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Understanding Mathematical Learning Disabilities: A Case Study of Errors and Explanations

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When solving basic number fact problems (e.g., 4x5=20), students with mathematical learning disabilities (MLDs) make qualitatively different kinds of errors than both typically achieving and low-achieving students. This study explores the origin of these qualitative differences through a detailed analysis of one student with an MLD. Data were drawn from a longitudinal study of weekly videotaped tutoring sessions during the student's eighth through twelfth grade years. An error analysis was conducted on the student's responses to a computerized multiplication fact program (Math Master), which confirmed that she demonstrated a similar error profile to that documented in students with MLDs. A microgenetic analysis of a subset of tutoring sessions (19 sessions total) focused on the student's math fact usage while she was engaged in more complex problem solving, specifically fraction reduction. Although her math fact errors appear on the surface to be unrelated, the detailed analysis revealed the origin of these errors to be her atypical and cognitively cumbersome strategy for solving multiplication problems. An intervention, which functioned to decrease the cognitive load of her multiplication strategy, increased her accuracy on fraction reduction problems. This research suggests that a detailed analyses of students' problem solving can reveal the origins of the qualitative differences that typify MLD, and provide information that can contribute to strategies for remediation.

Research on MLDs is in its infancy as compared to research on language-based learning disabilities (Fletcher, Lyon, Fuchs, & Barnes, 2008; Mazzocco, 2007). Mathematical learning disabilities (MLDs) have been commonly characterized as cognitive impairments resulting in poor automaticity of math facts; e.g., 4x5=20 (Gersten, Jordan, & Flojo, 2005; Swanson & Jerman, 2006). Currently there is no consensus operational definition of MLDs, and researchers rely on the insufficient proxy of low-achievement to identify students with MLDs (Murphy, Mazzocco, Hanich, & Early, 2007). To address this methodological issue, Mazzocco, Devlin and McKenney (2008) designed an innovative subject identification procedure that relied on longitudinal data of math performance to classify students. They explored eighth grade students' number fact errors on a timed math facts assessment and determined that students with MLDs make *qualitatively different kinds of errors* than their low-achieving or typically achieving peers. This finding is critically important because it suggests that these qualitative differences might be the key to differentiating MLDs from more general low math achievement. However, it remains unclear what the origin of these error patterns are and *why* these qualitative differences emerge.

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To extend the findings of Mazzocco et al. (2008), a detailed case study of one student with an MLD was conducted. The purpose of this study is to identify the source of the qualitatively different error patterns documented in students with MLDs. The analysis focuses on the student's problem-solving processes and explanations, both while she answered math fact problems and when these basic arithmetic problems were embedded within more complex mathematical problems (e.g., fraction operations). This study addresses the following research questions:

- Does the student's error patterns match those documented for students with MLDs in Mazzocco et al. (2008)?
- How does the student solve multiplication problems, and what does this reveal about her patterns of errors?
- In what ways can remediation address these kinds of math fact errors?

Analysis indicates that the student demonstrated the same kinds of qualitatively different errors documented by Mazzocco et al. (2008), and a detailed analysis of her explanations indicates that her atypical way of retrieving math facts may explain these error patterns. A remedial intervention, designed to build upon her atypical strategy, was implemented and appeared promising.

Methods

Sample Description

The case study student, "Emily," was a participant in a larger longitudinal study spanning her eighth through twelfth grade years (age: 13–18 years). Achievement test scores, classroom observations, interviews, and weekly videotaped tutoring sessions were used to establish that the case study student, "Emily," meets the qualifications for having a mathematical learning disability. Emily's STAR[1] achievement test scores from sixth to eleventh grade indicate that she demonstrates persistent low math achievement (her math achievement was rated "below basic" or "far below basic" across all five years). Percentile scores were reported for the sixth and seventh grade tests and were at 23rd and 11th percentile respectively. Her percentile scores place her in the lowest 25th percentile, which is the most commonly used criterion for identifying students with MLDs (Geary, 2004). Observations of Emily in her eighth grade math class indicate that she does not exhibit attentional or behavioral problems. Emily is a Caucasian, native English speaker from an upper-middle class background, eliminating confounding linguistic and socio-economic factors. Despite weekly tutoring sessions, Emily continues to have persistent difficulties in nearly all domains of mathematics. A lack of confounding factors generally correlated with low math achievement (see Hanich, Jordan, Kaplan, & Dick, 2001), suggests that Emily's low math achievement test score is indicative of a MLD.

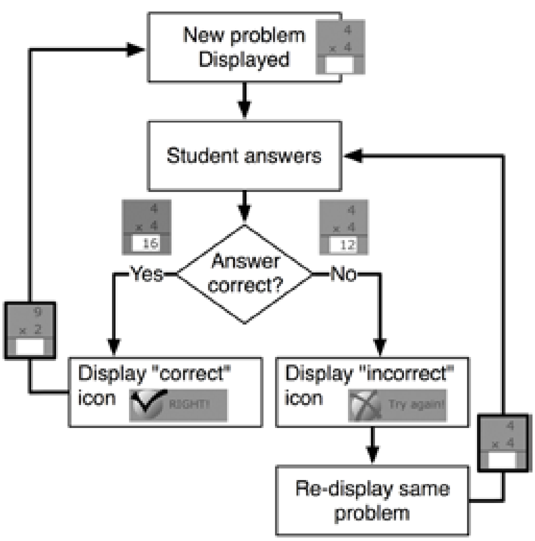
1 The STAR (Standardized Testing and Reporting) test is the California state mandated achievement test administered each year to students in second through eleventh grade (see http://www.cde.ca.gov/ta/tg/sr/ for more information).

Procedures

Tutoring sessions. These data were drawn from a larger corpus of data collected during weekly video-taped tutoring sessions with Emily, where the author provided one-on-one math instruction on topics including fraction operations, integer operations, and algebra. The tutoring sessions did not have a pre-established protocol, but instead were negotiated between the student and tutor, and provided an opportunity for the student to work through challenging problems and explain her reasoning and rationale for her answers. For the purpose of this paper, 30 tutoring sessions were included in the analysis—11 sessions in which a computerized number facts program was used; 19 sessions (focused on fraction reduction) in which a number facts intervention was implemented. The larger tutoring data corpus serves to validate and corroborate the prevalence of her strategies documented in this paper.

Computerized number facts program. A computerized number facts program, Math Master (available at http://www.cs.berkeley.edu/~colleenl/kelewis/mathFacts.swf) was used during 11 tutoring sessions. Math Master randomly generates 21 multiplication problems on the 10x10 multiplication table.[2] The student types the answer to the multiplication problem and hits the enter key. If the student answers correctly, a "Right!" icon is shown, and the next problem is displayed. If the student answers incorrectly, the student receives a "Try again!" message and the screen re-displayed the same problem until the student answers the problem correctly (see Figure 1 for a flowchart with screen shots). The program records the elapsed time per problem and the student's responses and stores the data as semi-colon delimited list. The program was used a total of 52 times during the course of the 11 tutoring sessions.

***Figure 1*. Overview of the math facts program.**



2 Although Math Master has functionality for addition, subtraction, multiplication, and division, only the multiplication component was used.

Number facts intervention. Of interest in this study is how these documented number fact errors occur during the course of more complex mathematical problem solving and potential remedial approaches to address these errors. To this end, fraction reduction was chosen as a focal topic because it is an authentic problem-solving context in which math facts are used as part of the solution process and not as the sole focus. Based on the diagnostic analysis it was hypothesized that the math fact errors Emily experienced when reducing fractions were partially due to the cognitive strain of her atypical multiplication fact strategy. A simple intervention strategy was designed in which Emily was asked to either rewrite the fraction as a multiplication problem or write down the amount by which she was dividing both the numerator and denominator (see Figure 2 for an example). The goal of the intervention was to have Emily reduce the cognitive load of the task by writing down the factor of reduction. Nineteen tutoring sessions were analyzed with respect to the remediation intervention. The intervention was implemented during the thirteenth session. Results from the student's accuracy before and after this intervention approach will be presented.

***Figure 2.* Example of two strategies for solving fraction reduction problems used during the tutoring intervention.**

Intervention Strategy 1:
Rewrite as a multiplication problem

$$\frac{8}{12} = \frac{\cancel{4} \times 2}{\cancel{4} \times 3} = \frac{2}{3}$$

Intervention Strategy 2:
Write down the factor of reduction

$$\frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

Analytic approach. Three different strands of data analysis were conducted: (1) error analysis of multiplication facts from the computerized program, (2) a qualitative analysis of the video data from the tutoring sessions focused on the student processes and explanations, and (3) an evaluation of the effectiveness of the remediation approach. Each will be discussed in turn.

In the first strand of analysis, data from the computerized multiplication fact program was analyzed. An error analysis based on the coding scheme used by Mazzocco et al. (2008) for multiplication math facts was conducted to determine whether Emily's error profile matches that of the students with MLDs. Two kinds of error types occurred in Emily's data: *operand errors,* in which the answer given is calculated using a different operand (e.g., 7x4=21), and *table errors,* in which the answer is on the 10x10 multiplication table (e.g., 7x4=27;(see Mazzocco et al.) for a complete description of all error types). As in Mazzocco et al., all errors, regardless of type, were classified with respect to whether the given answer was *decade-consistent*, an error in which the tens digit of the answer matches that of the correct answer, (e.g., 4x5 = 24) or *decade-inconsistent*, an error in which the tens digit of the answer does not match the correct answer (e.g., 4x5=16). The decade consistency of the answer was a particular focus because Mazzocco et al. found that students with MLDs made significantly more errors that were decade-inconsistent, and therefore differentiated the performance of students with MLDs from that of their low achieving peers. In addition to the coding scheme used by Mazzocco et al., Emily's errors were analyzed for other error patterns that might be unique.

In the second strand of analysis, the video data of the tutoring sessions were content-logged with respect to (1) the problem posed, (2) the student's process, answer, and explanation, and (3) any instruction that occurred. These content logs were coded for evidence of math facts usage, in particular those in which she provided an explanation of her math facts solution process. These data served to provide both insight into the nature of her math fact solution process and evidence of the typicality of a given solution process during the course of more general problem solving.

Third, the nineteen tutoring sessions, which serve as the basis for the evaluation of an implemented intervention approach, were analyzed. Each fraction reduction problem answered during the tutoring session served as a data point, and the student's accuracy in reducing fractions was evaluated both before and after the intervention.

Results

Results are presented in three parts. First, a general analysis of Emily's error patterns relying on Mazzocco et al.'s (2008) coding scheme is presented demonstrating that Emily's errors are consistent with those made by students with MLDs. Second, Emily's atypical strategy for solving multiplication problems is illustrated, providing a plausible common origin for the error patterns documented. Third, the remedial intervention is described and evaluated with respect to its effectiveness in improving Emily's accuracy on fraction reduction problems.

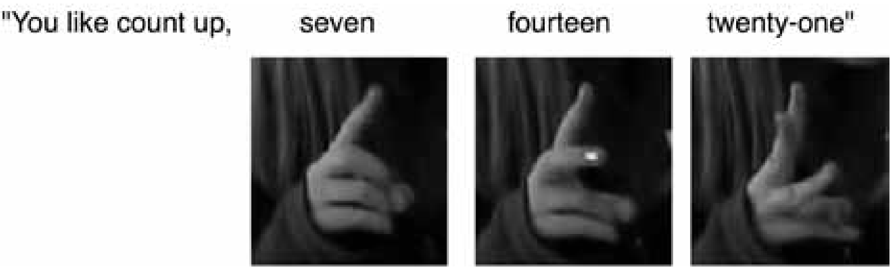
The Nature of Emily's Errors as Compared to Students with MLDs

Emily's errors were similar in nature to the students with MLDs in Mazzocco et al. (2008). Seventy-five percent of Emily's errors were *operand errors*, answers in which the problem was solved using the wrong operand (e.g., 7x5=30). The remaining 25% of Emily's errors were *table errors*, answers that are on the 10 x 10 multiplication table, but are not a multiple of either of the operands (e.g., 9x6=56). Mazzocco et al similarly found that operand errors were the most common errors made by students with MLDs. In addition, 79% percent of all of Emily's errors, regardless of error type, were decade-inconsistent (e.g., 4x5=16). This is similar to Mazzocco et al.'s (2008) finding that decade-inconsistent errors accounted for 70% of the errors made by students with MLDs as opposed to only 34% of the errors made by students without MLDs. Finally, one additional error pattern was identified, which was not reported in the Mazzocco et al.'s study and may be unique to Emily. Sixty-seven percent of Emily's errors on the computerized multiplication program involved problems in which one of the two operators was either a 6 or a 9. Each type of error is discussed in relationship to the explanations given by Emily during the tutoring sessions.

Atypical Strategy as a Plausible Origin of Errors

Although these errors are different in nature, an in-depth analysis of Emily's solution process indicates that these errors may all stem from an atypical strategy for solving multiplication problems. To solve multiplication problems, Emily recites a list of multiples while keeping track of her ordinal position in the list on her fingers. For example, in her explanation for how she solved the problem "7 x 3 = ", she said: "You like count up 7, (raises 1 finger) 14 (raises another finger), 21 (raises another finger)" (see Figure 3). The following excerpts highlight how Emily's atypical retrieval of math facts may leave her susceptible to the operand errors, decade-inconsistent errors, and errors involving 6 or 9 as an operand.

Figure 3. **Emily's demonstration of how to solve the problem 7x3.**



Operand errors. Although her multiples list strategy was often a productive way to solve multiplication problems, it was vulnerable to *operand errors*, which accounted for 75% of her errors. In her case, an operand error would result when she miscounted as she recites her multiples list. Indeed, 60% of her operand errors occurred when her operand was one more or one less than the correct operand (e.g., 7x3=14 or 7x3=28).

Decade-inconsistent errors. A potential underlying cause of Emily's decade-inconsistency can also be attributed to her use of multiples lists. As in the "7x3" example problem, Emily typically iterated through the multiples list corresponding to the larger of the two numbers (i.e., the *sevens* list as opposed to the *threes* list). In using the multiples list for the larger of the two operands, she increased the likelihood that miscounting would result in an answer that was not in the same decade as the correct answer (decade-inconsistent). Her tendency to choose the larger of the two operands may partially explain why 79% of her errors resulted in decade inconsistency.

Her tendency to iterate the larger of the two operands was particularly evident as she attempted to solve the problem "what times 2 is 16?"

Student: You can't do it… cause 2 doesn't go into 16, evenly.

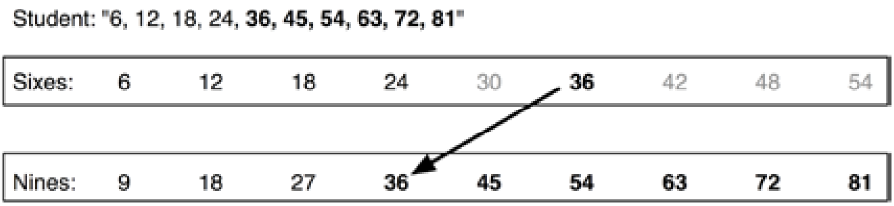
Tutor: Are we sure about that?

Student: Yeah. 5, 10. 6, 12. 7, 14. Oh! 8. Duh!

Rather than iterating through her twos list, which was the given operand in the problem, she opted to access the second number on her *fives*, *sixes*, *sevens*, and *eights* lists. Although it may appear more cognitively cumbersome to list the second element of many lists, Emily preferred this option to that of iterating through her *twos* list.

Errors when multiplying 6 or 9. Emily's tendency to make errors on problems that involved either a 6 or 9 operand may also result from her reliance on multiples lists. In the following example, Emily was attempting to find a common multiple for 6 and 12. The tutor suggested that she start by listing some multiples of 6. As she recites her *sixes* she inadvertently switches from the *sixes* to the *nines* list (see Figure 4).

Figure 4. **Visual depiction of Emily's transition from her sixes to her nines multiples list.**



Tutor: Lets start listing out some multiples of 6.
Student: 6, 12, 18, 24, **36, 45, 54, 63, 72, 81.**
Tutor: (writes 6, 12, 18, 24, 36 on paper) Do you do it in your head 6 times 1, 6 times 2, 6 times 3?
Student: (shakes head no) I just memorize them, and I don't even know if it's right. 6, 12, 18, 24, **36, 45, 54, 63, 72, 81**. Is that right or is that wrong?
Tutor: I don't know.
Student: 6, 12, 18, 24, **36, 45, 54, 63, 72, 81**? Or is that nines?
Tutor: No those are sixes (pointing to the paper).
Student: 9, 18, 27, 36, 45, 54, 63, 72, 81—no that's nines.
Tutor: Uh huh.
Student: 6, 12, 18, 24, **36, 45, 54, 63, 72**, I'm confused, they are both nines.

It is significant that Emily repeats this incorrect sequence *four* times during this brief episode. Emily was clearly unable to determine whether she was reciting her *nines* or her *sixes*. Her difficulty in listing her *sixes* in her attempt to solve this problem may suggest why 67% of Emily's errors on the computerized multiplication program involved problems in which one of the two operators was either a 6 or a 9. It is possible that sixes and nines were particularly susceptible to this type of error because of the linguistic similarity between the *sixes* list:

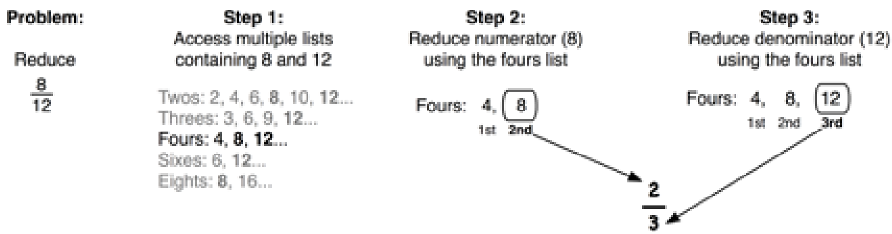
" . . . eighteen, twenty-four, thirty . . . " and the *nines* list: " . . . eighteen, twenty-seven, thirty-six . . . "

Although on the surface Emily's tendency to make operand errors, decade inconsistent errors and errors when the operand is a 6 or 9 may seem unrelated, Emily's atypical multiples lists strategy provides a plausible origin for all these errors. Given the proposed atypical multiples list strategy: what are the implications for her solution process in the context of solving more complex mathematical problems?

Multiples Lists in Fraction Reduction

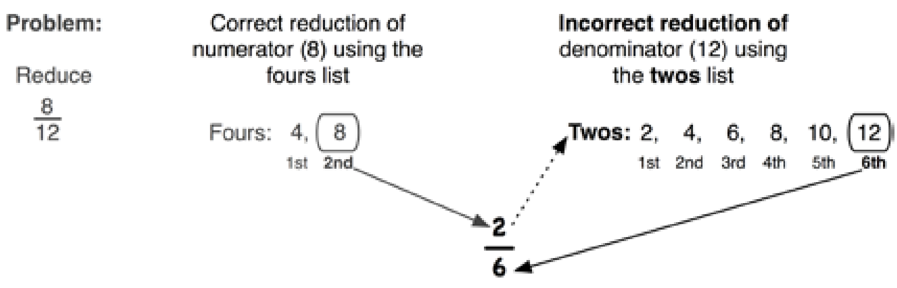
Fraction reduction was chosen as a focal topic to consider Emily's multiple lists strategy. To reduce a given fraction, like 8/12, using her strategy, several different strings of multiples could be accessed to determine what list contains both the value of the numerator (8) and the value of the denominator (12). The reduced fraction would be determined by the ordinal position of the numerator and denominator in the list (see Figure 5). For example, on the *fours* list, 8 is the second number, and 12 is the third number, which would result in the reduced fraction of 2/3. It is interesting to note that when Emily reduced fractions, she often recited the list twice, once for the numerator value and once for the denominator value, keeping track of her position in the list on her fingers. Because she was potentially accessing and reciting several multiples lists during this solution process, and must cycle through her chosen multiples list twice, this approach is considered to be cognitively cumbersome.

Figure 5. **Depiction of Emily's solution process for reducing the fraction 8/12.**



Errors in Fraction Reduction. Just as with the number facts program, this atypical strategy led to specific kinds of errors. A common error was reducing the fraction 8/12 to 2/6 or 4/3. In both instances the numerator of the reduced fraction disrupted Emily's ability to attend to the appropriate multiples list. The process of reducing the numerator seemed to cue a new list corresponding, not to the factor of reduction, but to the numerator of the reduced fraction (see Figure 6). This error type is considered to be the result of cognitive strain stemming from her atypical multiples list strategy.

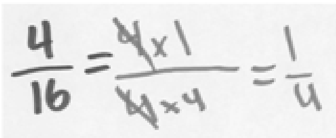
Figure 6. **Depiction of Emily's common error when reducing the fraction 8/12.**

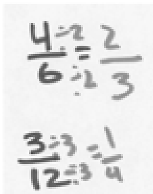


Remediation Approach

To address the math fact-based errors that occurred during her fraction reduction, a remediation approach was designed which attempted to help Emily improve her accuracy by decreasing the cognitive strain of her multiples list strategy. The intervention required Emily either to rewrite the fraction as a multiplication problem or write down the amount by which she was dividing both the numerator and denominator (see Figure 7 for example artifacts from the tutoring sessions). By cognitively offloading the factor of reduction, it was hoped that Emily could use her multiples list strategy more effectively in this context.

***Figure 7.* Written artifacts of the intervention approach for fraction reduction, in which Emily records the factor of reduction.**





Evaluation of Remediation. Nineteen tutoring sessions focused on fraction reduction were considered in the analysis of the intervention approach. In the thirteen tutoring sessions before the implementation of the intervention, Emily had 70% accuracy on fraction reduction problems. In the last six sessions, after the implementation of the intervention, Emily's accuracy rose to 85%. In fact, 74% of her errors after the intervention occurred on a problem where she did not write down the factor of reduction. On problems where she *did* write down the factor of reduction, she had a 93% accuracy rate.

Discussion

This case study sought to determine whether a detailed analysis could shed light on *why* students with MLDs make qualitatively different errors on math fact problems. Emily's errors when solving multiplication fact problems were consistent with those found in students with MLDs (Mazzocco et al., 2008). An in-depth analysis of her solution process and explanations revealed that she relied upon an atypical multiples list strategy. This strategy provides a plausible origin for her pattern of errors. Although the intervention implemented in this study was not sophisticated, it served as an existence proof—once the origin of the errors was understood, it was possible to help the student more effectively employ her atypical strategy. This is a case study of one student and consequently the findings cannot be generalized to all students with MLDs. It is possible that Emily's atypical strategy is unique, and does not explain why other students with MLDs make qualitatively different errors. Instead, this study suggests that atypical strategies may be the origin of seemingly disparate error patterns. Future research is warranted to investigate other atypical strategies and how prevalent these strategies may be in students with MLDs. Understanding the origin of these qualitative differences may ultimately provide the key to differentiating low achievement from MLDs.

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