Exploring Fourier series and Gibbs phenomenon using Mathematica

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Increasingly, in some parts of the world the use of CAS in mathematics instruction has been the subject of various research studies such as Heid (1988), Kutzler (1999) and Stacey (2001). Various issues in relation to CAS use have concerned mathematics educators, especially those related to the relevance of paper and pencil techniques in mathematics learning. Herwaarden (2001) gives a detailed description of a course in calculus and linear algebra for first year university students where an attempt was made to integrate paper and pencil techniques with computer algebra. This integration helped to enhance conceptual understanding as students were required to solve their assignments in two different ways: first using paper and pencil and then using computer algebra. The paper-pencil assignments focused only on developing the basic concepts while the computer algebra assignments were carefully designed so as to create a connection between the paper-pencil methods and computer manipulations. Mathematics educators have expressed concern over the issue that CAS use may lead to deterioration of 'by-hand' skills. Arnold (2004) reports that research on the classroom use of computer algebra strongly points to better understanding of concepts and does not lead to any loss of algebraic manipulations on the part of students. According to him, CAS gives students more control over the mathematics they are learning and the ways in which they learn it. The effective use of CAS requires greater mathematical understanding and concept development. Knowing which operations to choose, which processes to follow and interpreting the result are all higher level cognitive functions than knowing the steps of a manipulative process. Arnold describes CAS as "the ultimate mathematical investigative assistant" (p. 19) which allows the student to engage in "purposeful and strategic investigation of problems" (p. 21). Heid (2001) describes CAS as a cognitive technology which makes higher level mathematical processes accessible to students. According to her, CAS plays a dual role: that of an amplifier and a reorganiser. On one hand, CAS plays the role of an 'amplifier' by making it possible to generate a larger number and greater range of examples and thus can be used to extend the curriculum. On the other hand it serves the role of a 'reorganiser' by changing the fundamental nature and arrangement of the curriculum. This refers to a resequencing of concepts and procedures, reallocation of time traditionally spent on the refinement of paper and pencil methods to the interpretation of symbolic results and application of concepts. CAS provides an ideal environment for developing a multi-representational understanding of mathematics through its numeric, symbolic and graphic capabilities. Lagrange (1999) also suggests that CAS helps to set a balance between skills and understanding. It lightens the technical work thus allowing students to focus on concepts and applications. According to him, students using CAS can build deep links between their enactive knowledge and computable representations. Thomas and Hoon (2004) describe how some students in their study used CAS to check their procedural work while others used it for performing a procedure within a complex process to reduce cognitive load. Lindsay (1995) highlights the ability of CAS to extend student's understanding through transfer between algebraic and graphical representation of problems and in enabling visualisation as a facilitator in the learning process. He claims that the proper use of CAS can augment learning and provide a rich and motivating environment to explore mathematics.

The laboratory module described in this paper draws heavily from the above research studies. It shows how the use of CAS played the role of an 'amplifier' by making higher level mathematical concepts accessible to students of year 12. Using Mathematica students were able to visualise Fourier series of functions and explore Gibbs phenomenon which is usually a part of college mathematics. During the module, it was ensured that students acquire sufficient proficiency in calculating the Fourier series and manipulating integrals before resorting to Mathematica. Thus paper and pencil methods helped students to understand the calculations while Mathematica added meaning to the calculations by providing graphical and numerical representations. Once the technical work was taken over by Mathematica, students were free to focus on the behaviour of the graphs of the functions and this enabled them to visualise Gibbs phenomenon. They were able to write the codes, with occasional help from the teacher and the outputs of the codes helped them to observe patterns and make conjectures. The power of Mathematica in the module was not merely in the computational aspect but also in the fact that it enabled the students to engage in a purposeful and strategic investigation of the problem at hand. It helped to create a link between the symbolic expressions and the graphical representations.

The study

The study described in this paper was conducted at the Mathematics Laboratory and Technology Centre at the school where the author has been a teacher. The Centre's primary objective is to supplement regular classroom teaching (for grades 6 to 12) with innovative teaching methods many of which integrate technology. It is equipped with computer algebra systems such as Mathematica, software packages, for example, Autograph and

Geometer's Sketchpad and graphics calculators like the Casio FX 9860. It offers an optional course on Applicable Mathematics to students of grades 11 and 12, which focuses on concepts and applications and is primarily driven by technology. The study is based on a laboratory module on exploring Fourier series and Gibbs phenomenon which was undertaken with 32 year 12 students (as a part of a project to investigate the effect of integrating computer algebra with traditional teaching.) The module took five one hour sessions. Students' explorations were guided by worksheets consisting of various tasks and Mathematica codes. The worksheets required each student to show their paper and pencil calculations as well as their observations from the Mathematica outputs. At the end of the module students were required to respond to a short questionnaire and give a written feedback describing their experience in the module. The objective of the study was to reflect upon the following issues:

- Does the use of CAS give students more control over the mathematics they learn?
- How does CAS help students access higher level mathematical concepts?
- Is it really possible to balance conceptual understandings and 'by hand' skills?
- How does CAS help to sustain student's interest?
- How does CAS effect student's perception of paper and pencil skills?

Educational setting

The school, where the author was a teacher, follows the curriculum prescribed by the Central Board of Secondary Education (CBSE) which is the national board for school education in India. The CBSE does not prescribe the use of technology for teaching mathematics or permit its use in examinations. Individual schools, however, have the freedom to integrate technology in their classrooms or mathematics laboratories. Calculus forms an integral part of the syllabus as prescribed by the CBSE for the mathematics curriculum in year 12 (CBSE, 2009). This includes the topics of functions, limits, continuity, differentiation, application of derivatives, indefinite and definite integration and differential equations. The focus is on the development of paper and pencil skills. Examinations at the end of year 12 also emphasise the same. The author designed the course Applicable Mathematics to enable the student to visualise important concepts in Calculus, Linear Algebra and Probability and Statistics through the use of technology. The course was optional and was specially designed for students who planned to pursue an engineering discipline or a Bachelor degree in mathematics for their undergraduate studies. The 32 year 12 students who were a part of the study, had chosen Applicable Mathematics, which required them to complete five exploratory laboratory modules based on real world applications. These students were also undertaking a Calculus course as a part of the regular curriculum which was taught in a traditional 'chalk and board' manner, without any technology. However, while studying Applicable Mathematics, the same students were given access to technology and had undertaken a few calculus laboratory sessions in which technology played a vital role. The laboratory module on exploration of Fourier series and Gibbs phenomenon was one of the five modules which the students were required to complete. The rationale for including Fourier series is that they are used to model various types of problems in physics, engineering and biology and arise in many practical applications such as modelling air flow in the lungs, electric sources that generate wave forms that are periodic and frequency analysis of signals. In contrast to Taylor series which can be used only to approximate functions that have many derivatives, Fourier series can be used to represent functions that are continuous as well as discontinuous. The partial sums of the series, approximates the function at each point and this approximation improves as the number of terms are increased. However, if the function to be approximated is discontinuous, the graph of the Fourier series partial sums exhibits oscillations whose value overshoots the value of the function. These oscillations do not disappear even as the terms are increased. This phenomenon is referred to as Gibbs phenomenon (Libii, 2005). Thus the approximation of functions by Fourier series, near the points of discontinuity are inaccurate and this limits the use of Fourier series in some cases.

The topic of Fourier series is an integral part of mathematics courses at most undergraduate programmes in Engineering and Science. However CAS enables students to visualise the series and perform computations quite easily as will be evident in this study. The study and exploration of Fourier series have also been suggested as a theme for an application task for Specialist Mathematics (VCAA, 2002) in Victoria, the Australian state where the use of CAS is most widespread. These application tasks recommend teachers to 'use investigative, modelling and problem solving approaches that involve the use of mathematics in real life contexts.' The study described in this paper goes on to show how the investigative approach was used to make year 12 students visualise and explore Fourier series. The draft National Curriculum (ACARA, 2009), in the section on mathematics for years 11 and 12, recommends four types of courses for senior secondary mathematics. While the first two types of courses will cater to the needs of students who will pursue tertiary studies in the university with a moderate demand on mathematics, the third and especially the fourth type of course is intended for students with a keen interest in mathematics as well as those expecting to study mathematics in college. The topic of Fourier series is relevant for students of this category. The draft also recommends that digital technologies be embedded in the curriculum as they allow 'new approaches to explaining and presenting mathematics'. Using these tools the student can readily explore various aspects of the behaviour of a function numerically, graphically, geometrically and algebraically. The draft further goes on to say that 'digital technologies can make previously inaccessible mathematics accessible, and enhance the potential for teachers to make mathematics interesting to more students'. The study described in this paper fully supports this argument which clearly indicates that the study of Fourier series using a CAS or a CAS enabled calculator can be a part of the curriculum for the fourth type of course as suggested by the draft National Curriculum.

The laboratory module

This section includes a detailed description of the laboratory module in which 32 Year 12 students explored Fourier series using Mathematica. The aim is to highlight the role of Mathematica in enabling the learning process through five sessions. Mathematica helped students to verify their paper and pencil work, visualise symbolic expressions graphically, explore functions numerically and make conjectures leading to their understanding the concept of Gibbs phenomenon.

Exploring trigonometric integrals

The module began with an exploration of trigonometric integrals. Students had access to Mathematica 2.2. Typically, Mathematica commands begin with an uppercase alphabet while all other alphabets are lowercase. Also all commands use the [ ] brackets rather than the ( ).

In the first exercise of the worksheet, students were required to evaluate the following integrals by hand and then verify them using Mathematica's **Integrate** command.

$$\int_0^\pi \cos nt\ dt = 0 \text{ for } n \neq 0,\ \int_0^\pi \sin^2 nt\, dt = \frac{\pi}{2} = \int_0^\pi \cos^2 nt\, dt \text{ for } n = 1, 2, 3\ldots$$

$$\int_0^\pi \cos nt \cos mt\, dt = 0 \text{ and } \int_0^\pi \sin nt \sin mt\, dt = 0 \text{ for } n \neq \pm m$$

Integrate[Cos[n*t],t] gave the output

$$\frac{\sin[n\,t]}{n}$$

By specifying the limits of integration as t varying from 0 to π using the command **Integrate[Cos[n*t],{t,0,Pi}]** (inserting different values for n) students were able to confirm that their 'paper and pencil solution', that is, 0, was correct. Similarly **Integrate[{Cos[n*t]}^2,{t,0,Pi}]** returned

$$\frac{\pi}{2}$$

for different values of n and **Integrate[Cos[n*t]*Cos[m*t],t]** gave the output

$$\frac{m\sin[mt-nt]+n\sin[mt-nt]+m\sin[mt+nt]-n\sin[mt+nt]}{2(m-n)(m+n)}$$

which confused the students since it was different from their paper and pencil result. Some students tried to combine the sine terms in the above output manually to obtain the paper-pencil result while others tried different Mathematica commands such as **Simplify** to 'reduce' the output to the 'expected answer'. However, after a few attempts **Apart[%]** returned the expected output

$$\frac{\sin[(m-n)t]}{2(m-n)}+\frac{\sin[(m+n)t]}{2(m+n)}$$

Since Mathematica acted like a 'black box', students had to use their manipulative skills to check the validity of their own paper and pencil methods.

The above outputs can be obtained using a later version of Mathematica.

The calculations done using Mathematica 7 are as follows

$$\int \cos[n\,t]\,dt$$

$$\frac{\sin[n\,t]}{n}$$

$$\int_0^{\pi} \cos[n\,t]\,dt$$

$$\frac{\sin[n\,\pi]}{n}$$

$$\int \cos[n\,t]\cos[m\,t]\,dt$$

$$\frac{\sin[(m-n)t}{2(m-n)}+\frac{\sin[(m+n)t}{2(m+n)}$$

In Mathematica 7, the $\int$ sign can be obtained using the keyboard by pressing the Esc key, followed by typing 'int' and then pressing the Esc key again. Similarly the 'd' of 'dt' within the integral can be obtained by pressing the Esc key, followed by typing 'dd' and then pressing the Esc key.

Study of the basics of Fourier analysis

After evaluating the trigonometric integrals students were introduced to the following results related to Fourier Series.

Result 1

Any function $g(t)$ defined and continuous on the interval $(0, \pi)$ can be written as $g(t) = b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots + b_n \sin nt\ldots$ which is referred to as the *Fourier sine series.*

Result 2

The Fourier coefficients b_n are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\sin nt\ dt$$

The proof was illustrated as follows:

$$\int_0^{\pi} g(t)\sin nt\ dt$$

$$= \int_0^{\pi} b_1 \sin t \sin nt\ dt + \int_0^{\pi} b_2 \sin 2t \sin nt\ dt + \int_0^{\pi} b_3 \sin 3t \sin nt\ dt \ldots + \int_0^{\pi} b_n \sin^2 nt\ dt$$

Using the results

$$\int_0^{\pi} \sin nt \sin mt\, dt = 0,\ m \neq \pm n \ \text{ and } \int_0^{\pi} \sin^2 nt\, dt = \frac{\pi}{2},$$

all the integrals vanish except the one with coefficient b_n. Thus,

$$\int_0^{\pi} g(t)\sin nt\ dt = b_n \int_0^{\pi} \sin^2 nt\ dt = \frac{\pi}{2} b_n \quad \text{or} \quad b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\sin nt\ dt$$

Result 3

A function $f(t)$, continuous on the interval $(0,\pi)$, with period 2π can be written as

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt,\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt,\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

Evaluating the Fourier series of some elementary functions

After a preliminary study of Fourier coefficients, students were asked to evaluate the Fourier Series for $f(t) = 1,\ t,\ t^2$ and t^3 using the results illustrated in the previous section.

For $f(t) = 1$ where $0 < t < \pi$, the Fourier coefficients were obtained as

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \\ &= \frac{2}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{2}{\pi}\left[\frac{-\cos nt}{n}\right]_0^{\pi} \\ &= -\frac{2}{n\pi}\left[\cos n\pi - \cos 0\right] \\ &= \frac{2}{n\pi}\left[1-\cos n\pi\right] \end{aligned}$$

Since $\cos n\pi = 1$ when n is even and -1 when n is odd ,

$$b_n = \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Substituting these in $f(t) = \sum_{n=1}^{\infty} b_n \sin nt \ = b_1\sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots$ we get

$$1 = \frac{4}{\pi}\left[\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \ldots\right] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$$

For $f(t) = t^2$ $(0 < t < \pi)$ the students calculated the Fourier coefficients as

$$b_n = \begin{cases} \dfrac{2}{n} & \text{if } n \text{ is odd} \\ -\dfrac{2}{n} & \text{if } n \text{ is even} \end{cases}$$

and the Fourier Series for $f(t) = t$ was obtained as

$$t = 2\left[\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \ldots\right] = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$$

Similarly the Fourier series for $f(t) = t^2$ and $f(t) = t^3$ were respectively obtained as

$$t^2 = \frac{\pi^2}{3} - 4[\cos t - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \ldots] = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nt}{n^2}$$

$$t^3 = -\left(12 - 2\pi^2\right)\sin t + \left(\frac{12}{2^3} - \frac{2\pi^2}{2}\right)\sin 2t - \left(\frac{12}{3^3} - \frac{2\pi^2}{3}\right)\sin 3t + \left(\frac{12}{4^3} - \frac{2\pi^2}{4}\right)\sin 4t -$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)\sin nt$$

Plotting the partial sums of Fourier series

After evaluating the Fourier series of the above functions, exercise 3 required students to visualise how the series actually approximates the function by plotting the partial sums of the Fourier series. Students used the following Mathematica code to visualise the partial sums of $f(t) = 1$.

```
fourierseries1[k_]:=
(4/Pi)*Sum[(Sin[(2n+1)t])/(2n+1),{n,0,k}];
Plot[{1,fourierseries1[10]},{t,0,Pi},
PlotRange->{{0,Pi},{.75,1.25}}]
```

In line 2 of the above code the **Sum** command has been used to define the Fourier Series where k denotes the number of terms of the series. In lines 3 and 4 the **Plot** command is used to plot k terms of the series (here $k = 10$). Students experimented by inserting different values of k in line 3 (by varying k from 1 to 100) and observed that the Fourier series plot 'oscillated' around the actual function $f(t) = 1$. Students further modified the above program to obtain a better visual output.

```
fourierseries1[k_]:=
(4/Pi)*Sum[(Sin[(2n+1)t])/(2n+1),{n,0,k}];
Table[Plot[{1,fourierseries1[k]},{t,0,Pi},
PlotRange->{{0,Pi},{0.75,1.25}},
PlotStyle->{{Thickness[0.009],
RGBColor[1,0,0]},RGBColor[0,0,1]}],{k,1,100,10}]
```

Here the **Plot** command has been enclosed in the **Table** command in line 3 so that the output generates the plots of the partial sums as k varies from 1 to 100 in steps of 10. Also **PlotStyle** with **Thickness** and **RGBColor** options display the actual function $f(t) =1$ in red and the Fourier series plot in blue (see Figure 1).

In exercise 4 students were asked to summarise their observations from the outputs of the above program.

Graphical analysis of the Fourier series partial sums

The exercise led students to visualise and conclude that the function $f(t) = 1$ can be approximated by the terms of its Fourier series for t lying in the interval $(0, \pi)$. The larger the number of terms, the better is the approximation. The plots of the first 1, 11, 21, 31, 41, 51, 71 and 91 terms between 0 and π shown in Figure 1 (a) to (h) reveal that the approximation gets better as the number of terms are increased. The graph of the Fourier series partial sums (FSPS) approaches the constant function $f(t) = 1$. Similarly the plots for the Fourier series of $f(t) = t$ (see Figure 2) and $f(t) = t^3$ (see Figure 3) show that as the number of terms increase the plot of the FSPS comes closer to the graph of the actual function.

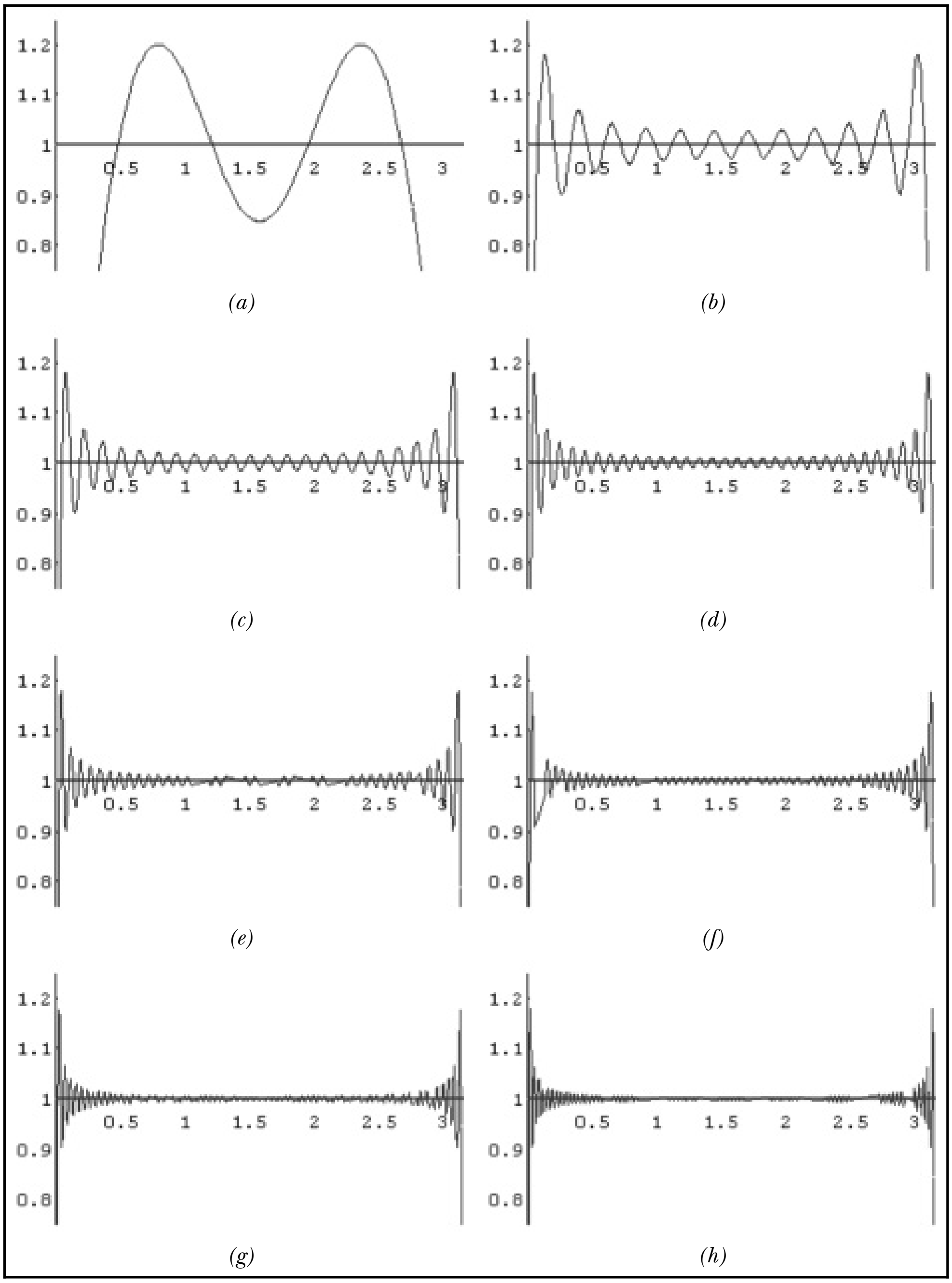

Figure 1. Plots of the FSPS of f(t) = 1 for 1, 11, 21, 31, 41, 51, 71 and 91 terms.

Students observed that the Fourier series plots tend to oscillate towards the end points of the interval $(0, \pi)$ and the peaks overshoot the function value. These oscillations persist and seem to approach the end points as the terms are increased. This was introduced as the Gibbs Phenomenon. The Fourier series gives a good approximation of the function only within the interval $(0, \pi)$.

Students used the following Mathematica code to plot the Fourier series partial sums of $f(t) = 1$, closer to the endpoint 0 (Figure 4). The **Table** command (last line of the code) was used to generate a table of values to measure the height of the highest peak.

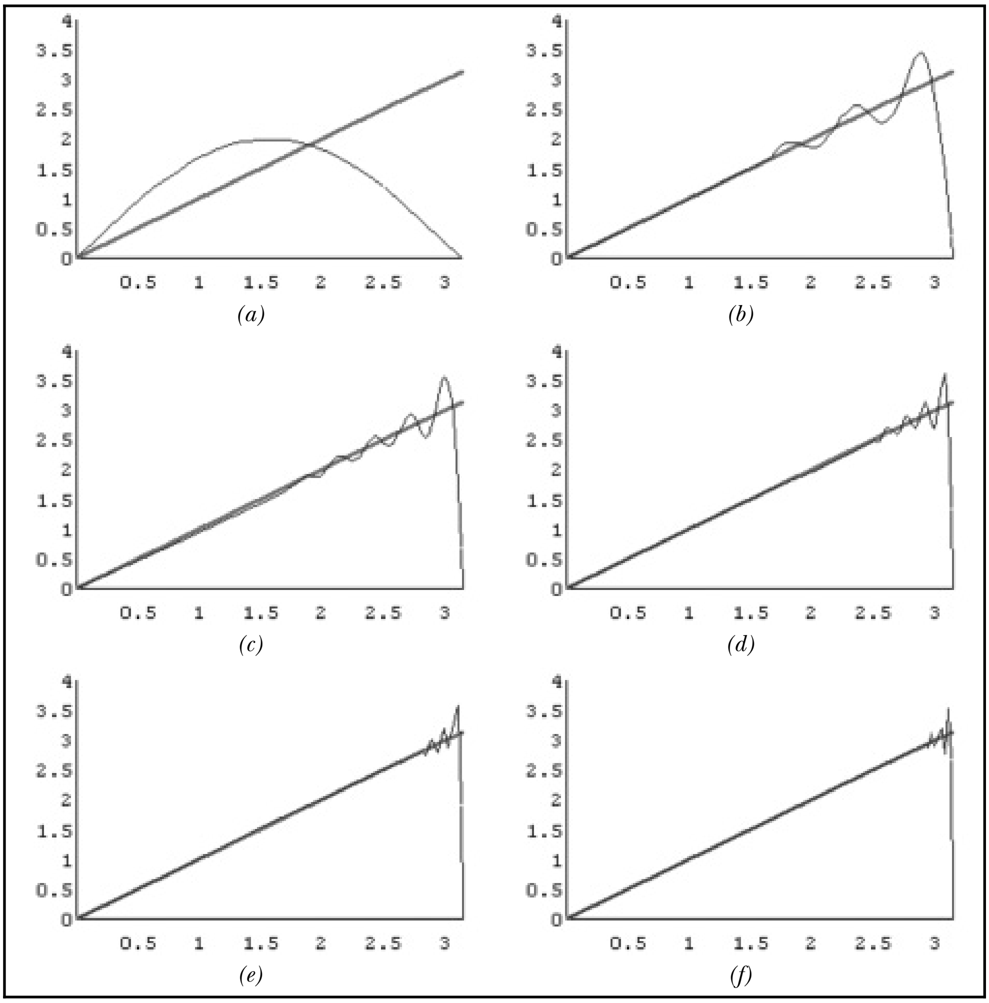

Figure 2. Plots of the FSPS of f(t) = t for 1, 11, 21, 41, 71 and 91 terms.

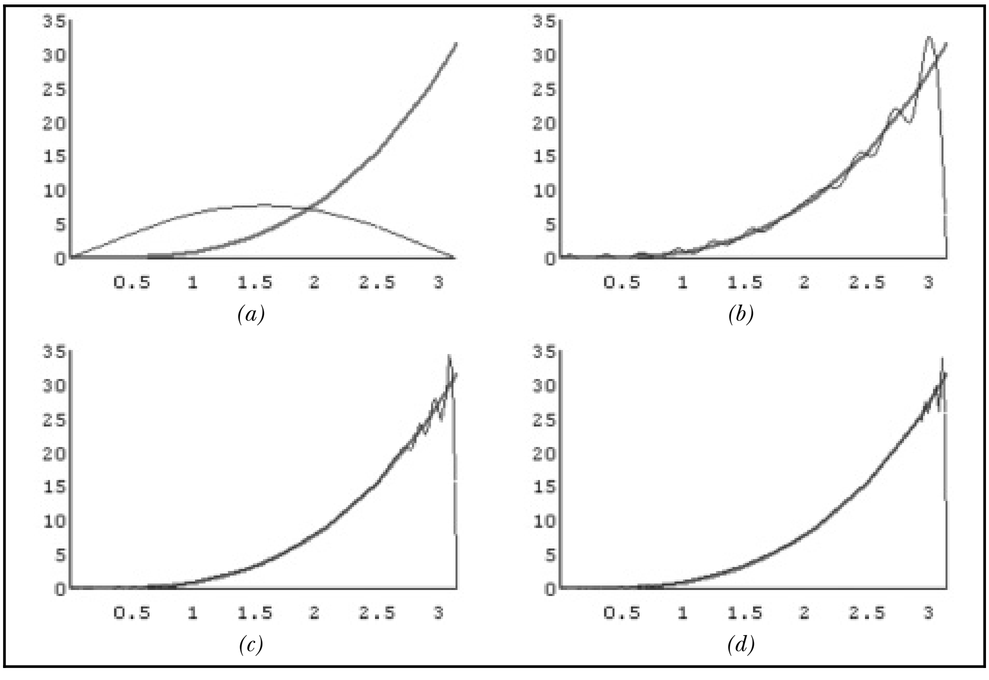

Figure 3. Plots of the FSPS of f(t) = t3 for 1, 11, 51 and 91 terms.

```
fourierseries1[k_]:=
(4/Pi)*Sum[(Sin[(2n+1)t])/(2n+1),{n,0,k}];
plota=Plot[{1,fourierseries1[10]},{t,0,Pi},
PlotRange->{{0,1},{0.75,1.25}},
PlotStyle-> {{Thickness[0.009],RGBColor[1,0,0]},RGBColor[0,0,1]}]
Table[{t,N[fourierseries1[10]]},{t,0.1,0.2,0.001}.
```

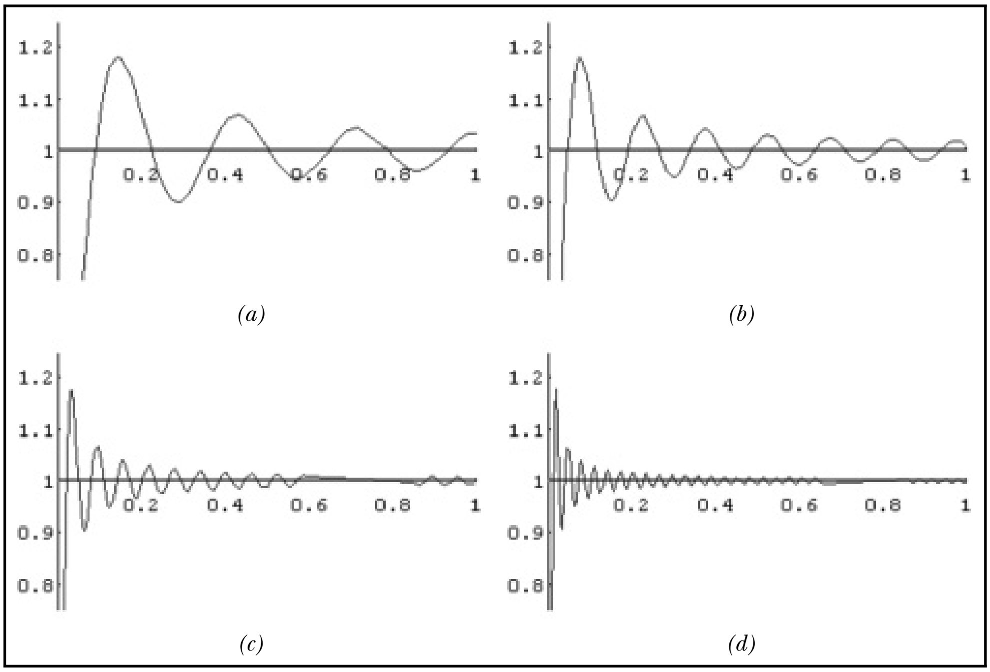

Figure 4. Plots of the FSPS of $f(t) = 1$ for 11, 21, 51 and 91 terms closer to the endpoint $t = 0$ of the interval $(0, \pi)$.

The Fourier Series partial sum values of the highest peaks of the function $f(t) = 1$ are summarised in the following table.

Table 1. Calculations of the FSPS for $f(t) = 1$.

k (no. of terms of the Fourier Series)	FSPS value of the highest peak	Value of t at which the highest peak is attained
10	1.1796	0.144
20	1.1791	0.075
30	1.1790	0.051
50	1.1789	0.031
100	1.1781	0.016

The output of the program as tabulated in Table 1 helped students to conclude that the overshoots approach the y-axis as number of terms is increased. Also the value of the overshoot remains constant at 0.179. The above code was modified to plot the Fourier series partial sums (FSPS) of $f(t) = t$, closer to the endpoint π (as shown in Figure 5).

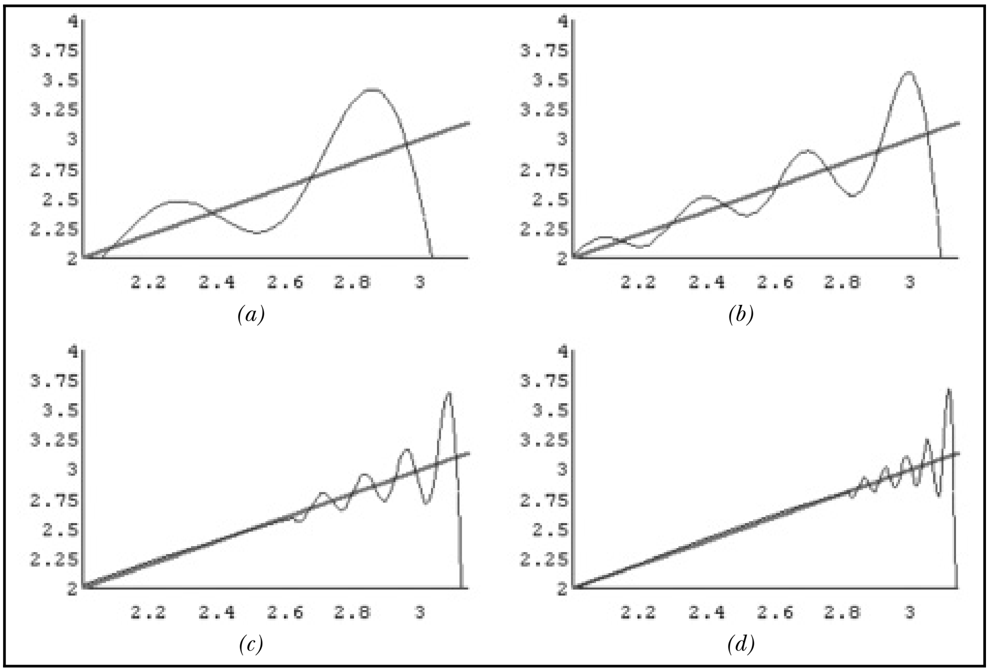

Figure 5. Plots of the FSPS of f(t) = t for 11, 21, 51 and 91 terms closer to the endpoint t = π of the interval (0,π).

Table 2. Calculations of the FSPS for f(t) = t.

k (no. of terms of the Fourier Series)	FSPS value of the highest peak	Actual function value	Value of t at which the highest peak is attained	Value of over-shoot
10	3.413	2.856	2.856	0.548
20	3.553	2.992	2.992	0.561
30	3.6018	3.041	3.041	0.561
50	3.6420	3.08	3.08	0.562
100	3.6719	3.11	3.11	0.562

The plots and results obtained in Table 2 revealed that the overshoots approach π as the number of terms is increased. Also the value of the over-shoot (approximately 0.56 up to two decimal places) remained constant with increase in the number of terms. Similar calculations for $f(t) = t^3$ are shown in Table 3. The overshoot value remained constant at 5.55 (approximately) as the number of terms was increased.

Table 3. Calculations of the FSPS for $f(t) = t^3$.

k (no. of terms of the Fourier Series)	FSPS value of the highest peak	Actual function value	Value of t at which the highest peak is attained	Value of over-shoot
10	28.532	22.979	2.843	5.552
20	32.227	26.677	2.989	5.550
30	33.616	28.066	3.039	5.549
50	34.738	28.189	3.0795	5.548
100	35.626	30.080	3.11	5.546

Results of the study

The entire module took five one hour sessions. At the end of the module students were asked to respond to a short questionnaire by indicating one of the following: Strongly Agree (SA), Agree (A), Not Sure (NS), Disagree (D), and Strongly Disagree (SD). The results are shown in Table 4. They were also asked to give a written feedback in terms of specific comments describing their impressions regarding how the module helped (or did not help) them.

Table 4. Responses of year 12 students (N = 32) to questionnaire.

Item No.	Item	SA	A	NS	D	SD
1.	This laboratory module helped you to visualise and understand the application of trigonometric integrals to Fourier series.	22	10	—	—	—
2.	Your confidence level in evaluating integrals of trigonometric functions by hand has increased after going through this laboratory module.	17	9	3	3	—
3.	Mathematica played a vital role in helping you visualise and explore graphically the concept of Gibbs phenomenon.	25	7	—	—	—
4.	Mathematica played a vital role in helping you to numerically explore the concept of Gibbs phenomenon.	22	8	2	—	—
5.	Similar laboratory modules in which Mathematica is the primary vehicle of exploration should be made a part of the regular curriculum.	20	9	3	—	—

About 69% of the students strongly agreed that the lab module helped them to visualise and understand the application of trigonometric integrals to Fourier series (item 1). Also about 81% of the students felt that their confidence level in evaluating integrals had increased after this module (item 2). One student commented, “In the first session we were required to work out the trigonometric integrals by hand and then check our answers using Mathematica. Later we had to work out the Fourier coefficients which was when it became clear why we needed the trigonometric integrals after all.”

Another student wrote, “We had to work out the Fourier series of some functions by hand and this involved a bit of hard work. We had to use integration by parts which we had also learnt in our regular class so this module gave us some extra practice in evaluating integrals.”

All students felt that Mathematica played a vital role in enabling them to visualise and explore Gibbs phenomenon (items 3 and 4). Some of the students’ comments were as follows:

"Although we were familiar with trigonometric integrals (from our regular class), in the module we had to work out the Fourier coefficients and Fourier series expansions of some simple functions. All of this did not make much sense until we plotted the partial sums using Mathematica."

"Mathematica made the Fourier series come alive… although writing the programs (codes) took some getting used to. Finally we could actually see Gibbs phenomenon."

"Mathematica's Table command helped to calculate the values of the overshoots of the functions at the end points of the intervals. This was very revealing and I don't think it [the calculations] would have been possible without Mathematica."

About 91% of the students were of the opinion that similar laboratory modules should be created for various topics of the curriculum (item 5). However, three students were unsure about this as they felt that it is important to acquire sufficient expertise in using Mathematica before attempting such modules. One of them commented: "I had a hard time keeping up with the syntax to figure out the codes. One needs to be really good at Mathematica programming to benefit from such laboratory modules."

Concluding discussion

This paper describes a laboratory module on Fourier series and Gibbs phenomenon which was undertaken by 32 Year 12 students. This module was one among five modules which the students were required to complete as a part of the course on Applicable Mathematics. The module took five one hour sessions. In each session students were given a worksheet which posed several tasks designed to enable their exploration. In the first session, students were required to evaluate the trigonometric integrals by hand and then verify their answers using Mathematica. In some cases the Mathematica output was different from their paper and pencil solution and they had to manipulate the output using commands such as **Simplify** to obtain their solution. In the second session, after being introduced to the basics of Fourier series, the students had to manually evaluate the Fourier series of $f(t) = 1$, t, t^2 and t^3. This was followed by plotting the Fourier Series partial sums using Mathematica. In subsequent sessions, students used Mathematica extensively for plotting partial sums, calculating function values at the peaks and finding values of the overshoots. Mathematica facilitated the computational process by helping to quickly generate the table of values without which it would not have been possible to observe Gibbs phenomenon. Thus paper and pencil methods helped students to understand the computations while Mathematica gave meaning to the computations. This supports Herwaarden's study (2001) which concluded that the integration of CAS helped to create a connection between the paper-pencil methods and computer manipulations. The extensive use of Mathematica in the last three sessions helped to lighten the technical work so that students could focus on making observations from the graphical and numerical outputs. Thus as suggested by Lagrange (1999), Mathematica helped to balance 'by hand' calculations and conceptual understanding. Student feedback, discussed in the previous section, revealed that they began to perceive the paper and pencil tasks (evaluation of integrals, in this case) as more meaningful after using Mathematica. Mathematica helped to illustrate concepts and processes, which would be difficult to explain using only chalk and board. For example, in this module it would be impossible to visualise the Fourier Series Partial Sums of the functions without graphing them. Students created their own Mathematica codes to produce the desired outputs and in creating these codes they used their mathematical understanding of Fourier series. In fact Mathematica served the purpose of a 'mathematical investigation assistant' as proposed by Arnold (2004) and gave students control over what they were learning. Initially the Fourier series partial sum was only a symbolic expression but after graphing, it became a physical entity which could be modified and manipulated. The integration of Mathematica permitted students to understand the concepts through the three modes of representation, namely, symbolic, graphic and numeric and also understand the concept of Fourier series and Gibbs phenomenon which is far beyond the scope of the regular curriculum. This supports Heid's (2001) theory that CAS facilitates a multi-representational approach to learning mathematical concepts and also acts like an 'amplifier' giving students access to higher level mathematical concepts.

One of the drawbacks of this module was that students needed sufficient knowledge of basic Mathematica programming to be able to benefit from it. Some students felt they needed more familiarity with Mathematica since during the laboratory sessions their focus was more on the syntax and its output rather than understanding the actual concepts. Also these modules were time consuming and required students to work beyond their regular school hours.

It may be concluded that integrating Mathematica in the laboratory module led to a very satisfying combination of technology use and 'by-hand' skills. The study supports Lindsay (1995) that, when properly used, CAS can augment learning and provide a rich and motivating environment to explore mathematics. A similar exploration of Fourier series and Gibbs phenomenon can also be done on a CAS enabled handheld calculator such as TI-89 or Casio Classpad. The encouraging feedback from students led the author to design similar laboratory modules for other topics.

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