Capitalising on inherent ambiguities in symbolic expressions of generality

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Introduction

The exploration of number patterns as a pedagogical approach to introducing algebra has been advocated by many mathematics educators. French (2002) comments that the introduction of algebra through what is potentially a wide range of pattern generalisation activities may be effective in assisting pupils to see algebra as both meaningful and purposeful right from the earliest stages. Generalisation is after all one of the core components of mathematical activity. In addition, from a pedagogic point of view, pattern generalisation activities are a meaningful way of arriving at and exploring algebraically equivalent expressions of generality.

The purpose of this article is briefly to explore the generalisation of patterns set in pictorial contexts, with specific focus on the ambiguity inherent in the algebraic expressions as they relate to the pictorial pattern.

Figural patterns

There are numerous pictorial and practical contexts in which pattern questions can be set, among the most obvious being dot patterns, matchstick patterns, as well as two- and three-dimensional building block patterns. Far from simply being a visual representation of a numeric pattern, number sequences presented in a pictorial/practical context allow for a potentially deeper appreciation of the underlying structure of the pattern, as the pictorial/practical context allows for both a greater depth and scope of interpretation. In essence, the use of a pictorial context aims to exploit the visual decoding of the pictorial sequence to give meaning to the symbolic expressions constructed.

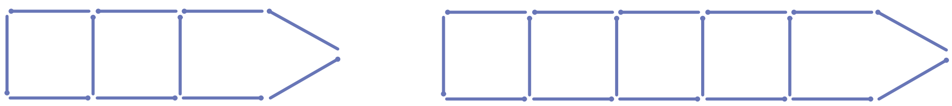
Based on their studies, Orton, Orton and Roper (1999) nonetheless caution that placing a pattern in a pictorial context must not automatically be assumed to be helpful. In addition, some contexts are more difficult than

others and the perceived relationship between pattern and context may also be problematic. Furthermore, English and Warren (1998) comment that one of the potential limitations of using pictorial patterning contexts as an approach to algebra is that only positive integer values can be assigned to the variable.

There are thus a number of important considerations to take into account when using figural pattern generalisation questions in the classroom. The first relates to the importance of task design/presentation and the notion that design features may well impact on the generalisation strategies pupils adopt when engaging with such tasks (Samson, 2007a; Chua, 2009). The second relates to the notion that different contexts will resonate differently with different pupils and that while problems presented in a pictorial and/or practical context have the potential to widen the scope of solution strategies for some learners, for others this may well create additional complications.

Pedagogical approaches to figural pattern generalisation

Notwithstanding these important issues surrounding figural pattern generalisation, such tasks have become a common feature of many school mathematics curricula. Furthermore, numerous mathematics education researchers (Bishop, 2000; Mason, Graham & Johnston-Wilder, 2005; Rivera & Becker, 2005; Samson, 2007b) have advocated a multiple representational view of pattern generalisation within the classroom - not only to explore the notion of equivalence, but to encourage pupils to critically engage with the underlying physical structure of the pictorial context as seen from alternative viewpoints. By way of example, a typical pedagogical strategy is to suggest various equivalent algebraic expressions for the general term of a pictorial pattern. Pupils are then encouraged to arrive at plausible explanations for each expression by referring to the physical structure of the provided context. Figure 1 illustrates a typical example. While the task shown in Figure 1 presupposes a capacity to express generality using algebraic expressions, a similar task could be formulated using purely numeric considerations.



Shape 3 Shape 5

Look at the diagrams shown above. Shape 3 requires 11 matchsticks while shape 5 requires 17 matchsticks.

(a) Write down an algebraic formula that will determine how many matchsticks are needed to build the nth shape.

(b) Terry says the formula is $2(n + 1) + n$, Jeff says the formula is $1 + 3(n - 1) + 4$, and Fran says the formula is $3n + 2$. By referring to the diagram, explain how Terry, Jeff and Fran arrived at their respective formulae.

Figure 1. A typical question encouraging a multiple representational view of pattern generalisation.

The purpose of this article is briefly to explore an often overlooked aspect of such multiple representation tasks, namely the semantic ambiguity inherent in the structure of the algebraic expressions themselves. Consider Terry's expression of the scenario represented in Figure 1. While $2(n + 1) + n$ is written in standard algebraic format, the $2(n + 1)$ portion of the expression could be interpreted as representing either two multiples of $(n + 1)$ matches or alternatively as $(n + 1)$ multiples of two matches. These two interpretations are shown diagrammatically in Figure 2.

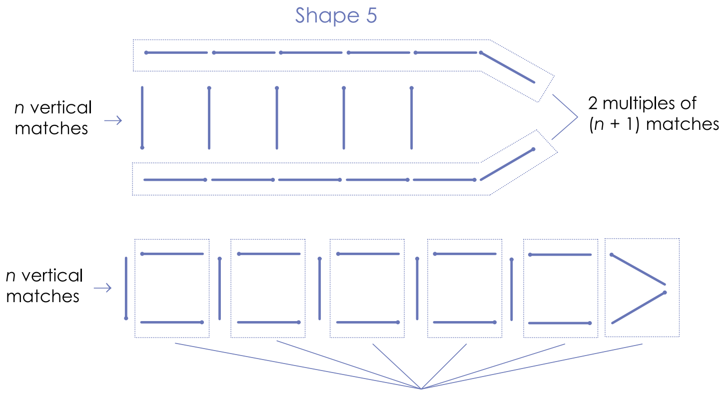


Figure 2. The semantic ambiguity inherent in Terry's algebraic expression $T_n = 2(n + 1) + n$.

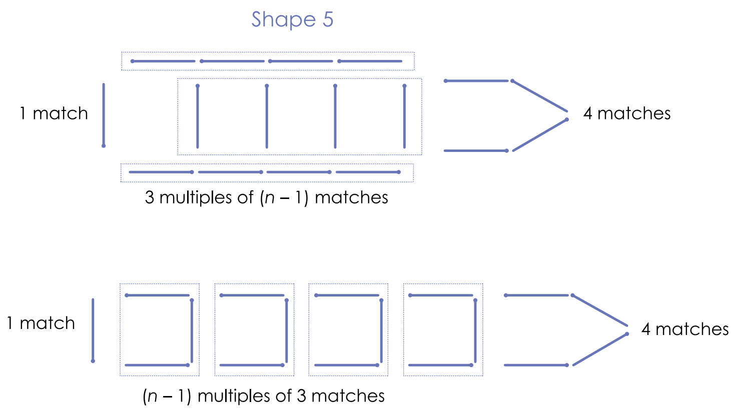


Figure 3. The semantic ambiguity inherent in Jeff's algebraic expression $T_n = 1 + 3(n - 1) + 4$.

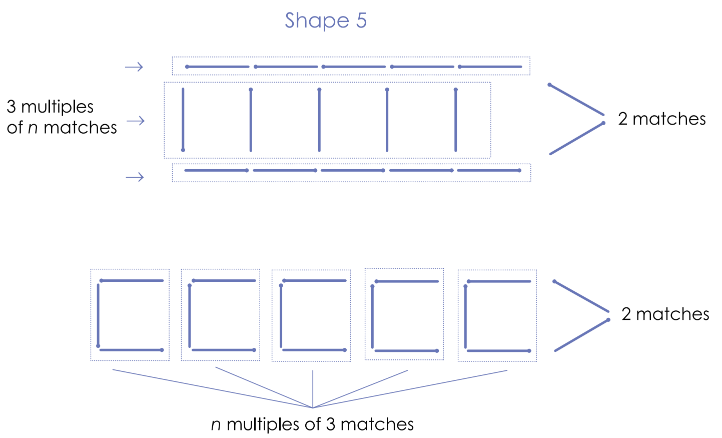


Figure 4. The semantic ambiguity inherent in Fran's algebraic expression $T_n = 3n + 2$.

An important implication of this observation is that even though Terry's algebraic expression is given in non-simplified form and thus retains what Radford (2002) refers to as a *symbolic narrative*, this narrative by no means has a unique interpretation.

A similar situation arises with both Jeff's and Fran's expressions. The $3(n - 1)$ portion of Jeff's expression could be interpreted as either representing three multiples of $(n - 1)$ matches or $(n - 1)$ multiples of three matches, while the $3n$ portion of Fran's expression could be interpreted as either three multiples of n matches or n multiples of three matches. These interpretations are shown diagrammatically in Figure 3 and Figure 4 respectively.

Discussion

Consider the two interpretations of the expression $T_n = 1 + 3(n - 1) + 4$ represented in Figure 3. In both interpretations there are five constant matches: one on the left and four on the right. In the lower diagram the remaining matches are subdivided into $(n - 1)$ multiples of three matches, each forming a backward C-shape. However, in the upper diagram the remaining matches are visualised in terms of horizontal

and vertical groupings. There are two horizontal rows of $(n - 1)$ matches and a single grouping of $(n - 1)$ vertical matches thus giving three multiples of $(n - 1)$ matches in total. Each of these two interpretations thus represents a different mode of visualisation. The former could be termed a *local generalisation*, since it relies on an appreciation of the step-by-step process of constructing one term from the previous. The latter mode of visualisation could be termed a *global generalisation* since the visual decomposition of the figure is more holistic in nature and is not reliant on a recursive process of construction. The upper and lower diagrams of Figure 2 are similarly based on global and local visualisations respectively, as are the upper and lower diagrams in Figure 4.

Although there are no doubt other modes of visual deconstruction within the context of figural pattern generalisation (Samson & Schäfer, 2010), the distinction between *local* and *global* is sufficient for the purposes of this discussion. Consider the three general expressions suggested in Figure 1, $T_n = 2(n + 1) + n$, $T_n = 1 + 3(n - 1) + 4$ and $T_n = 3n + 2$. All three expressions are symbolically more aligned with a global visualisation strategy in the sense that the usual semantic interpretations of $2(n + 1)$, $(3n - 1)$ and $3n$ are "two multiples of $(n + 1)$", "three multiples of $(n - 1)$", and "three multiples of n" respectively. Nonetheless, the important point here is that each of these three expressions of generality can be interpreted in two different ways, each with its own associated mode of visualisation—local or global.

The semantic ambiguity inherent in such expressions of generality, although perhaps somewhat subtle, nonetheless has the potential to open up interesting spaces for classroom discussion. However, a prerequisite to capitalising on this potential is for the classroom teacher not only to be aware of this semantic ambiguity, but also to have appropriate strategies for discussing it and using it to pedagogical advantage.

Imagine a classroom scenario where pupils are working on the exercise shown in Figure 1. While working individually, students are unlikely to be aware of any inherent semantic ambiguity. Once they have found a visual explanation for the given algebraic expression they are unlikely to continue to dwell on the task and are thus unlikely to realise that the same algebraic expression could have been interpreted in two different ways. However, should a classroom discussion reveal that two different students have found different visual deconstructions for the same algebraic expression then the notion of semantic ambiguity will need to be addressed. In such a situation the teacher will not only need to be able to validate both students' interpretations, but also be able to help the class reconcile the two different interpretations in a meaningful manner.

This highlights a number of important requirements of the teacher when dealing with pictorial pattern generalisation in the classroom. Not only does the teacher need to have the visualisation and/or algebraic capacity to verify students' responses, but there is a need for teachers to be able to carefully and critically engage with students' explanations of their generalisation processes.

Concluding comments

The purpose of this article was to highlight an often neglected aspect of patterning tasks, particularly in relation to a multiple representational view of pictorial pattern generalisation. Teachers should be aware of the subtle nuances of semantic interpretation inherent in algebraic expressions since

such awareness will enable them to engage with students' expressions of generality on a far deeper and more meaningful level. However, a critical requirement in terms of capitalising on this potential richness is for classroom teachers to be not only aware of such inherent semantic ambiguity, but to have appropriate strategies for maximising its potential pedagogical value.

While it is true that a given pictorial pattern can be interpreted in a variety of ways through different modes of visual deconstruction, each leading to its own algebraic expression of generality, it is equally true, though often left unexplored, that a single algebraic expression could conversely have arisen from different modes of visualisation. It is this latter consideration, with its potential pedagogical richness, that this article hopes to have highlighted.

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