Mathematics Teachers' Covariational Reasoning Levels and Predictions about Students' Covariational Reasoning Abilities*

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Abstract

Various studies suggest that covariational reasoning plays an important role on understanding the fundamental ideas of calculus and modeling dynamic functional events. The purpose of this study was to investigate a group of mathematics teachers' covariational reasoning abilities and predictions about their students. Data were collected through interviews conducted with five secondary mathematics teachers to reveal about their covariational reasoning abilities as they worked through a model-eliciting activity, predictions about their students' possible approaches to solve the given problem, possible mistakes in solving the problem, and misconceptions they possibly held. The results showed that not only the teachers' covariational reasoning abilities were weak and lack depth but also their predictions about students' reasoning abilities bounded by their own thoughts related to the problem.

Key Words

Covariational Reasoning, Functions, Mathematical Modeling, Mathematics Education, Secondary Mathematics Teachers.

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The notion of change is fundamental for the major concepts of calculus, a critical course for students majoring in mathematics, physics, engineering, and others (Carlson, Larsen & Lesh, 2003; Carlson & Oehrtman, 2005; Cottrill et al., 1996; Kaput, 1994; Saldanha & Thompson, 1998; Thompson, 1994a; Zandieh, 2000). Studies show that students' covariational reasoning abilities have a major role to construct and interpret the models of continuously changing events (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Kaput, 1992, 1994; Monk, 1992; Rasmussen, 2001). These studies also demonstrated that this type of reasoning (i.e., covariational reasoning) ability plays an important role on modeling dynamic functional situations. However, even high performing students have difficulties in modeling dynamic functional situations (Carlson et al., 2002; Carlson & Oehrtman, 2005; Köklü, 2007; Monk & Nemirovsky, 1994). In particular, students' difficulties are associated with lack of imaging and coordinating the simultaneous changes of variables, namely their covariational reasoning abilities (Carlson & Oehrtman, 2005). Although there are various definitions of covariational reasoning in the literature (e.g., Carlson et al., 2002; Confrey & Smith, 1995; Saldanha & Thompson, 1998), the common trait in these definitions is that covariational reasoning is imagining and coordinating the changes in two quantities simultaneously. In this study, we adopted the definition provided by Carlson et al. (2002) to describe covariational reasoning: "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354).

Researchers emphasize that the concept of function should be introduced as covariation, that is coordinating changes in one variable with the other variable, and student should be provided with more opportunity to explore the concept of rate of change in earlier grades (Confrey & Smith, 1995; Kaput, 1994; Köklü, 2007; Thompson, 1994b). In *Principles and Standards for School Mathematics,* NCTM (2000) emphasizes the need for more function related tasks in the classrooms and stresses the importance of the students' understanding of the concept of rate of change in real-world situations. In this context, modeling activities provide opportunities to achieve these aims (Carlson et al., 2003; Lesh & Doerr, 2003b; Lesh, Hole, Hoover, Kelly & Post, 2000). For effective instruction, on the other hand, it is important for teachers to know the concepts students have difficulty with and the ways to overcome the common misconceptions or misunderstandings. However, research studies suggest that although it is an important component of pedagogical content knowledge (Shulman, 1987) there are discrepancies between teachers' predictions about students' difficulties and students' actual difficulties (Even, 1993; Hadjidemetriou & Williams, 2002).

Some research conducted in Turkey on the concept of function (e.g., Aydın & Köğçe, 2008; Karataş & Güven; 2003; Türkeli-Şandır, 2006; Ural, 2006), were not concerned with the covariational approach to functions. Moreover, studies focusing on in-service teachers' covariational reasoning abilities have been neglected in the literature compared to that focuses on prospective teachers and elementary and secondary students in international studies. Thus, this study aimed to explore covariational reasoning abilities of in-service mathematics teachers and their predictions about students' covariational reasoning abilities by using a model-eliciting activity.

Method

In this study, we used the covariation framework developed by Carlson (1998) and Carlson et al. (2002) to describe and interpret the covariational reasoning abilities of students (see Table 1). This framework contains five distinct developmental levels: Coordination (Level 1); Direction (Level 2); Quantitative Coordination (Level 3); Average Rate (Level 4); and Instantaneous Rate (Level 5) (for details see Carlson et al., 2002).

Table 1.
Mental Actions and Corresponding Behaviors in the Covariation Framework *

Mental Action	Description of Mental Action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other.	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x).
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable.	• Constructing an increasing straight line. • Verbalizing an awareness of the direction of the change in the output while considering changes in the input.

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Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other.	• Plotting points/constructing secant lines. • Verbalizing an awareness of the variable amount of change of the output while considering changes in the input.
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain. • Verbalizing an awareness of the rate of change of the output (with respect input) while considering uniform increments of the input.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of function.	• Constructing a smooth curve with clear indications of concavity changes. • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct).

* Carlson et al., 2002, p. 357.

Participants and Data Collection

Data were collected using qualitative research methods through interviews conducted with five secondary school mathematics teachers, of whom two were teaching in Anatolian high schools; one was teaching in a foreign language intensive high school; one was teaching in a *dershane* that is a type of private education institute offering specialized preparation for nationwide standardized tests like the university entrance exam; and one was teaching in a vocational high school. The modeling activity used in this study was based on the concept of functions, and particularly increasing and decreasing functions, a topic in mathematics curriculum for 12th grade (See Figure 1).

Imagine that the bottle shown below is being filled with water.

a) Draw a volume-height graph (the height of the water given the amount of water that is in the bottle). Explain your reasoning while drawing the graph.

b) Explain how one can draw a graph for any type of bottle.

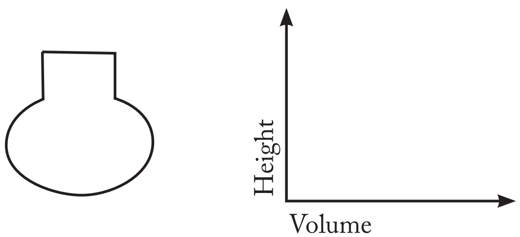

Figure 1. *Model Eliciting Task Utilized in the Study [adapted from Carlson (1998) and Carlson et al. (2003)].*

All participating teachers had experience in teaching 12th grade students. During the interviews, teachers were first asked to read and answer the mathematical modeling task shown in Figure 1. They were also asked a question related to covariational reasoning and their views and predictions about students' covariational reasoning abilities. The following topics were covered in the interviews: (a) Teachers' covariational reasoning abilities; (b) Students' mathematical knowledge; (c) Students' possible approaches, or strategies to solve the problem; (d) Thoughts those students had during the solution; (e) Possible mistakes students made to solve the problem; and (f) Misconceptions students' may have. Teachers were asked to elaborate and rationalize their answers to the questions directed during the interviews.

Data Analysis

In analyzing the data gathered, teachers' solution sheets for the given modeling task and transcriptions of the interviews were evaluated together. Verbal explanations and drawings on the solution sheets were analyzed and coded according to mental actions and corresponding behaviors described in the theoretical framework. To determine the teachers' predictions of students' covariational reasoning abilities transcriptions of interviews were coded to assist in categorizing and summarized to be compatible with the research question. The next step in the data analysis involved identifying recurring patterns in the coded data. As reading through the data, certain words, and phrases about teachers' predictions about students' ways of thinking repeated and stand out. Finally, whole data set was put into several categories, helping to understand the teachers' predictions about students' covariational reasoning abilities. While we essentially adopted the titles of subcategories from the literature based on the theoretical framework (e.g., treating time as the input variable as reported in Carlson et al., 2003), we occasionally constructed some based on our understanding of the data.

Results

In the modeling task, teachers were expected to construct the graph of a dynamic situation in which the rate changing from decreasing to increasing. Thus, we tried to identify the teachers' thought processes as well as how successful they were in constructing the graphs of increasing, decreasing, and linear functions. Also, we attempted to assess to what extent they predict the students' covariational reasoning abilities.

The Levels of Teachers' Covariational Reasoning

Participating teachers' answers to the modeling activity revealed that all participants had difficulty to solve the problem correctly. Their understanding of the relationship between two varying quantities was poor and implied that they might have misconceptions. Interviews confirmed the weaknesses in their ability to represent and interpret continuously changing rate.

The first teacher (T1) demonstrated covariational reasoning ability in Level 3 (i.e., quantitative coordination). He used the words such as "accelerating", "faster", "slower", "at gradually decreasing speed" while constructing and interpreting the graph. He claimed that the flow rate of the water is the main factor (i.e., variable) in constructing the graph of the function. Although he constructed the graph correctly, when prompted to explain the rationale for it, he expressed that he might have considered the axes inversely. That is, he considered the volume axis as height, and height axis as volume.

The second teacher (T2) demonstrated covariational reasoning ability in Level 2 (i.e., direction). He first drew the graph as line segments with a positive slope and explained that time was the effective factor so he drew the graph according to "time". When prompted to elaborate more on the graph, he changed the graph, yet again could not draw the correct one.

The third teacher (T3) constructing the correct graph demonstrated covariational reasoning ability in Level 4 (i.e., average rate). He attempted to construct the graph by considering the uniform increments in volume axis. But, similar to T2, he explained that he considered and treated the volume as "time". When asked about the flow rate of the water, he explained that he could not construct the graph in that way if the flow rate of water is not constant.

The fourth teacher (T4) constructing a wrong graph demonstrated covariational reasoning ability in Level 3 (i.e., quantitative coordination). She had difficulties in solving the task like how to find the volume. She also demonstrated some misconceptions about linear functions. There were inconsistencies between her explanations and her drawings. But, contrary to other teachers, she explained that the flow rate of the water has no impact on the graph of the function.

Demonstrating covariational reasoning ability in Level 2 (i.e., direction), the fifth teacher (T5) drew the same curve, not changing it for different parts of the bottle, in her first attempts. First, she drew a line segment with a negative slope, and then drew a curve resembling the graph of $y = x^2$ in the first quadrant. Then, she substituted different values for x and tried to find corresponding values of y. Although she thought that she had to find the volume of the bottle, she could not remember the formula. Once again, she could not construct the correct graph.

Teachers' Predictions about Students' Covariational Reasoning Abilities

All the teachers stated that the modeling activity that they had worked on would also be difficult for their students to solve. They all expressed that their students could not construct the graph correctly. Their predictions did not go beyond their own thoughts related to the problem. Some expressed that students will consider the axes as "time" versus "height". Moreover, teachers have pointed out that these types of activities are hard to put into practice in Turkish context due to reasons such as the ÖSS (Student Selection Exam) and limited time they have for teaching so many topics. Four of the teachers stated that students are more accustomed to working multiple-choice items and they face with difficulties when they are confronted with open-ended questions and asked to make generalizations like in this activity. Also, teachers suggested that students assume the flow rate of the water as constant to simplify the task as it is not mentioned in the task.

Discussion

Even though this study was conducted with in-service teachers, the results support the findings of those carried with students and preservice teachers and introduction of functions through situations involving covariation would be useful (e.g., Carlson et al., 2003; Confrey & Smith, 1995; Kaput, 1994; Köklü, 2007; Saldanha & Thompson, 1998; Thompson, 1994a, 1994b). The results show that the participant teachers had poor covariational reasoning abilities and difficulties in representing and interpreting graphs involving covariation. The results showed that the teachers think functions as correspondence instead of covariation. Their inability to identify being concave up or concave down or inflection points of a function while constructing the graphs would be perceived as signs of their deficient understanding of the major concepts of calculus. Teachers' answers to questions revealed that their explanations for the question and their conception did not differ from those found in other studies conducted with pre-service mathematics teachers and even high school students (Even, 1993; Norman, 1993). Such findings also reveal the complexity of covariational reasoning (Carlson et al., 2002, 2003; Confrey & Smith, 1995). On the other hand, participant teachers treated "time" as the independent variable instead of "volume". This might have been the result of their attempts to reduce the cognitive load caused by difficulty of thinking two variables at the same time (Carlson et al., 2003). They also mixed up the role of independent and dependent variable (Carlson et al., 2002).

Although some of the teachers (T1,T2, and T4) demonstrated some of the behaviors particular to mental actions, when we analyzed their explanations it was seen that their behaviors took place without understanding. These behaviors defined by Vinner (1997) as "pseudo-analytical behaviors" which might look like analytical behaviors, but which in fact do not characterize conscious behaviors.

This study provides evidence that visual features in the problem might be an effective factor leading the solver into mistakes by making correspondence between the shapes in the problem (e.g., curved shape of a bottle) and the shape of their graphs (Leinhardt, Zaslavsky & Stein, 1990). Teachers considered the flow rate of the water is important for the shape of the curve even though it was not. As the participants attempted to recall and apply their knowledge in related physics concepts in the mathematical situation at hand, this study indicates that mathematical modeling activities may provide students and teachers with opportunities to develop both conceptual and procedural understanding of particular mathematical and physical concepts simultaneously.

In this study, teachers' behaviors and explanations as the signs of their mental actions differed at some points from the descriptions denoted within the covariation framework we utilized. These variations might be helpful for further development of a framework for defining teachers' covariational reasoning abilities. The results also suggest that developing and implementing modeling activities are necessary to promote teachers' understanding of major calculus concepts. Furthermore, implementing modeling activities, "thought-revealing" in nature, in the classrooms would help teachers to develop pedagogical content knowledge, particularly the knowledge of students' ways of thinking (Carpenter & Fennema, 1996; Lesh & Doerr, 2003a).

References/Kaynakça

Aydın, M., ve Köğçe, E. (2008). Öğretmen adaylarının "denklem ve fonksiyon" kavramlarına ilişkin algıları. *Yüzüncü Yıl Üniversitesi Eğitim Fakültesi Dergisi, 5*(1), 46-58.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education, III* (CBMS Issues in Mathematics Education, Vol. 7, pp. 114–162). Washington, DC: Mathematical Association of America.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating models and modeling perspective with existing research and practice. In R. Lesh &H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 465-478). Mahwah, NJ: Lawrence Erlbaum Associates.

Carlson, M., & Oehrtman, M. (2005). Key aspects of knowing and learning the concept of function. *Research Sampler Series, 9, The Mathematical Association of America Notes Online*. Retrieved September 16, 2009, from http://www.maa.org/t_and_l/sampler/rs_9.html.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*, 66-86.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior, 15*, 167-192.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94-116.

Hadjidemetriou, C., & Williams, J. (2002). Teachers' pedagogical content knowledge: Graphs, from a cognitivist to a situated perspective. In E. Nardi (Ed.), *Proceedings of the 26th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 57-64). Norwich, UK: University of East Anglia.

Kaput, J. J. (1992). Patterns in students' formalization of quantitative patterns. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes* (Vol. 25, pp. 290-318). Washington, DC: Mathematical Association of America.

Kaput, J. J. (1994). Democratizing access to calculus: New routes to old roots. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77-156). Hillsdale, NJ: Lawrence Erlbaum Associates.

Karataş, İ. ve Güven, B. (2003). Fonksiyon kavramının farklı öğrenim düzeyinde olan öğrencilerdeki gelişimi. *Eğitim Araştırmaları, 4*(16), 64-73.

Köklü, Ö. (2007). *An investigation of college students' covariational reasoning*. Unpublished doctoral dissertation, The Florida State University, Florida.

Leinhardt, G., Zaslavsky, O., & Stein, M. S. (1990). Functions, graphs and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Lesh, R., & Doerr, H. (2003a). A modeling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 125-139). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Doerr, H. (2003b). Foundations of models and modelling perspective on mathematics teaching, learning and problem solving. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Hole, B., Hoover, M., Kelly, E., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591- 645). Mahwah, NJ: Lawrence Erlbaum Associates.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes* (Vol. 25, pp. 175-193). Washington, DC: Mathematical Association of America.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Shoenfeld & J. J. Kaput (Eds.), *Research in collegiate mathematics education I* (pp. 139-168). Providence, RI: American Mathematical Society.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Norman, F. A. (1993). Integrating research on teacher' knowledge of functions and their graphs. In Romberg, A. T., Fenemma, E., & Carpender P. T. (Eds.). *Integrating research on the graphical representation of functions* (pp. 159-187). Hillsdale, NJ: Erlbaum.

Rasmussen, C. L. (2001). New directions in differential equations: A framework for for interpreting students' understandings and difficulties. *Journal of Mathematical Behaviour, 20*, 55-87.

Saldanha, L., & Thompson, P. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen & W. N. Coulombe (Eds.), *Proceedings of the twentieth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 298-304). Raleigh, NC: North Carolina State University.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Türkeli-Şandır, Y. (2006). *Fonksiyon kavramı hakkında öğretmen adaylarının görüşleri üzerine bir fenomenografik çalışma*. Yayımlanmamış yüksek lisans tezi, Gazi Üniversitesi, Eğitim Bilimleri Enstitüsü, Ankara.

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*, 229-274.

Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education I* (pp. 21-44). Providence, RI: American Mathematical Society.

Ural, A. (2006). Fonksiyon öğreniminde kavramsal zorluklar. *Ege Eğitim Dergisi, 7*(2), 75-94.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics, 34*, 97-129.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education IV* (pp. 103-127). Providence, RI: American Mathematical Society.