## Visual aids

We employ three visual aids to help students visualise Earth geometry; a globe, Quinlan's (2006) three-dimensional model of the Earth (see Figure 1), and Ruangsuwan & Arayathanitkul's (2009) transparent spherical model made from readily available hemispherical coffee-cup lids together with angle measuring scales (see Figure 2). Quinlan's model should help students visualise the latitude and the longitude and the transparent model is used to help them build up their understanding by letting them locate specified coordinates on the model. In fact, Quinlan demonstrated the difference between paths (and distances) along a great circle and along a circle of latitude by adjusting the way great and small circles are put together. However, he did not explain how to calculate the great-circle distance.

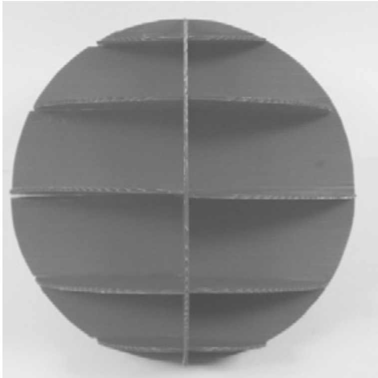

*Figure 1. Quinlan's (2006) three-dimensional model of the Earth.*

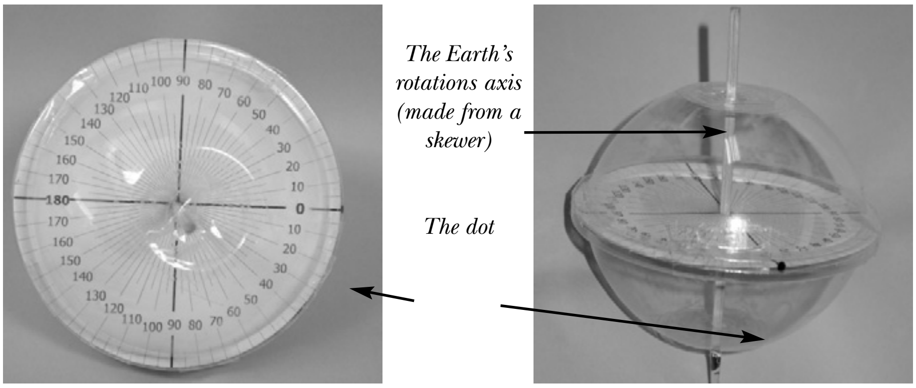


*Figure 2. A transparent model of the Earth with the dot located at latitude and longitude 0.*

## Angle measure

Students are divided into groups of three to five. The instruction begins with an activity that aims at constructing a coherent concept of angle measure. We should ask students to position the transparent model so that the skewer is vertical with the circular protractor face up, and mark the point two radians to the right of the dot using a string and the radian ruler (see Figure 3(a)) whose scale matches that of the transparent model (one radian is equal to the radius of the model) and another point one radian above the dot. The (longitudinal) angle of the first point can be read from the circular protractor and the (latitudinal) angle of the second point can be measured using the properly scaled quarter-circular protractor (see Figure 3(b)). Students should be asked to relate the angle obtained from the protractor to the length measured from the radian ruler. The degree side of the ruler should help them realise that an angle can be measured by its subtending arc length. In fact, this is the definition of an angle when one uses radian as the unit of measurement. Measuring an arc length along the surface of the model is easier and more accurate than doing it on a sheet of paper. Students should be asked to apply their understanding of angle measure to the construction of the radian ruler for the globe and repeat the measurements using the newly made radian ruler (and the globe). They should see that, with the properly scaled radian ruler, the angles remain unchanged.

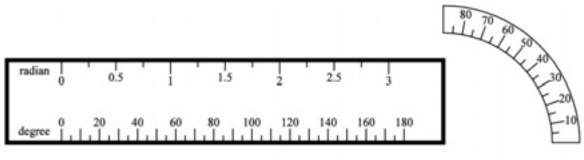


*Figure 3. (a) The radian ruler and (b) the quarter-circular protractor.*

## Latitude and longitude

From our experience, even though most students are familiar with latitudinal and longitudinal lines and the *names* of these lines (e.g., latitude 32 degrees south, longitude 128 degrees east), they usually do not realise that the word *degree* refers to the angle measure, and do not know how to measure it even among the ones with the realisation. So we should let them draw a few latitudinal lines on the transparent model and ask them how to measure the latitudinal angles of these lines. If students seem to struggle, we could remind them of the way they measure angles in the previous activity, or we could recommend using Quinlan's model as a visual aid. If students still cannot do the measurement, we could show them Figure 4 (the one without $\phi$). We should then ask them about the shape of a latitudinal line (circular), the relative sizes of these lines (decreasing from the equator to the Poles), and the characteristics of latitudinal planes (they are parallel to the equatorial plane). We should make sure that students' understanding of the latitude encompasses the angle measure in Figure 4.

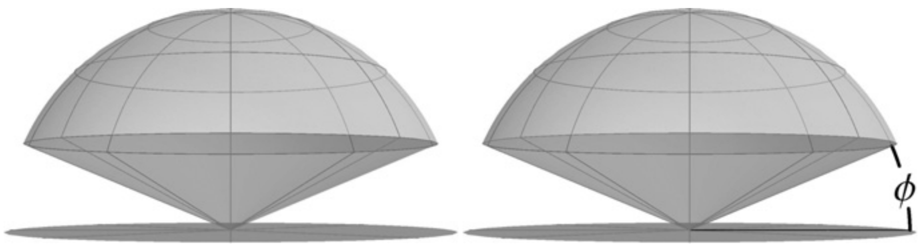


*Figure 4. Measuring a latitudinal angle.*

The activity for the understanding of longitude is similar to that for the understanding of latitude, with Figure 5 replacing Figure 4. However, keep in mind that longitudinal angles are measured relative to the Prime Meridian, which passes through Greenwich, England, and longitudinal lines are semicircles whose diameter is the line joining the North and South Poles. To provide students with an opportunity to apply their knowledge of Earth geometry, we should specify the coordinates of a number of points and ask them to locate and mark the points on the transparent model. Some of these points share the same latitude while others share the same longitude. We should then ask students to compare the distances between two pairs of points along different lines of longitude sharing the same latitudinal difference (they are equal) and vice versa (the pair closer to the equator is farther apart). We should inform them that semicircles of longitude are parts of *great circles*,

circles on the Earth surface centred at the centre of the Earth, and opposite semicircles form a great circle, whereas circles of latitude besides the equator are *small circles*.

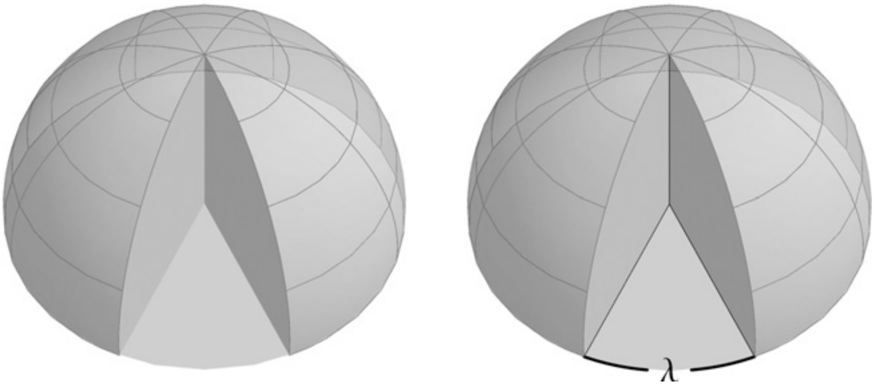


*Figure 5. Measuring a longitudinal angle.*

## Distance along a circle of longitude

Having gone through the first two activities, students should realise that they can apply the definition of angle measure to find the distance between two points along a circle of longitude (multiply the (smaller) latitudinal difference, in radians of course, by the radius of the Earth). We should use this opportunity to tell them that one *nautical mile* is the length of the arc along a great circle subtending an angle of one minute and that one *knot* equals one nautical mile per hour, and ask them about the way to calculate the distance along a circle of longitude, in nautical miles, without converting the latitudinal difference into radians (the latitudinal difference in minutes is the distance in nautical miles). Another question is to find the radius of the Earth in kilometres given that one nautical mile equals 1.852 kilometres ($1.852 \times 60 \times 180/\pi$) or to find the number of kilometres in one nautical mile given that the radius of the Earth is 6367 kilometres. A more applied problem could be to find the time it takes to fly around the Earth along the equator provided that the plane's ground speed is 500 knots ($60 \times 360/500$ hours). Of course, the solution is not exact because the Earth is not a perfect sphere and the plane has to travel at a certain altitude. However, since the radius of the Earth along the equator is around 6378 kilometres and a typical cruising altitude for a jumbo jet is about 10 kilometres (or more than 30 000 feet, already higher than Mount Everest), the answer would be off by only about a third of one per cent or less than ten minutes.

Essential Mathematics teachers should change from distance-related to time-related questions at this point. They could begin by asking about the exact time difference (if there is no time zone) between two adjacent degrees of longitude (24/360 hours or 4 minutes) before introducing the time zones, followed by time-zone-related questions like: If the match between Fulham and Everton starts at 2.30 pm on 26 December (local time), at what time should Sydneysiders watch their favourite Australian soccer players live on cable television? (On this Boxing day, England uses standard time while Sydney observes daylight saving time.)

## Distance along a circle of latitude

So far, the instruction does not involve trigonometry and thus can be adopted in all four courses. However, to find the distance along a circle of latitude, one needs to know the radius of that small circle, which can be derived from Figure 4. If $R$ denotes the radius of the Earth and $\phi$ denotes the latitude, the radius of the circle of latitude would be $R \cos \phi$. Figure 5 or Quinlan's model should help students see that the longitudinal differences are the same across all circles of latitude. The problem is then reduced to just another application of the definition of angle measure. However, they may not realise that travelling along a circle of latitude is not optimal because it is not a great circle. The next activity is designed to help them on this issue.

## The great-circle path

Students should be asked to mark on the transparent model two points that are on latitude 60° N and on two opposite longitudes, for example, 60° N 30° E and 60° N 150° W, and calculate the distances along the circles of latitude and longitude ($\pi R/2$ and $\pi R/3$). Since a circle of longitude is a great circle, the latter distance is the shortest between the two points along the Earth surface. To convince students that this is the case, we may utilise circular sectors whose radii are the same and whose central angles are 60°, 90°, and 180°. The cone made by joining the two radii of the last sector can be perfectly bisected by inserting the pointed end of the first sector into it (see Figure 6(a)). The arc of the bisecting sector represents the path along the circle of longitude while that of the ninety-degree sector inserted into the same cone represents the path along the circle of latitude (see Figure 6(b)).

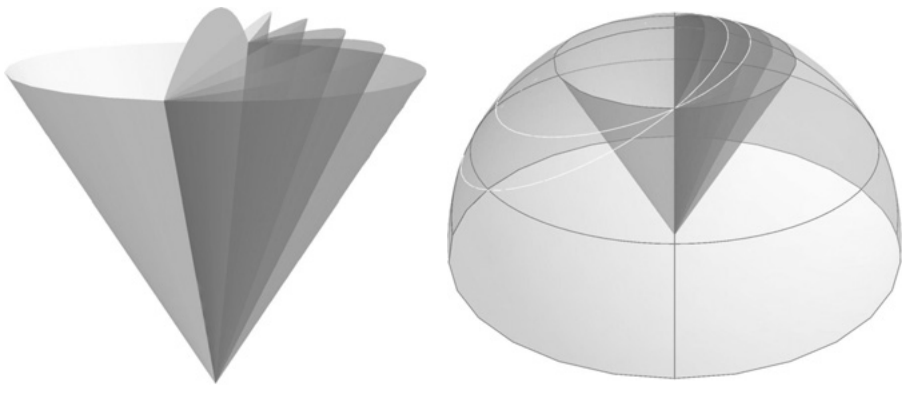

*Figure 6. (a) Circular sectors inserted into the same side of the cone and (b) the cone inserted into the enclosing hemisphere.*

Fixing the bisecting sector in the middle and inserting common-radius sectors whose central angles are 62°, 66°, 72°, and 80° into the same side of the cone gives us the cone in Figure 6(a). (A better alternative would be to use a more durable plastic cone and adjust the circular sectors accordingly.) They should see that the surface of the bisecting sector is flat while those of the others are curved, indicating that the great-circle path is the shortest one, comparable to a straight line being the shortest path between two points. Students should also notice that the arcs of the sectors outline a sphere's surface, indicating that these arcs represent paths on the surface. Since the arc lengths can be directly compared and the bisecting sector has the shortest arc, the same conclusion can be reached more concretely.

The arcs of the circular sectors other than the sixty-degree one are parts of the small circles, some of which are depicted by the white circles in Figure 6. The circumferences of these small circles are the same as those of the circles at latitudes 37° 34' 16", 50° 38' 8", 56° 40' 47", and 59° 18' 7". So if we tilt the cone until one of the arcs is horizontal, the arc of the sixty-degree sector would represent the great-circle path (and distance) between the two points on the corresponding latitude. From our experience, students can mentally construct the circular sector whose arc represents the great-circle path between any two points. Thus the problem of finding the great-circle distance between points A and B is reduced to finding the central angle, θ, of the sector in Figure 7.

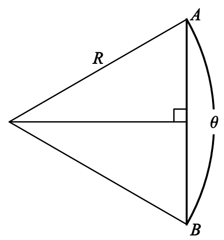


*Figure 7. The circular sector whose arc represents the great-circle path between A and B.*

## The great-circle distance

Given the distance $\overline{AB}$ (through the Earth), simple trigonometry tells us that

$$\sin\frac{\theta}{2} = \frac{\overline{AB}}{2R}$$

which implies that $$\theta = 2\arcsin\frac{\overline{AB}}{2R}$$

and since the required distance can be found from the Cartesian coordinates of $A$ and $B$, we should turn students' attention to the task of converting latitude-longitude coordinates into Cartesian ones after informing them that the origin of the Cartesian coordinate system is at the centre of the Earth, the positive part of the $x$-axis goes through latitude 0 and longitude 0, and the positive part of the $z$-axis goes through the North Pole. For the groups that cannot accomplish the task, we should tell them to first consider only the plane of latitude of point $A$ (or $B$) and try to find its $x$- and $y$-coordinates with the help of Figure 5. They should see that the longitudinal angle is the same across all parallels of latitude and thus the coordinates depend on the radius of the circle of latitude of that point, which is already known from a previous activity. The $z$-coordinate can be obtained from the height of the parallel of latitude, which can be derived from Figure 4 or from Quinlan's model. If they still have difficulties, we should show them Figure 8. In conclusion, the Cartesian coordinates of the point at latitude $\phi$ and longitude $\lambda$ are $(R\cdot\cos\phi\cdot\cos\lambda, R\cdot\cos\phi\cdot\sin\lambda, R\cdot\sin\phi)$.

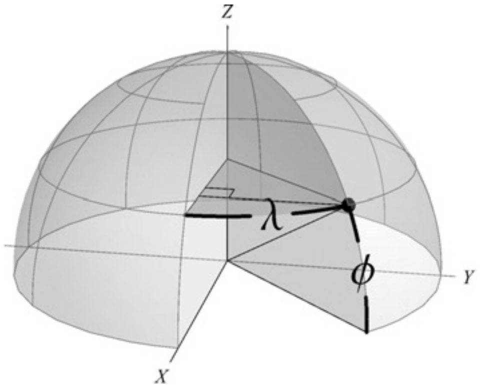


*Figure 8. Converting latitude and longitude to Cartesian coordinates.*

Suppose students find that the Cartesian coordinates of the two points are $(x_A, y_A, z_A)$ and $(x_A, y_A, z_A)$ and they seem to struggle with finding the distance between the points, we can suggest considering the horizontal distance first (between, say, $(x_A, y_A, z_A)$ and $(x_A, y_A, z_A)$) and ask whether they see the right triangle whose hypotenuse is the required distance. Given the chord length and still they have trouble finding the central angle (see Figure 7), we should draw the figure for them. Make sure that they measure the angle in radians and multiply it by the radius of the Earth to get the great-circle distance.

## Pythagoras versus the great circle

We have devised a problem to test students' understanding of great circles and to help them realise that the Earth surface covering a relatively small area can be estimated by plane geometry whereas that covering a relatively large area cannot. First, let students locate Adelaide, Canberra, and Mackay on a rectangular map wherein a line of latitude and a line of longitude are always straight and perpendicular to each other, for example, a Mercator projection, and provide them with the cities' latitudes and longitudes. We should round off and/or modify the coordinates slightly so that the three cities form a perfect right triangle on the map. Use the modified coordinates to calculate the distances between Adelaide and Canberra, and between Canberra and Mackay. Supply these distances to students and ask them to find the distance between Adelaide and Mackay. Repeat the activity using Adelaide, Tokyo, and Athens. Let students report their results to the class and discuss the resulting distances. Typically, some students will use Pythagoras' theorem to calculate the distances and the discrepancies will be small for the first group of cities but much larger for the second group. If all students happen to use the same method, the teacher would have to play the devil's advocate by showing the calculation and the result of the other method to stimulate the discussion.

## A closed-form formula for the great-circle distance

Specialist Mathematics teachers may challenge their students to find a closed-form formula for the great-circle distance between A and B using trigonometric identities. (The formula is easier to obtain using the scalar product of vectors from the origin to the two points but students should not have learnt about that yet.) If the (latitude, longitude) of $A$ and $B$ are $(\phi_A, \lambda_A)$ and $(\phi_B, \lambda_B)$, their Cartesian coordinates would be $(\cos\phi_A \cdot \cos\lambda_A, \cos\phi_A \cdot \sin\lambda_A, \sin\phi_A)$ and $(\cos\phi_B \cdot \cos\lambda_B, \cos\phi_B \cdot \sin\lambda_B, \sin\phi_B)$ after normalising the radius of the Earth to 1. Applying Pythagoras' theorem twice yields
Expanding the first parenthesis yields

$$\overline{AB} = \sqrt{(\cos\phi_A \cos\lambda_A - \cos\phi_B \cos\lambda_B)^2 + (\cos\phi_A \sin\lambda_A - \cos\phi_B \sin\lambda_B)^2 + (\sin\phi_A - \sin\phi_B)^2}$$

$$\cos^2\phi_A \cos^2\lambda_A + \cos^2\phi_B \cos^2\lambda_B - 2\cos\phi_A \cos\lambda_A \cos\phi_B \cos\lambda_B \qquad (1)$$

and the second parenthesis yields

$$\cos^2\phi_A \sin^2\lambda_A + \cos^2\phi_B \sin^2\lambda_B - 2\cos\phi_A \sin\lambda_A \cos\phi_B \sin\lambda_B \qquad (2)$$

(1) + (2) yields

$$\begin{aligned} &\cos^2\phi_A + \cos^2\phi_B - 2\cos\phi_A \cos\phi_B (\cos\lambda_A \cos\lambda_B + \sin\lambda_A \sin\lambda_B) \\ &= \cos^2\phi_A + \cos^2\phi_B - 2\cos\phi_A \cos\phi_B \cos(\lambda_A - \lambda_B) \qquad (3) \end{aligned}$$

Expanding the third parenthesis yields

$$\sin^2\phi_A + \sin^2\phi_B - 2\sin\phi_A \sin\phi_B \qquad (4)$$

Putting (3) + (4) inside the radical sign yields

$$\overline{AB} = \sqrt{2\left(1 - [\cos\phi_A \cos\phi_B \cos(\lambda_A - \lambda_B) + \sin\phi_A \sin\phi_B]\right)}$$

Since $R$ in Figure 7 has been normalised to 1, we have $\sin\frac{\theta}{2} = \frac{\overline{AB}}{2}$, which implies that

$$\theta = 2\arcsin\frac{\overline{AB}}{2} = 2\arcsin\sqrt{\frac{1 - \left[\cos\phi_A \cos\phi_B \cos\left(\lambda_A - \lambda_B\right) + \sin\phi_A \sin\phi_B\right]}{2}}$$

From the half-angle formula for sine (in our case $0 \le \theta \le \pi$),

$$\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}} \Rightarrow \theta = 2\arcsin\sqrt{\frac{1 - \cos\theta}{2}}$$

we have $\cos\theta = \cos\phi_A \cos\phi_B \cos(\lambda_A - \lambda_B) + \sin\phi_A \sin\phi_B$. Thus the great-circle distance between $A$ and $B$ is

$$R\arccos[\cos\phi_A \cos\phi_B \cos(\lambda_A - \lambda_B) + \sin\phi_A \sin\phi_B].$$

Keep in mind that students may obtain other equivalent formulas using different identities. This activity should help them realise that trigonometric identities can be useful in real-life problems whose solutions are not predetermined, unlike a typical textbook question specifying the final expression into which another expression has to be transformed.

## Conclusions

From our experience, we find that teachers of trigonometry usually do not incorporate several aspects of geography discussed in this article, even as an engagement step, into their instruction, nor do those teaching geography incorporate trigonometry into theirs. We have seen that, in order to understand latitude, longitude, distance, area, and other related topics in geography, students should at least be able to recall certain basic aspects of trigonometry. Likewise, Earth geometry provides an excellent opportunity to integrate real-life experience into trigonometric education. The integrated approach to science education in general, and to mathematics education in particular, helps bring various topics to life and heighten students' attention to an otherwise potentially difficult and uninteresting topic.

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