The logical heart of a classic proof revisited: A guide to Gödel's 'incompleteness' theorems

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In a recent newspaper article Polster and Ross (2010) decry the absence of "reasoning involving proof—the one compelling argument for teaching mathematics" in the new national draft curriculum (ACARA, 2010). The rationale for the subject Queensland Senior Mathematics B (Queensland Studies Authority, 2008) includes the aim that students should appreciate the "nature of proof" and "the contribution of mathematics to human culture and progress".

Although the study of proofs is not specifically recommended in it, the draft curriculum (ACARA, 2010) advocates as an aim the development of "thinking skills" and "creativity" in students. The study of Kurt Gödel's proof of the "incompleteness" of a formal system such as *Principia Mathematica* (Russell & Whitehead, 1910–1913) is a great way to stimulate students' thinking and creative processes and interest in mathematics and its important developments.

This paper describes salient features of the proof together with ways to deal with potential difficulties for students. It recommends the study of the logical-skeletal structure before students attempt the proof itself. It describes how students can be introduced to the proof with a documentary highlighting its importance (Malone & Tanner, 2008); two books for the 'general reader', Nagel and Newman (2001) and Frantzen (2005) are evaluated and the best description of its logical core written in clear English (Feferman, 2006c) is given. The author also suggests a prior discussion about paradoxes in mathematics with students, in particular the Richard paradox, the Liar's paradox—"This sentence is false,"—and Russell's set-theoretical paradox in the theory of classes (Hersh, 1998).

Bertrand Russell and A. N. Whitehead's *Principia Mathematica* (1910–1913), hereafter designated as PM, contained a proof that the whole of mathematics can be developed on the basis of set theory. With it they hoped to prove that all mathematics is founded on logic.

Kurt Gödel's proof (1931/1986) of the 'incompleteness' of formal systems such as PM is important for many reasons. It is important in the history of mathematics and for further developments in mathematics such as: the theory of algorithms and the theory of formal systems which has led to the development of computers and computer languages, and advances towards artificial intelligence (Hofstadter, 1999); for the evolution of mathematical proof and proof theory; and for the development of logic as it is taught today. It is interesting because to master it an understanding of language is as important as knowledge of mathematics.

A superb logician, Gödel improved the standing of logic within mathematics and was the first mathematician to prove a number theoretical truth, not by deducing it formally from the axioms and rules of a formal system, but through meta-mathematical argument (Gödel, 1931; Nagel & Newman, 2001; Feferman, 2006c). The theorems make one reflect on: the nature of mathematics, the notion of proof, and the fascinating interface of language and mathematics.

The difficulties students have in understanding proofs have been studied (Padula 2003, 2006) but not exhaustively, and although Feferman (2006b) remarks that Gödel's incompleteness paper is a "classic … elegantly organised and clearly presented", at first reading its 46 definitions and 11 propositions, the difficulties of understanding formal language, and, in part, the somewhat different symbolism Gödel uses (German letters, terms) can combine to intimidate students. It poses a challenge for teachers. How do you allow students to appreciate its logical core while not allowing them to become daunted by its technical complexities? However Solomon Feferman, editor of Gödel's collected works (1986) has written a good structural summary of Gödel's theorems (Feferman, 2006c). Feferman starts with an explanation of formal systems and Gödel's motivation in writing the proof.

Description of a formal system

To understand the proof it is essential to have knowledge of what constitutes a formal system in mathematics. A formal system S, with a formal language L, is an idealised model of mathematical reasoning. It is described as *complete* if for every sentence A of L either A, or its negation –A, is provable. It is described as *consistent* if there is no sentence A, such that both A and –A are provable. (Feferman's more detailed account is given in Appendix A.)

The incompleteness theorems

Gödel's initial aim was to provide a consistency proof of an axiomatic system, furthering the mathematician David Hilbert's program. Hilbert wanted to secure mathematics against paradoxes that had emerged at the turn of the century by axiomatising it in formal systems representing the various parts of the subject: number theory, analysis, geometry, set theory, etc. and establishing the consistency of each axiomatic system. He challenged mathematicians to prove that the axioms (assumptions, postulates) of (formal) arithmetic are consistent – that a finite number of logical steps based on them can never lead to contradictory results. Gödel's plan was to reduce the consistency of analysis to the consistency of number theory and then to prove the consistency of number theory by finitary[1] means as Hilbert recommended (Feferman, 2006a).

Gödel's results were unexpected by Hilbert and the mathematicians of the day (circa 1931). He found that formal arithmetic was neither complete nor consistent—in any formal system there always exists a statement that cannot be proven within the system even though its truth is apparent—the first incompleteness theorem. Moreover, if formal arithmetic is consistent then that consistency cannot be proven within formal arithmetic itself—the second incompleteness theorem (Hawking, 2005). The first incompleteness theorem showed that on the assumption that the system of PM satisfies the condition that Gödel called ω-consistency (omega consistency), it is *incomplete*, meaning that there is a statement in the language of the system that can neither be proved nor disproved in the system. Such a statement is *undecidable* in the system. The second incompleteness theorem established that if the system is *consistent*—meaning that there is no statement in the system that can be *both* proved and disproved—the consistency of the system cannot be shown within the system (Franzén, 2005).

Gödel's proof uses a creative slant on a paradox from philosophy called Epimenides', or the Liar's Paradox, "This sentence is false."

1. 'Finitary' means the small set of reasoning methods usually accepted by mathematicians (Hofstadter, 1980). In more detail, "[O]nly such procedures as make no reference either to an infinite number of structural properties of formulas or to an infinite number of operations with formulas" (Nagel & Newman, 2001, p. 33).

The Liar's Paradox, the paradox analogous to the diagonal lemma

The paradox that forms the basis of a crucial step in the proof, the diagonal 'lemma' or argument, is what is commonly known as the Liar's Paradox. It is altered somewhat by Gödel in the process.

In PM, Russell and Whitehead tried to remove paradoxes from mathematics. They invented an elaborate (and infinite) hierarchy of levels in a desperate search for a way to circumvent paradoxes of self-reference in mathematics (Hofstadter, 1999). Gödel tried a different approach: he used a well-known paradox intuitively, translating an ancient paradox in philosophy, Epimenides', into mathematics.

> Gödel realised he would need to formally express the concept of truth for number theoretical sentences in the language of number theory itself, but if he could do that, he would be able to produce a form of the Liar Paradox (a statement that asserts its own falsity, e.g., This sentence is false) within number theory. This would be a contradiction so his plan could not be carried out. (Feferman, 2006a, p. 6)

However, Gödel realised that since the concept of provability *could* be formally expressed in the system, a number theoretical statement called G (or A, or D, etc.) could be formed that asserts its own unprovability. Now *if it were possible to prove G in the system for number theory* it would contradict its own statement of unprovability; therefore it is indeed unprovable, and number theory is incomplete (Feferman, 2006a). As Hawking (2005) remarks, Gödel formalises the paradox in his proof. Instead of stating: "This sentence is false," he re-forms it as: "This statement is unprovable," a subtle variant that avoids the trap of absurdity.

Note that whether you assume that the statement is true or false you end up with an inconsistency, and therefore the statement is neither true nor false (Singh, 2005); it is simply "undecidable".

Gödel acknowledges the similarity of the paradox to the argument in his proof and compares it to an analogy with the Richard 'antinomy' (correctly drawn inferences, each of which is supported by reason). It is closely related to the 'Liar' too, he continues (Feferman, 1988). In 1934, three years after writing his incompleteness theorems, Gödel, along with Alfred Tarski, came to the realisation that the (Richard or Liar's) paradox can be considered a proof that a 'false statement in A' cannot be expressed in the language A:

> … a complete epistemological description of language A cannot be given in the same language A, because the concept of truth of sentences of A cannot be defined in A. It is this theorem which is the true reason for the existence of undecidable propositions in the formal systems containing arithmetic. (Feferman, 1988, pp. 104–105)

"Epistemological" means of that branch of philosophy which investigates the origin, nature, methods, and limits of human knowledge.

Many people, however, including famous mathematicians and authors, think that the most remarkable aspect of the proof is Gödel's numbering system.

Gödel's numbering system

Gödel found that numbers could replace the patterns in PM and that the patterned formulas of PM could be seen as saying things about themselves and each other. His coding enables statements of number theory to be understood on two levels: as statements of number theory, and also *statements about statements* of number theory (Hofstadter, 1999), or meta-mathematical statements. As well as symbols for constants, and variables based on the product of primes (Nagel & Newman, 2001), Gödel's numbering system includes mapping and the arithmetisation of syntax (Franzén, 2005).

Mapping

Gödel introduced an ingenious method of mapping. Since every expression of PM is associated with a particular Gödel number, a meta-mathematical statement about formal expressions and their *typographical* relations to one another may be construed as a statement about the corresponding Gödel numbers and their *arithmetical* relations to one another. In this way meta-mathematics becomes ‘arithmetised’. Each meta-mathematical statement about strings of symbols and how they are *typographically* related corresponds to a statement about the strings’ Gödel numbers and how those numbers are *arithmetically* related (Nagel & Newman, 2001).

Arithmetisation of syntax

The representation of sentences and proofs as numbers and expressing statements about sentences and proofs as arithmetical statements about the corresponding numbers is known as the *arithmetisation of syntax* and was first carried out by Gödel in his proof. A method of representing syntactical objects (such as sentences and proofs) as numbers is called *Gödel numbering*. For example, “*n* is the Gödel number of a proof in S of the sentence with the Gödel number *m*” can be defined in the language of arithmetic. Also, it is necessary that the Gödel number of any sentence or sequence of sentences can be mechanically computed and that computable properties of syntactic objects correspond to computable properties of Gödel numbers (Franzén, 2005).

According to Douglas Hofstadter (1999) Gödel’s numbering system was his “great stroke of genius” in that he realised that numbers are a universal medium for the embedding of patterns of any sort; statements supposedly about numbers alone can encode statements about other universes. He saw beyond the surface level of number theory and realised that numbers could represent any kind of structure. Computers, for example, basically manipulate numbers, and because they are a universal medium for the embedding of patterns of any sort they can deal with many different patterns.

Ordinary people use computers for: word processing, game playing, communication, animation, designing, drawing, etcetera, without ever thinking about the basic arithmetical operations going on deep in the hardware (Nagel & Newman, 2001). Gödel’s numbering system and work in logic, followed by Alan Turing’s papers and efforts at code-breaking at Bletchley Park during the Second World War, led ultimately to the development of computer languages and digital computers.

Let us now consider the logical structure, or ‘heart’ of the proof by studying Feferman’s summary of Gödel’s proof, part of a lecture at the Princeton Institute, Gödel Centenary Program.

Feferman’s summary

Feferman (2006c) presumes some previous knowledge of mathematics and his explanation is written in a mixture of good mathematical English and symbols. He uses words such as “function” and “hypothesis”. It explains Gödel’s proof concisely and it is well laid out. Once its symbols are understood it may be easier to follow for some students, especially perhaps those students whose command of English is not as good as those from an English-speaking background but who have an understanding of the relevant technical terms. Footnotes have been added by the author because Feferman does assume prior understanding of some formal-language concepts, for example: "Peano Arithmetic" (PA) and "recursive". They should help clarify (or extend) certain points for students. He precedes his summary with a brief description of the symbols he has used; they are not exactly the same as Gödel's. His symbol for "not" is "¬", not the tilde, for example.

Feferman's explanation of Gödel's incompleteness theorems:
Feferman's symbols for the formal language are:

> A symbol for 1, symbols for addition, +, and multiplication, ×, symbols for equals, =, and the less-than relations <, and symbols for the logical particles 'and' (&), 'or' (∨), 'not',(¬), implies (⇒), 'if and only if' (⇔); '*n*' and '*m*' that act as variables interpreted as referring to arbitrary (positive) integers and what are called the quantifiers 'for all *n*' (*n*) and 'there exists *n*' (∀*n*), as well as parentheses to avoid ambiguous expressions…
>
> **The first incompleteness theorem**. If S is a formal system such that
> (i) the language of S contains the language of arithmetic,
> (ii) S includes Peano Arithmetic[2]
> (iii) S is consistent
> then there is an arithmetical sentence A which is true but not provable in S.
> Gödel proved it in the following manner. He firstly showed that a large class of relations that he called recursive[3] (and we now call *primitive recursive*) can all be defined in the language of arithmetic. As well, *every numerical instance of a primitive recursive relation is decidable in PA*. Similarly for primitive recursive functions. Among these functions are: exponentiation, factorial(s), and the prime power representation of any positive integer.
> He then attached numbers to each symbol in the formal language L of S and, using the product-of-primes representation[4], attached numbers as codes to each expression E of L, considered as a finite sequence of basic symbols. These are now called the Gödel number of the expression E. In particular, each sentence A of L has a Gödel number. Proofs in S are finite sequences of sentences, and so they too can be given Gödel numbers. He then showed that the property:
>
> n is the number of a proof of A in S,
>
> written
>
> Proof_{s} (nA)
>
> is primitive recursive and so expressible in the language of arithmetic.
> (The) sentence

2. A formal version of the axioms proposed for arithmetic by the Italian mathematician Giuseppe Peano in the 1890s; its axioms assert some simple basic facts about addition, multiplication and the equality and less-than relations (Feferman, 2006c, p. 7).
3. "Recursive" means effectively computable, may be computed by a (Turing) machine. For more about recursion, read Feferman (2007) and Gödel's (1931/1986) paper (pp. 157, 159, 163) where he lists the functions, $x+y$, $x.y$ and x^y, as well as the relations $x<y$ and $x=y$ as recursive.
4. The variables in a formula are assigned Gödel numbers in accordance with the following rules: (i) each distinct numerical variable (numerals, such as ss0, or the successor of the successor of 0) is associated with a distinct prime number greater than 12; (ii) each distinct sentential variable (sentence or formula) is associated with the square of a prime number greater than 12; (iii) each distinct predicate variable (predicates such as: "is prime", "is greater than") is associated with the cube of a prime number greater than 12 (Nagel & Newman, 2001, p. 74).

(∃n) Proof $_s$ (nA)

expresses that A is provable from S.[5] Moreover if it is true, it is provable in PA. So we can also express directly from this that A is *not* provable from S, by ¬Prov. Gödel used an adaptation of what is called *the diagonal method* to construct a specific sentence, call it D, such that PA proves:

D ⇔ ¬Prov s (D).

Finally, he showed:

(*) If S is consistent then D is not provable from S.

The argument for (*) is by contradiction[6]: suppose D *is* provable from S. Then we could actually produce an n which is a number of a proof S of D, and from that we could prove in PA that "n is the number of a proof of D in S", from which follows "D is provable in S". But this last is equivalent in S to ¬D, so S would be inconsistent, contradicting our hypothesis.[7] Finally, the sentence D is true because it is equivalent, in the system of true axioms PA, to the statement that it is unprovable from S.[8]

It should be clear from the preceding that the statement that S is consistent can also be expressed in the language of arithmetic, as ¬Prov s (A & ¬A), for some specific A (it does not matter which)[9]; we write Con $_s$ for this. Then we have:

The second incompleteness theorem. If S is a formal system such that
(i) the language of S contains the language of arithmetic,
(ii) S includes PA, and
(iii) S is consistent,
then the consistency of S, Con $_s$ is not provable in S.

The way Gödel established this is by formalising the entire preceding argument for the first incompleteness theorem in Peano Arithmetic. It follows that PA proves the formal expression of (*), i.e. it proves

(**) Con $_s$ ¬Provs (D).

But by the construction of D, it follows that PA (and hence S) proves

(***) Con $_s$ ⇒ D.

Thus if S proved Con $_s$ (consistent) it would prove D, which we already know to be not the case.[10] (Feferman, 2006c, p. 5 & pp. 7–10)

Thus, S proves the consistency of S (Con $_s$) implies the statement D, and since D is not provable in S if S is consistent, it follows that the consistency of S is not provable in S under the same conditions (Feferman, 2006a). Actually Gödel (1931) designates two types of consistency in his proof: *consistency* of

5. To be more precise Proof s (nA) is here written for Proof s (n, m) where m is the Gödel number of A (Feferman, 2006c, p. 8).
6. Is D a theorem of PM? If so then it must assert a truth. But what in fact does D assert? Its own non-theoremhood. Thus from its theoremhood would follow its non-theoremhood: a contradiction. (With thanks to D. Hofstadter 1999, p. 448.)
7. D is valid in S, otherwise its negation 'This statement is provable' is valid in S; also, refer back to Feferman's (2006c, pp. 6–7) explanation of a formal system in the Appendix .
8. Now what about D being a non-theorem? This is acceptable, in that it doesn't lead to a contradiction. But D's non-theoremhood is what D asserts—hence D asserts a truth. And since D is not a theorem, there exists (at least) one truth which is not a theorem of PM. (With thanks to D. Hofstadter, 1999, p. 448.) See also Franzén (2005, p. 42) for further explication of 'equivalence'.
9. Remember, S is said to be *consistent* if there is no such sentence A such that A and not-A are provable in S (see Feferman's explanation of a formal system in the Appendix). ¬Prov $_s$ (A & ¬A) is just a symbolic expression of this.
10. For a fuller description with slightly different symbolism see Feferman (2006b, pp. 5–6) and Franzén (2005, p. 97).

the system implies non-provability of [R(q);q], the string of symbols described by him in a footnote (Gödel, 1931/1986) as merely a *meta-mathematical description* of the undecidable proposition; ω-*consistency* implies non-provability of its negation (Feferman, 1988).

It can be argued that if the consistency of the formal system could be demonstrated inside the system itself, then the informal argument could be formalised and the formalised version of the statement, "This statement is unprovable," would itself be proven, thereby contradicting itself and demonstrating the *inconsistency* of the system! (Hawking, 2005). The second incompleteness theorem shows that no sufficiently strong formal system (containing PA) that happens to be consistent can prove its own consistency. It is stated by Feferman (2006a) that the Second Incompleteness Theorem requires much more 'delicate' work than the First.

Two books for the 'general reader'

Nagel and Newman's (2001) book about Gödel's proof was written for the 'general reader'. Although it was valuable for many years it contains some errors, the most egregious being the misstatement of Gödel's first incompleteness theorem. After constructing a formula G which is true if and only if it is not provable, the authors fail to give the proof that G is not provable. They state in section C (ii) on page 93: "G is demonstrable if and only if ~G is demonstrable." As Putnam (1960) notes, this statement could have been replaced by the following argument: suppose G were demonstrable, then ~G would be demonstrable. Therefore, if arithmetic is consistent, G is not demonstrable. But the meta-mathematical statement: "G is not demonstrable" is mapped by Gödel's mapping onto the statement G itself, and moreover the mapping maps truths onto truths. So G is true, and therefore ~G is not demonstrable, assuming no falsehoods are provable in number theory (Putnam, 1960; Feferman, 2011).

Feferman's choice

Feferman (2006c) recommends a different book as an introduction to the proof for the 'general reader', Torkel Franzén's (2005): *Gödel's Theorem: An Incomplete Guide to its Use and Abuse,* and another of Franzén's books: *Inexhaustibility: A Non-exhaustive Treatment* (2004) for readers with 'a moderate amount of logical and mathematical background' (Feferman, 2006c). Franzén's (2005) book reflects his experiences over the years of reading and commenting on references to the incompleteness theorem on the Internet. Although the author himself recognises that his book will be, in part, 'heavy going' for readers not used to mathematical proofs and definitions, it does reward the persistent reader with a description of ideas such as 'provable fixpoints'.

Provable fixpoints

As Franzén explains in Section 2.7 of Chapter 2 of his book the general fixpoint construction is widely used in logic to prove various results. Gödel used it to prove his first incompleteness theorem, by applying it to the "property of not being a theorem of *S*", *S* being a formal system (like PM). By a Gödel sentence for *S* is meant a sentence *G* obtained through the general fixpoint construction such that *S* proves

G if and only if *n* is not the Gödel number of a theorem of *S*,

where *n* is the Gödel number of *G* itself.

He follows this with a description of the reasoning behind Gödel's proof (after noting that *G* is *equivalent* to the statement that no number *p* is the Gödel number of a proof of *G* in *S*, and the property of being such a number *p* is a computable one, given the general requirements on formal systems and Gödel numberings):

> First, if *G* is in fact a theorem of *S*, then it is provable in *S* that *G* is a theorem of *S* (that is, that *n* is the Gödel number of a theorem of *S*). The reason for this is that being a theorem of *S* is a property that can be verified by exhibiting a proof in *S*, and since being a proof in *S* is required to be a computable property of sequences of sentences, the verification can be carried out within *S*. So if *G* is a theorem of *S*, this is provable in *S*, but since *G* is a provable fixpoint of the property of *not* being a theorem of *S*, the negation of *G* is then also provable in *S*, so *S* is inconsistent. (Franzén, 2005, p. 42)

Implications for teaching

These days, when there are very many students in our high schools and universities from non-English-speaking backgrounds (NESB) it is incumbent upon teachers to consider the difficulties these, and indeed, even many students of English-speaking background (ESB) encounter when reading that hybrid language, mathematical English. (The new national draft curriculum requires that students be *taught* to "interpret mathematical symbols" and "understand the meaning of the language of mathematics" (ACARA, 2010, p. 5). How do they do this if they do not study good examples of mathematical English such as Feferman's summary and Franzén's book?)

Padula (2006) showed that the mix of mathematical symbols and words in a proof (often, as in this case, a mixture of natural language and the English mathematical register) can be more difficult for students than well-known mathematical symbols alone (or with few words). One reason for this may be that 'simple' everyday language is not an ideal means of communication: Gödel is reported as saying that he marvels that we ever understand each other (Goldstein, 2005); another, that mathematical English itself is complex (Padula, 2001, 2002)—like 'ordinary' language it consists of definitions whose meanings have to be assimilated, and whose grammar provides a structure for ordered thinking and communication (Arianrhod, 2003). (It can be even more difficult where paradoxes in mathematics are concerned. Paradoxes, as Goldstein (2006) relates, are catastrophes of reason where the mind is compelled by logic itself to come to contradictory conclusions. Many are self-referential—and problematical because a linguistic item refers to itself. The meaning of the words is implicit, not explicit (Padula, 2001) and it is not clear what is the referent for the phrase, “This sentence,” or Gödel’s: “This statement”.) Understanding it “seems simple, but it depends on our very complex, yet totally assimilated ability to handle English” (Hofstadter, 1999, p. 495).

Yet mathematical symbolism can convey more than words; symbolism enables you to see at a glance similarities and differences that may not be obvious if you think only in words (Arianrhod, 2003). So some students, particularly perhaps, those of non-English-speaking background, may find that a mostly verbal explanation, although clearly written like Franzén’s (2005), is not as accessible as that of Feferman (2006c), who uses formal symbolism and fewer words.

NESB students who know the technical terms may prefer Feferman’s (2006c) summary, but many ESB students will wish to extend their understanding of the proof with Franzén’s (2005) detailed explanations. Teachers may decide that both are necessary for complete understanding, are indeed complementary, and may wish to refer all their students to them. And, although it is certainly outdated and has faults, Nagel and Newman (2001) may still provide a good introduction to formal language for some students; its introductory chapters (1-6) are adequate, but with some philosophical reservations as noted by Putnam (1960). Overall, Feferman (2006c) delivers a clear and brief summary of the underlying logic of the proof.

High-achieving Year 12 students doing Specialist Mathematics and first- or higher-year university students studying mathematics, or history of mathematics, will enjoy studying Gödel’s proof—if it is introduced thoughtfully. Teachers may wish to start with the BBC Four documentary (Malone & Tanner, 2008) which shows how Gödel built upon the work of Hilbert and Russell and paved the way for Alan Turing’s papers on computers and computability. This could be followed by giving students a good summary of a formal system, such as the Appendix. (Students can be advised to keep this description in mind when studying the summary and to refer back to it as they follow the logic of the proof.) At this point a discussion about Russell’s theory of sets, the paradox he found, and the Liar’s Paradox would set the historical scene.

Lecturers and teachers may find Feferman’s (2006c) version suitable as a template for the following lecture or lesson, and students can be encouraged to read Franzén’s (2005) book for more detail. Undergraduate students can then proceed to study the original proof (Gödel’s introduction is very clear), while for Specialist-Mathematics students a more detailed study may be left till they are at university.

The Boolos proof, an alternative approach

There is no great theorem with only one proof (Hersh, 1998). Interested students would benefit from a look at a new, brief proof of Gödel's incompleteness theorem by George Boolos, a librarian at Oxford University. Its premise: *There is no algorithm whose output contains all true statements of arithmetic and no false ones.* Like Gödel's, the proof is obtained by the substitution of a name for a number in a certain crucial formula, but as Hersh comments it does not use diagonalisation (from his Notes at the back of the book).

Importance of the theorems-conclusion

Gödel's incompleteness proof is an elegant classic. He has constructed a proof which is technically innovative with its use of a unique numbering system, the ingenious mapping of the Correspondence Lemma (Proposition V) of statements of number theory with meta-mathematical statements, the arithmetisation of syntax, proof of a number-theoretical statement through meta-mathematical argument, and the diagonal lemma (with the Liar's Paradox by analogy). It is also of great significance: PM and related systems are incomplete in that there are true statements of number theory that their methods of proof are too weak to demonstrate (Hofstadter, 1999). Moreover, it shows that not just PM, but any related formal system, is *essentially incomplete.* If more axioms are added, the augmented system would still not formally yield all arithmetical truths (Feferman, 2006c). With his proof Gödel showed that provability is a weaker notion than truth, no matter what axiomatic system is used (Hofstadter, 1999).

Gödel eliminated Russell and Whitehead's dream of proving that all mathematics is based on logic, with logic, and showed that logic could be pursued mathematically with results as decisive, important and interesting as those from other branches of mathematics. In doing this he laid the groundwork for the subject of mathematical logic as we know it today (Feferman, 1988).

As Gödel showed, there is no limit to the creativity of mathematicians in devising new methods of proof (Hofstadter, 1999), and indeed, (pure) mathematics itself. No matter how many problems are solved, there will always be other problems that cannot be solved within existing rules (Feferman, 2006c).

The idea of proof has completely changed the way we think: philosophically, scientifically, medically and legally. Only in pure mathematics can it be fully achieved—beyond any doubt—because of the self-consistency of mathematical language (Arianrhod, 2003). The new Australian national curriculum should include the study of the great historical proofs and Gödel's incompleteness proof in particular—it is historically, culturally, and pedagogically so important. It is never too early to excite students about proof, the very essence of mathematics; after all, Gödel was a young man (just 25) when he accomplished his ground-breaking work and today both the subject and our understanding of it have developed to the point where the proof is not considered difficult (Franzén, 2005).

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Appendix

An introduction to formal systems. A formal system is an idealised model of mathematical reasoning. It is said to be complete if each sentence, A, in its language, either A or its negation –A, is provable. It is said to be incomplete, if for some sentence A, both A and not –A are unprovable.

A minimal criterion for the acceptance of a formal system S is that it should be consistent. If S is to be used to obtain truths about numbers and other kinds of mathematical objects, it should start from true axioms and use rules of inference that invariably lead from truths to truths. (Once we have a formal language L, we can specify a formal system S in L by telling which sentences A of L are axioms and which relations between sentences are rules of inference.) The sentence A is said to be provable in S, if there is a proof in S which ends with A. S is said to be consistent if there is no sentence A such that both A and not-A are provable in S. There are two concepts of completeness related to Gödel's theorem: (i) S is said to be formally complete for L if for every sentence A of L, either A is provable in S or not-A is provable in S. (In Gödel's terminology, every sentence L is decided by S. If S is not complete then there are undecidable L-sentences in S.) (ii) S is truth complete for L if every true sentence of L is provable in S. If S is consistent and truth complete for L it is formally complete, because each sentence A of L is either true or false, i.e. its negation is true. But S may be consistent and formally complete and not truth complete, because it may prove false sentences. It is of course stronger to prove of a given S that it is not formally complete, than that it is not truth complete, and that's what Gödel did in his original formulation. (Feferman, 2006c, pp. 6–7 & p.11)