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Modelling Problem-Solving Situations into Number Theory Tasks: The Route towards Generalisation

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This paper examines the way two 10th graders cope with a non-standard generalisation problem that involves elementary concepts of number theory (more specifically linear Diophantine equations) in the geometrical context of a rectangle's area. Emphasis is given on how the students' past experience of problem solving (expressed through interplay among different modes of thinking and actions that show executive control and decision-making skills) supported them in their route towards generalisation.

Generalisation according to Mason (1996) is "a heartbeat of mathematics" (p.74) and mathematical thinking takes place only when the students work at expressing their own generalisations. Even though there are numerous studies on generalisation, we agree with Sriraman (2003) that there is a lack of research on generalisation in the context of higher-order mathematical processes such as problem solving at high school-level ages, and this is why this paper focuses on this perspective. More specifically our thesis is that modelling problem-solving situations into generalisation tasks related to number theory is useful for learning mathematics and includes two stages: modelling and solving the number theory task that emerges. On the one hand, solving generalisation tasks dealing with number theory serves as a tool for developing patterns, as a vehicle towards appreciation of structure, as a gateway to algebra, and as a rich domain for investigating and conjecturing at any level of experience (Zazkis, 2007). Despite their significance, number theory-related concepts are not sufficiently featured in mathematics education. Consequently, many issues related to the structure of natural numbers and the relationships among numbers are not well grasped by learners (Sinclair, Zazkis, & Liljedahl, 2004). On the other hand, according to Mamona-Downs and Papadopoulos (in press), when students have an accumulated experience in problem solving they can affect changes in approach and are able to take advantage of overt structural features appearing within the task environment. Moreover, students with experience in problem solving can show a deeper understanding of the nature of mathematical generalisations (Stacey, 1989).

In this paper, an extended version of a paper presented in CERME-6 (Iatridou & Papadopoulos, 2010), we follow two 10th graders (15-year-olds) during their effort to cope with a non-standard generalisation problem-solving activity relevant to elementary number theory concepts. Both students had experience in problem solving due to their participation in a project which lasted for three years (from the 5th grade to their 7th grade) and emphasised problem-solving techniques relevant to area. This project conducted by Mamona-Downs and Papadopoulos (in press) addressed technique elaboration associated with a single concept, that is, area, leaving open the issue of addressing other concepts or working in other domains in the same sort of spirit. The current case is interesting since it displays executive control skills related to the way the students proceed when they have to work on a new domain and to the handling and establishment of a "model" that could lead to the generalisation. We explore the interplay between students' approaches during their problem-solving path towards generalisation and at the same time refer to the actions of the students concerning decision making and executive control. In the next section we present a review of the relevant research literature that constitutes our theoretical framework concerning generalisation, number theory and problem solving. We then present the task and describe the students' backgrounds, the procedure of the study, and our data collection and analysis. Then we describe the problem-solving approaches followed by our students (Katerina and Nikos). These are followed by a discussion section trying to shed light on how these two axes (i.e., the interplay and the control issues) facilitate generalisation. We end with the conclusions section.

The Route Towards Algebraic Generalisation ...

The ability to generalise has been considered by Krutetskii (1976) as one of the building blocks of mathematical structure. This ability could refer to mathematical objects, relations and operations as well. But what exactly is generalisation? Polya (1957), viewed generalisation as a gradual "passing from the consideration of one object to the consideration of a set containing that object" (p. 108). According to Polya there is a tentative generalisation which facilitates understanding of the observed objects, making analogies, and testing special cases. However, after that, follows a finer generalisation which could be considered as final only when a mathematical proof takes place. Polya refers to these types of generalisation, respectively, as induction (related to generalisation) and mathematical induction (related to rigorous proof). Harel and Tall (1991) define generalisation as a process of applying a given argument in a broader context, and they discriminate between expansive, reconstructive, and disjunctive generalisation. For Kaput (1992) generalisation in tandem with formalisation is intrinsic to mathematical activity and thinking. We opt for the induction (related to generalisation) of Polya in combination with the approach of Ellis (2007), who refers to the generalisation taxonomy as a generalisation level that includes: forming an association between two or more mathematical objects, searching for similarities and relationships, and extending a pattern, relationship or rule into a more general structure.

An important part of the relevant research studies concerns primary school students. Cooper and Warren (2008) analysed 3rd and 5th graders' ability to generalise in a variety of situations using a variety of representations, and to switch between these representations. Stacey (1989) reports responses of students aged between 9 and 13 on finding and using patterns in linear generalising problems. Amit and Neria (2008) focused on the generalisation methods used by 6th and 7th graders in solving linear and non-linear pattern problems. In a similar spirit, we find results in the work done by Becker and Rivera (2004), Carraher, Martinez, and Schliemann (2008) and Ishida (1997). We could mention much more of the existing studies but since our interest is in secondary school students (15 years old) we will restrict ourselves to presenting analogous studies concerning similar ages. Thus, refer to the work of Balacheff (1988) who, working with 13-14-year-old students using generalising patterns (the number of diagonals of any polygon), found that most students made conjectures about generality by looking at only a few cases (relevant findings found in Cooper and Sakane, 1986). Lee and Wheeler (1987), working with 10th graders on generalising linear and quadratic problems, found that the students in general did not check their generalisations in order to see whether they were correct in particular cases. Rico (1996), working on the same topic but using three symbolic representational systems (figures, decimal number progressions, and arithmetic number sequences), found that the students used the numeric patterns to make generalisations rather than analysing the diagrams for relationships. Orton, Orton, and Roper (1999), working with middle school students in finding pattern generalisations, observed that even though the students were given a geometrical context in which to work, they added or multiplied to identify common differences in number sequences, thus ignoring the diagrams and studying the numbers instead. Finally, Steel and Johanning (2004) and Steel (2008) investigated 7th graders' development of algebraic thinking concerning – among others – identifying and generalising patterns. Their findings demonstrated that the students recognised patterns in related problems which enabled them to describe generalised quantitative relationships in the problem. Even from this small part of the literature it can be seen that the process of generalisation lies within the context of number concepts, arithmetic, and algebra. As already mentioned earlier, there is a lack of research on generalisation in the context of higher-order mathematical processes such as problem solving at high school-level ages (Sriraman, 2003), and this is why we try to focus on that perspective in this case study.

... via Number Theory Tasks ...

According to Zazkis's (2007) plenary lecture in the International Symposium Elementary Maths Teaching (SEMT),

> number theory is useful for teaching and learning mathematics because it provides a powerful introduction to the essence of mathematical activity: It serves as a tool for developing patterns, as a vehicle towards appreciation of structure, as a context for developing proofs and as a gateway to algebra. (p.52).

Unfortunately, number theory in mathematics education has not yet taken the place that it deserves in the Greek official curriculum as well as in the curricula of many countries (Zazkis, 2007). On the other hand, it is worth mentioning that at a research level and since Ball (1988) there has been an increasing interest in number theory as a means of teaching and learning mathematics. We could also mention here studies on topics such as divisibility (Zazkis & Campbell, 1996), even and odd numbers (Zazkis, 1998), factors and divisors (Zazkis, 2000), multiples and the least common multiples (Brown, Thomas, & Tolias, 2002), and prime numbers and prime decomposition (Zazkis & Liljedahl, 2004). The majority of the aforementioned studies are relevant to preservice teachers. An exception is the work of Ginat (2006) concerning 11^{th} and 12^{th} graders and Kieran and Guzman (2006) concerning 12-15-year-olds, in a computer environment. We are also interested in these ages but our topic is the Diophantine linear equation, a topic that is completely unknown to students of secondary education.

... in the Context of Problem Solving

Polya's well known problem-solving model consists of four phases: understanding, planning, implementing, and looking back. Lester (1985) modified it so as to include a cognitive and metacognitive component. In the cognitive components the four phases were relabelled as orientation, organization, execution, and verification, respectively. We will give emphasis on the execution phase which refers to monitoring progress and consistency of local plans and decision making; in other words, the issue of control in problem solving according to Schoenfeld (1985). He described thinking mathematically as developing a mathematical point of view, valuing the processes of representation and abstraction, and having the predisposition to generalise them. Generalisation is presented as a challenge action for early finishers during problem solving. According to Mamona-Downs and Papadopoulos (in press), when students have an accumulated experience of problem solving then they can show a rich output in terms of accessing an appropriate heuristic, forming conjectures, verification processes, and assessing, exploring, and displaying executive control. Stacey (1989) found that students experienced in problem solving used their methods more consistently and showed a deeper understanding of the nature of mathematical generalisation. Sriraman (2003) working in the problem-solving setting found that although his experienced students never found general solutions to the problems, they did consistently work their way up by beginning with simpler cases that modelled the given problem situation. He expressed his findings in terms of orientation and organisation phases. This gave us the opportunity to put forward a study that would emphasise the execution phase of the problem-solving process, highlighting issues such as executive control or interplay among different representations.

Description of the Study

This paper presents the findings of a study involving two students trying to cope with a generalisation problem-solving task that is relevant to Diophantine linear equation. We posed the following problem to the students:

Which of the rectangles below could be covered completely using an integer number of tiles each of dimensions 5cm by 7cm but without breaking any tile?
Rectangle A: dimensions 30cm by 42cm
Rectangle B: dimensions 30cm by 40cm
Rectangle C: dimensions 23cm by 35cm
Rectangle D: dimensions 26cm by 35cm.
For each rectangle that could be covered according to the above condition show how the tiles would be placed inside the rectangle.
Now, we want to cover a rectangle with an integer number of (rectangular) tiles. Each tile is of dimensions 5cm by 7cm. What could be the possible dimensions of the rectangle?

The mathematical problem is: Define a set of necessary and sufficient conditions on a, b so that there exists a rectangle of dimensions a by b, that can be covered completely with tiles of dimensions 5cm by 7cm.

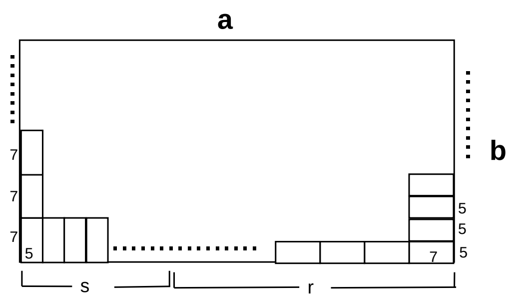

Figure 1. Putting tiles in a non uniform orientation.

Look at the side of length a: If there are s tiles that are placed on it along their 5cm-side and r tiles that are placed on it along their 7cm-side, then a = 5s+7r, where s and r are non-negative integers. If we start with the side of length b then the same reasoning applied gives b = 5s′+7r′, where s′ and r′ are non-negative integers. Now if c denotes the total number of tiles used, then the area ab of the rectangle should be 35c. Therefore 35 divides ab. Thus, there are three cases: i) 35 divides a, ii) 35 divides b, or iii) none of the previous; however, since 35 divides ab, 7 must divide a and 5 must divide b (or vice versa). Consequently, a and b should satisfy one of the following necessary conditions: i) b = 35n, a = 5s+7r, ii) a = 35m, b = 5s′+7r′, iii) a = 7k, b = 5t (or vice versa).

It is easy now to show that these conditions are also sufficient. In the first case (b = 35n, a = 5s+7r) we place s columns with 5n tiles, followed by r columns with 7n tiles as can be seen in Figure 1. In the second case (a = 35 m, b = 5s′+7r′) we place s′ rows with 5m tiles, followed by r′ rows with 7m tiles, and in the last case (a = 7k, b = 5t) we place k rows and t columns of tiles in the direction 7x5 or vice versa.

Thus, even though the context of the task seems to be geometrical because of its relevance to area, a crucial aspect in solving the task is the implementation of a Diophantine linear equation ax+by=c where the unknowns x and y are allowed to take only non-negative integers as solutions (this is why one could equally refer to it as positive integer linear combinations instead of a Diophantine equation).

This idea of an interdisciplinary approach attempting to put the elements of number theory on a geometrical basis (i.e., Diophantine linear equation and area of plane figures) can be understood if one takes into account the particular students' backgrounds. Our task consists of two parts.

In the first part four rectangles have been carefully selected to enable the solver upon finishing the first part to reach the generalisation requested in the second part. The four rectangles have been selected so as to prepare the route towards generalisation. This is in accordance with Polya's (1968) suggestion that generalisation starts from the simplest, most transparent particular case (p. 60).

Katerina and Nikos were 10th graders who had participated in an earlier two-phase research project conducted by Mamona-Downs and Papadopoulos (in press) aiming to explore and enhance the students' comprehension of the concept of area with an emphasis on problem-solving techniques for the estimation of the area of irregular shapes. The overall students' participation in the first phase of the project gave them, on the one hand, sufficient content knowledge that was prerequisite for problem solving ancillary to the concept of area and, on the other hand, experience in the application of various techniques enabling them to calculate the area of some irregular shapes. In the second phase, all the students were challenged to overcome the limitations of the techniques (in the form that they knew them) so as to tackle new problems related to area determination, thus stressing the students' command of their usage of the techniques. This led to a broad range of students' activity. For instance, there were observed cases of students altering the task environment in some way to allow an application, adapting the technique to allow a solution of a particular task, generating a new technique from an old one, or combining two known techniques in tandem. Consequently Katerina and Nikos acquired experience in problem solving concerning area and we tried to make this knowledge base the starting point of their attempts in this interdisciplinary approach to work in a completely new domain. They were above the average level of their classroom and were chosen because of their lively involvement in the stages of the project. There was anticipation that the results of this case study would support more systematic future research as well. The conceptual framework in this work mainly lies in number theory. However, in the official Greek curriculum for 10th graders the only reference to number theory concepts is a brief one commenting on the divisibility rules for the numbers 2, 3, 5, 9, and 10. The case of the Diophantine linear equation as well as the method for its solution is completely unknown to the students and this lack of prior knowledge is an important parameter for this study. Under this perspective this task could be considered as a challenge for the students since the only part from their knowledge base they could recall was the knowledge relevant to the concept of area.

The problem-solving session lasted one hour, without any intervention from the researchers. The students were asked to vocalise their thoughts while performing the task (for thinking aloud protocol and protocol analysis, see Schoenfeld, 1985). Protocol analysis gathered in non-intervention problem-solving sessions is considered especially appropriate for documenting the presence or absence of executive control decisions in problem solving and demonstrating the consequences of those executive decisions, and this minimises the interference of the researchers (the authors). The students' protocols were then parsed into episodes (according to Schoenfeld's (1985) proposed *Framework for Analysis of Problem-Solving Protocols*). Each episode was characterised as one of the following: reading, analysis, planning, implementation, exploration, verification, or transition (juncture between episodes). Emphasis was given to the transitions between episodes since these were the points at which the direction of the problem solution changed or the representation used by the students changed. More specifically, these were the points at which action at the control level or at the interplay between different modes of thinking might be considered (see questions below). These modes do not refer to a certain official term in mathematics education. It could be said that these modes include the various kinds of representations (visual, arithmetic, algebraic) as well as various approaches or ways of thinking.

The students' efforts were tape-recorded and transcribed for the purpose of the paper. Their worksheets and the transcribed protocols constituted our data. The students were prevented from erasing their work and this is why we provided them with more than one worksheet. Since the aim of the study was to investigate how the interplay between different modes of thinking and decision-making actions facilitated the students' problem-solving path towards generalisation, we coded the data according to the levels below:

- Recording the successive movements of the students between different modes of thinking, and
- Identifying the actions of the students that indicate executive control and decision-making skills.

Katerina's Path to Generalisation

Katerina's first criterion for deciding whether the four rectangles could be covered completely by the tile was based on whether the dimensions of the four rectangles were multiples of the dimensions of the tile. This is why her answer was positive only for the rectangle A (since 30=5*6 and 42=7*6) and negative for the remaining three. She used the quotient of their areas (E1/E2, where E1 is the area of rectangle A and E2 is the area of the tile) as a way to determine the number of the tiles required for the covering and not as a criterion to decide whether the tiling is possible. She then tried (according to the task) to show how the tiles would be placed within the rectangle. The visual aspect of this action made the student realise her mistake and re-examine the four rectangles:

K.1.23. The tiles could be placed in any orientation within the big rectangle.
K.1.24. It is not necessary to place all of them in a similar orientation.

After that she verified that the rectangle A could be covered according to the task's instructions. For rectangle B she worked with an interplay between an arithmetical and geometrical-visual approach and she realised that the case of tiles with different orientation could mean that she could work with an "equation" since she was not able to proceed geometrically. This was the first time a linear combination was involved and an attempt was made to express her conjecture symbolically:

K.1.37. It could be ….. 5x+7y=30
K.1.38. It must be a rectangle with length of 30cm and this has to be expressed with some tiles placed along their 7cm-side and some with their 5cm-side.

She was not able to express her thought using proper mathematical terms. Her intention was to say that this equation did not have integer solutions (she had excluded the case for an unknown to be equal with zero). Thus she decided to use terms such as "round numbers" (a primitive kind of symbolization) to show x and y must be positive integers:

K.1.42. However, this case (*the equation 5x+7y=30)* is not possible…
K.1.43. We could not have "round" numbers for x and y.

For the rectangle C she decided to rely on the question of whether the length of the side of the rectangle could be written as a linear combination of the dimensions of the tile, which meant finding the solutions of the specific Diophantine equation. The lack of relevant knowledge in this domain forced Katerina to work with successive multiples of 7 checking every time whether the remainder of the subtraction 23-7x could be expressed in terms of multiples of 5 and thus apply a certain technique for overcoming this difficulty. This could be considered as the first step towards the solving of the linear Diophantine equation, since before finding all the integer solutions of this equation it is necessary to find a particular solution (x_0,y_0). Given that the Euclidean algorithm was not known to the students, the preferred method to find the specific solution was that of working by trial and error. Katerina followed the same line of thought for the rectangle D. The criterion of the linear combination was already established and by applying the technique of the successive multiples she found that:

K.1.67. For the side of 26cm it is necessary to have 3 tiles of length 7cm and 1 tile of 5cm.

Immediately she turned to the visualisation in order to verify that indeed this could be done, working however independently on each dimension of the rectangle D (Fig. 2).

For the second part of the task she started with two steps that according to her opinion could help her:

K.1.74. I will use drawings because this seems easier to me.
K.1.76. How could I exploit the findings of the first part of the task?

And she continued:

K.1.77 From the first part I know that the rectangle A can be covered. It is 30x42.
K.1.78 and the rectangle D can be covered too.
K.1.84 I will draw a rectangle... and there is no restriction about the way I'll place the tiles.

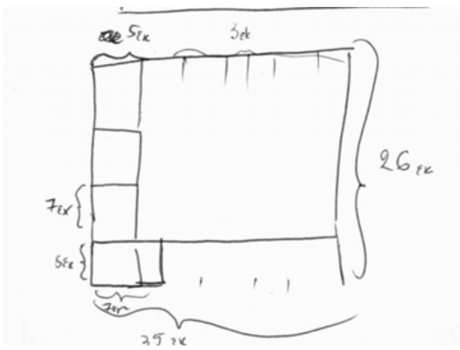

Figure 2. Visualizing the linear combination.

She started by focusing on the positive cases (A, D) of the first part, avoiding mentioning the negative ones (B, C). In order to respond to the demand of the dimensions of any rectangle according to the task's statement, she drew a random rectangle placing the tiles along its one dimension successively but in an accidental orientation (Figure 3, top). This was her way to guarantee the general case of "any" rectangle.

Now she had to face the required number of tiles needed for the coverage as well as to determine the rectangle's dimensions. She rejected the condition of E1 being an integer multiple of E2 as the unique criterion since:

K.1.87. …it might be necessary for a tile (or some tiles) to be split.

Her model for finding the possible dimensions of any rectangle that could be covered by tiling using an area unit (tile) with dimensions 5cm by 7cm included two cases exploiting her previous findings of the first part of the task.

The first case resulted from considering rectangle A:

K.1.92. If all tiles are oriented uniformly then the asked dimensions of the rectangle could be multiples of 5 or 7.
K.1.93. I will make a drawing (Figure 3, bottom).
K.1.94. It is a shape whose length is a multiple of 7 and width a multiple of 5.

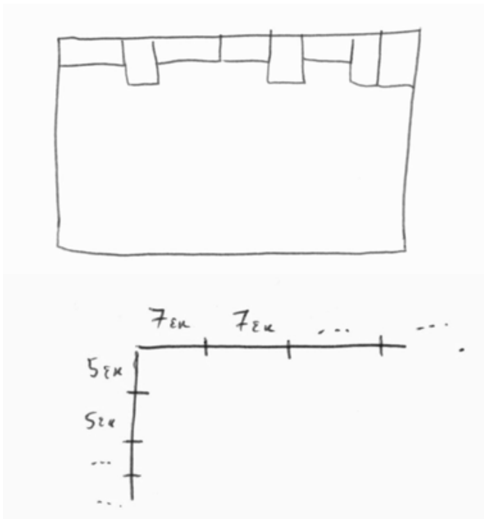

Figure 3. The general case of accidental orientation (top) and uniform orientation (bottom).

The second case resulted mainly as a consequence of the rectangle D, and two conditions must be satisfied: one side must be multiple of the least common multiple of the dimensions of the tile and the second side must be a linear combination of them as well.

K.1.96. Its length must be multiple of both dimensions of the tile
K.1.97. that is, 35, or 70, 140, etc.
K.1.98. whereas its width will be a sum...
K.1.99. for example x tiles will be placed along their 7cm-side and y tiles along their 5cm-side
K.1.100. This case is like rectangle D
K.1.101. Length must be common multiple of 7 and 5 whereas width must be sum of tiles of which some are oriented horizontally and others vertically.

She then tried to refine her model asking for a rule that governs the common multiples of 5 and 7 (i.e., of 35). For number 5 she knew the divisibility rule (the last digit must be 0 or 5) but she could not give any rule for the 7 or the 35. Finally, she finished with a recapitulation of her model trying to describe in a more formal way the second case:

K.1.110. The rectangle in the second case should have one of its dimensions expressed as a common multiple of both 7 and 5 and the other one as a sum of multiples of both 5 and 7.

Nikos's Path to Generalisation

Nikos's first step was to interpret the statement of the problem in relation to the correct tiling: a) there is a rectangular region that has to be covered and b) the tile is a structural element of the task:

N.1.5. Each rectangle must be covered and I must use an integer number of tiles
N.1.6. So I could consider this tile (5X7) as a measurement unit.

In his work we can distinguish a concrete line of thought. For the rectangle A, his criterion was (as in Katerina's case) the proportionality of the sides, that is, whether the dimensions of the rectangle were multiples of the tile's dimensions. We have to mention here that his way of reading the task was non-linear in the sense that he did not follow the instructions of the task in the given order. According to the task he had to answer whether the rectangles could be covered by an integer number of tiles and then to show the way the tiles would be placed in their interior. The fact is that he worked independently on each rectangle. When his answer was positive he immediately worked on the way the tiles could be placed in this rectangle. In case there was no proportionality among the lengths of the sides of the rectangle and the tile – as it happened in the rectangle B – he used the criterion of E1/E2 as a way to ensure a negative answer. In B this quotient was not an integer number and this meant that there could not be coverage according to the task's statement. As he explained:

N.1.20. Because the ratio of their areas is not an integer.

Now, in the rectangle C, the E1/E2 was an integer but the dimensions were not proportional. It is interesting to note the fact that Nikos's decision about E1/E2 is justified by the fact that E2(=35cm^2) is a factor of E1(=23*35), a property often overlooked even by pre-service elementary school teachers (Zazkis & Campbell, 1996). In their study and in an analogous quotient, teachers first calculated the product and then divided. At that point, Nikos asked for the linear combination that satisfied one of the dimensions (23cm) since the second (35cm) was a multiple of 5:

N.1.24. When the area is 23 by 35, then obviously this product is divided by 35 which is the area of the unit (*tile*)
N.1.27. The point is **the way** the tiles must be placed
N.1.29. We could have 3*7+2, 2*7+9
N.1.34. 5+5+5+8, 4*5+3,….
N.1.35. For the 23 cm I can't make any **combination** of 5s and 7s.

In the rectangle D, he applied directly the rule of the linear combination that could satisfy the side of 26cm since the other one (35cm) was a multiple of 5. Trying to describe how the tiling would take place he initially worked independently on each side. However, the way the tiles are placed in one dimension affects the way the tiles are placed in the second. This made Nikos turn towards a consideration of both dimensions at the same time and as a result he succeeded (Figure 4).

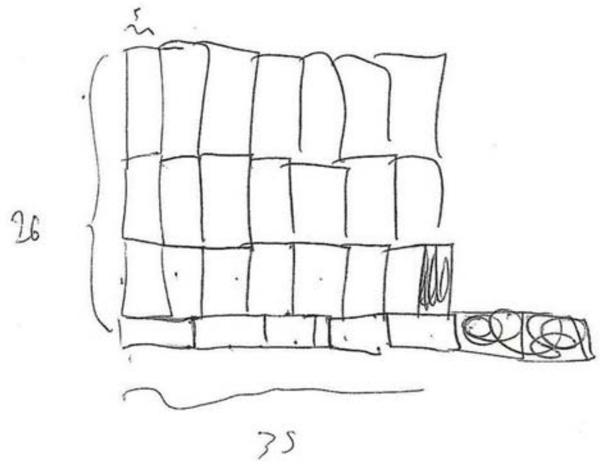

Figure 4. Considering both dimensions at the same time.

For the second part of the task the need to find the possible dimensions of any rectangle was translated by Nikos as a necessity to find a formula. Even though the task does not speak clearly about generalisation, the student confronted the task according to this perspective. For him, generalisation is equivalent with a symbolic form that holds for any rectangle.

Despite the fact that the method followed in the first part could be considered adequate for him to give a general answer for each rectangle, he preferred to re-check the given rectangles B and C. It is not accidental that he first reconsidered the negative cases since these are the ones that must be avoided.

He continued with an impressive conjecture:

N.1.83. Obviously, if we want to cover a rectangle with this specific unit of dimensions 5 by 7, then the rectangle's sides must be the sum of multiples of 5 and 7 at the same time.
N.1.84. The case of 0*5 and 0*7 must be included in this.

However, he still considered the two dimensions separately. Trying to determine what would be the general case for the asked dimensions of the rectangle, he created some arithmetical examples.

He considered the tile 5x7 as the first of a sequence of new rectangles. He created a series of rectangles that is, 5x7, 5x14, 5x21, et cetera, keeping one dimension constant (i.e., 5), and changing the other one obtaining successive multiples of 7.

N.1.95. I start from the 5x7 case which obviously is the simplest case.
N.1.96. Then I place one tile next to it and now the dimensions are 5 and 14 (=2x7).
N.1.97. Next I can put another tile and I can make it 5 by 21 (=3x7), and so on.

He fulfilled the need for a linear combination for each dimension, without considering the fact that there is interdependence among the two dimensions since the area of the rectangle must always be a multiple of 35:

N.1.102. We could say that a = 5x+7y (where "a" is one of the rectangle's dimensions)
N.1.103. and similarly b = 5z+7w ("b" is the other dimension)
N.1.104. The product of these dimensions a and b will be the area.
N.1.105. I can choose for a and b any sum of multiples. For example, a=5+14=19, b=15+28=43. So, the area is 19*43.

Using a rectangle with arbitrarily chosen dimensions he tried to handle a general case. The dimensions of this rectangle were linear combinations of 5s and 7s with arbitrary coefficients.

Soon, Nikos realised that this was not enough:

N.1.106. In that case I have for the area a number that is not divided by 35.
N.1.107. So, 35 must divide the product a*b which is the area of the rectangle.
N.1.112. Thus, a=5x+7y, b=5z+7w, and the quotient ab/35 must be an integer.

He continued searching for more examples to validate his conjecture for the general case.

Then, trying to establish a model that would describe all the possible cases he was further influenced by the four rectangles of the first part of the task. He considered rectangles A and D as positive cases, and B and C as negative ones that must be avoided. All of them contributed equally to generalisation. He decided that his model would include two types of rectangles:

N.1.141. The first type concerns rectangles with one side a multiple of 5 and the other one a multiple of 7. So, a=5x and b=7y, which is a=5x+0*7 and similarly b=0*5+7y.
N.1.142. Consequently the area of such a rectangle divided by 35 gives an integer number as quotient.

N.1.154. And this is in accordance with the general form I conjectured earlier.

For the second type he decided that:

N.1.159. One of the rectangle's sides will be a sum of multiples of both 5 and 7.
N.1.160. whereas the second side will be a multiple of 35
N.1.171. that is a=5x+7y and b=35z
N.1.172. I think that these latter conditions **outline the most general form** for the dimensions of any rectangle able to be covered with tiles 5 by 7.

After that, Nikos applied this most general form for each of the four given rectangles to check its validity. Furthermore, he made clear that the first type of rectangles could be incorporated in the second:

N.1.188. … to incorporate the first type which essentially is a special case of the second more general type …

Finally, Nikos refined his model determining the circumstances that do not allow a rectangle to be covered according to the task by giving a certain counterexample:

N.1.213. The second side must always be multiple of 35 and it can be constructed using either 5s or 7s.
N.1.218. This is the only solution because 35 is the least common multiple of 5 and 7.
N.1.219. This means that it is not possible to have a rectangle for which both its dimensions are linear combinations of 5s and 7s.
N.1.220. When I say that a is a linear combination of 5s and 7s, I mean that a=5x+7y but not a multiple of 5 or 7.

Discussion

In relevance to our research questions we could make some comments on our fieldwork.

Interplay among Different Modes of Thinking

The task was designed so as to allow solvers to move between different modes of thinking (i.e., different kinds of representations or different thinking ways and approaches). According to Douady and Parzysz (1998) an interplay between these different modes is caused during the problem-solving process. They claim that the effort of the solver to reach the solution results in the co-relations of these modes as well as in the usage of some tools that each of them employ. Additionally "...this interaction provides new questions, conjectures, solving strategies, by appealing to tools or techniques whose relevance was not predictable under the initial formulation..." (p. 176). The flexibility in the transition from one form of representation to another could be considered as an aspect of students' *mental* flexibility, as can be found in Amit and Neria (2008). In their work the authors present the students' shifting from graphical representations to numerical and later to verbal and symbolic ones for the sake of the continuation of the solution process. During their attempts to solve the problem, our students worked in tandem with two pairs of modes. The first pair included the arithmetical mode and visualisation. Both students started arithmetically even though the context of the task was relevant to area, which is geometrical. From the very beginning Katerina used the visual aspect as a tool. She started arithmetically but when she was unable to proceed with numbers she preferred to make drawings that would help her (K.1.74). In the same spirit sometimes she moved from the visual context to algebra. At some point she acknowledged that it is not necessary for the tiles to be placed with the same orientation (K.1.23-K.1.24). However, she was not able to proceed geometrically and she preferred to turn to algebra asking for an equation (K.1.37). Nikos did not choose to work with this pair of modes. He mainly worked arithmetically and he turned to the visual aspect only to show the way the tiles could be placed in the interior of the four rectangles in the first part of the task. This moving between representations, among other things, reveals the mathematical structure underlying the task, as has been shown by Polya (1957) or Mason, Burton, and Stacey (1982). However, what is interesting here is that the students used this interplay as a tool to overcome difficulties in their problem-solving process. Whenever they got "stuck" they tended to turn to another representation, one that was more suitable to cope with the certain difficulty.

The second pair of modes has to do with the way students dealt with the dimensions of each rectangle. Working with the first mode (arithmetical mode), dimensions were considered by the students separately as two unconnected objects. Thus, they made calculations (they summed, multiplied, divided) to determine the way the tiles should be placed in one dimension. In the second mode (geometrical mode, relevant to area) the dimensions were interdependent. The fact is that the way the tiles will be placed in the first dimension influences the way the tiles will be placed in the second dimension. Working independently in two dimensions does not guarantee that the total area of the rectangle will be integer multiple of 35 which is the tile's area. Both students made successive movements between these two modes. Their initial approach was to work separately for each dimension. This finally resulted in making the connection concerning the interdependence of the two dimensions. For example it is clear that Nikos (N.1.102-N.1.112), working separately on each dimension, produced a rectangle that could not be covered with integer number of tiles since its area was not a multiple of 35. Therefore, it is interesting to see how the students drew on these modes of working in their attempts to coordinate the interrelationships between height and breadth when generalising their solutions from specific examples. This interplay constitutes an effective tool in their problem-solving tool-bag and not merely the key for solving the problem.

As a consequence of this interplay emerges – for Nikos in particular – the issue of putting forward a set of conditions (N.1.112) that are evidently realised as being *necessary* and later an equivalent set of conditions (N.1.172) that are seen as *sufficient* (because the covering of the relevant rectangles can be explicitly constructed).

Executive Control and Decision-Making Issues

Both students had an accumulated experience in problem solving and consequently we anticipated that this could contribute to a successful generalisation since the ability to generalise is not an exclusive inherent attribute. On the contrary, as Sriraman (2004) claims, it can be developed via certain experiences that allow students to monitor and reflect on their work and this will finally enable students to become capable of making generalisations. We share Sriraman's perspective that this ability for generalisation could be the result of certain mathematical experiences. In this study we are more specifically interested in the part of the problem-solving experience that is relevant to executive control and decision-making issues. "Executive control" and "decision making" constitute in general the issue of control in problem solving. Executive control is concerned with the solver's evaluation of the status of his/her current working vis-à-vis the solver's aims. In Schoenfeld's (1988) words: "there is a feedback loop that consists of monitoring one's actions on line, assessing progress, deciding whether change needs to be made, and taking action if the situation is deemed problematic" (p.67). This requires mature deliberation in projecting the potential of the present line of thought, along with an anticipation of how this might fit in with the system suggested from the task. In other words, the solvers monitor and assess both the state of their knowledge and the state of the solution and avoid the kinds of "wild chases" that often guarantee failure in the evolution of the solution.

The students realised many actions that indicate interesting executive control and decision-making skills. This was probably connected with their past experience in problem solving as a result of their participation in the specific project. Katerina rejected her initial approach, which was based only on the criterion of proportionality between the rectangle's and the tile's dimensions, because her turn to visualisation made her realise that it was not necessary for the tiles to be placed in a uniform orientation (K.1.23-K.1.24). This turn was in effect an important act of control. The task's instructions did not give any direction concerning the way the tiles could be placed inside the rectangle. It was up to her to interpret correctly the instructions. When she tried to solve the Diophantine equation she applied the technique of the successive multiples according to which if one has to solve the equation ax+by=c he/she starts with positive multiples of a and then examines whether c minus ax can be expressed in terms of multiple of b or vice versa (i.e., one starts with multiples of b) (K.1.67). This is an act of control as well since the solving of the equation was related to the task's limitation of using an integer number of tiles without breaking any of them. When Katerina decided to deal with the second part of the task her first thought was to exploit her previous results (K.1.76) which is in accordance with Polya's: "Can you use the result?" (Polya, 1957). Moreover, an important act of control was the "model" she proposed for estimating the possible dimensions of any rectangle that could be covered with an integer number of tiles according to the statement of the task (K.1.92, K.1.110). She exploited her previous findings (the four rectangles of the first part), and progressively she established this "model" checking step by step its accordance with these rectangles and also with examples generated by herself. The choice of these examples depends on the specific context in which the task is set and one has to be cautious because not every set of examples will facilitate a successful generalisation. There are some features of examples that help more than others as a means towards generalisation (Zazkis, Liljedahl, & Chernoff, 2008).

Nikos also made an analogous proposition of a "model." He also relied on the four rectangles of the first part of the task. The steps followed by his line of thought reveal presence of control:

> First look if there is proportionality among the dimensions. See also whether E1/E2 is not an integer. In that case the rectangle cannot be covered with integer number of tiles and the answer has to be negative. It is not necessary always to make the long division E1/E2. Instead, see whether E2 is factor of the E1(N.1.24). Now if there is not proportionality among dimensions and E1/E2 is an integer, then construct the necessary Diophantine equation and apply a strategy to find integer solutions.

However, it was not enough for him just to establish a model. After finishing the description of his generalisation model he checked its consistency against particular examples. This is important given that in a study conducted by Balacheff (1988) (in a slightly different context, that of patterns and generalisation) concerning 13- and 14-year-old students it was found that most made conjectures about generality by looking at only a few cases. Obviously, we cannot draw conclusions since we have only two students in our case study but we have evidence from an ongoing study with students without prior experience in problem solving. Our findings so far seem to strengthen the hypothesis that this checking of the consistency is rather connected with the students' prior experience in problem solving, thus constituting a component of their problem-solving behaviour. Besides, the absence of efficient control behaviour would have sabotaged their attempts.

Nikos interpreted the second part of the task as a question for a formula giving the possible dimensions of the rectangle. He seemed to be convinced that the generality must be expressed in a symbolic form.

Rectangles B and C have been reported as the cases that cannot be covered by tiles 5X7. Since the task asks for the possible dimensions of any rectangle, these two negative cases served as the examples to be avoided. Nikos turned often to them (N.1.59 and N.1.77, 78) to ensure his understanding of why these examples are negative ones. He generated his own rectangles besides the four given ones (N.1.87) since he felt the need for more of them to reach the general case and trying to do this he followed two directions: First, he kept constant the length of one side while changing the length of the other one by successive multiples, thus checking a variety of cases (following the specific rule of the technique of successive multiples). Secondly, he translated the general case of "any" as placing the tiles in a random orientation (horizontally or vertically). He considered that he could reach the general case by checking a random and "complicated" case (N.1.123).

This continuous checking of their steps in the problem-solving procedure that both students showed is especially significant as an act of control since students do not usually check their generalisations (Lee & Wheeler, 1987; Stacey, 1989).

A capable problem solver recognises a correct approach and insists on it. This evaluation of a specific approach could also be considered as an act of control. Nikos recognised the applicability of the linear combination and used it to check the plausibility of his answers according to task conditions (N.1.154). This frequent turn to the tasks' instructions was a common pattern for both students. However, perhaps the most important act of control on their behalf was their effort to refine their model regardless of whether they succeeded. Katerina tried without success to achieve a condition for the second side to be a common multiple of 5 and 7. Nikos did manage to refine his "model" determining whether it is impossible for a rectangle to be covered according to the task's requirements (N.1.219). He did that by asking for a counterexample. This action of the student could be considered as especially important since according to the literature students usually do not think a counterexample to their generalisation is important (Cooper & Sakane, 1986) or they tend to doubt the data when a counterexample is uncovered (Stacey, 1989).

While negotiating the issue of control behaviour in this subsection, the mathematics of the protocols were presented and commented on in tandem with the problem-solving features of the students work. However, it would be worth making effort to distinguish and present them in a more systematic way. In Table 1 the two columns correspond to these two aspects of the students' work connecting every aspect of the problem-solving process (control features) with the mathematical part of the problem's solution.

Table 1
Mathematics of the Protocol and General Problem-Solving Features

Problem-solving features	Mathematics of the protocol
Rejecting a solution path (realising the uniform orientation is not required)	Proportionality of the lengths of the rectangle and the tile sides is not sufficient
Applying techniques (working systematically in trial and error)	Obtaining a particular solution for the linear Diophantine equation
Exploiting previous results	Obtaining the general case
Generating examples	Reaching progressively generalisation
Checking against examples	Ensures validity of the generalisation model
Recognising a correct approach	Linear combinations are used for verifying that the arithmetical data are in accordance with the task's statement
Describing sequence of steps	Determines whether a specific rectangle could be covered with an integer number of tiles according to the statement of the task
Refining model (looking back)	Asking for counterexample

Conclusions

In this work we tried to explore students' routes towards generalisation in the problem-solving context putting elements of number theory (Diophantine linear equation) on a geometrical basis (area of plane figures). The students' ways of working were examined taking into account a) students' accumulated experience in problem solving related to the concept of area and b) the fact that this domain was completely new and students had no prior experience in number theory. More specifically we tried to explore how the interplay among different representations and students' activities that show executive control and decision making supported their effort to make generalisations.

We found that both of our students were able to apply this interplay among two pairs of modes. In the first pair (arithmetical-visual) this interplay was used as a way to overcome difficulties about how to proceed or verifying the validity of an argument. In the second pair of modes one mode (arithmetical, working on one dimension) was indicative of a surface understanding of the structural elements of the task but it seemed that finally the students did show a deeper understanding of these elements through the other mode considering both dimensions at the same time (geometric, interdependent dimensions).

As far as the executive control and decision-making skills are concerned, we found that despite their age, these 15-year-old students showed considerable performance in relation to the task's requirements on the one hand, and the specification of the "model" they proposed for solving the task on the other hand. They were able to insistently explore promising perspectives while abandoning other ones on the basis of how these fit to the task's requirements or their potential contribution to the problem's solution. They developed certain techniques and established "models" that described the situation and facilitated the solution, at the same time checking their validity or their accordance with the task's instructions step-by-step. Equally important was their ability to generate examples and ask for counterexamples.

Last but not least we must refer to the students' decision to use a kind of symbolic language in order to make the distinction between the two types of rectangles. Algebraic symbolism according to Mason (1996) is the language that gives voice to algebraic thinking, the language that expresses the generality, but symbolic description does not necessarily entail the use of letters. Even though our students did not employ a formal way of using letters as a vehicle of symbolisation, the invention of terms such as "simple" rectangles and "complex" ones is a form of verbal symbolisation and this is an indication of *symbol sense* which actually is the essence of algebraic thinking (Arcavi, 1994). In his paper Arcavi describes symbol sense as the ability to understand how and when symbols can and should be used to display relationship and generalisations.

In conclusion we would like to refer to some final remarks that emphasise the significance of our results. It is common thesis that the task design is a crucial parameter for teaching and learning algebra at every level so, in reference to our work, we could claim that the setting of modelling problem-solving situations into number theory tasks allows students to:

- transfer previously-acquired knowledge about problem solving to other concepts (i.e., generalisation) or other domains (i.e., elementary number theory) during their successful interplay among different modes of thinking (algebraic thinking and geometrical thinking);
- construct and propose a "model" that possibly describes the situation and facilitates the solution;
- generate examples that check the consistency of their model; and
- generate counterexamples that result in the refinement of the proposed "model."

Obviously it would be an exaggeration for these conclusions to be generalised since we dealt with only two students and this study should be considered as a case study. However, these findings were encouraging enough to call for the design of future research on these aspects of problem solving that involve successful transference of knowledge about problem solving from one domain to another.

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