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How High is the Tramping Track? Mathematising and Applying in a Calculus Model-Eliciting Activity

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Two complementary processes involved in mathematical modelling are mathematising a realistic situation and applying a mathematical technique to a given realistic situation. We present and analyse work from two undergraduate students and two secondary school teachers who engaged in both processes during a mathematical modelling task that required them to find a graphical representation of an anti-derivative of a function. When determining the value of the anti-derivative as a measure of height, they mathematised the situation to develop a mathematical model, and attempted to apply their knowledge of integration that they had previously learned in class. However, the participants favoured their more primitive mathematised knowledge over the formal knowledge they tried to apply. We use these results to argue for calculus instruction to include more modelling activities that promote mathematising rather than the application of knowledge.

The history of mathematics is full of stories that tell how many mathematics topics grew out of real world problems: Probability theory grew out of gambling dilemmas, trigonometry had its origins in astronomers tracking planetary motion. In spite of this, many students' experiences of school mathematics have led them to regard the subject as dry, abstract, and irrelevant to the real world. A growing number of mathematics educators concerned with this negative view of mathematics have advocated the use of real world problems in classrooms as a way of connecting the *mathematical world* to the *real world* (Freudenthal, 1968; Pollak, 1968). Two of the most common ways of doing this are modelling activities and application problems. These two ways connect the real and mathematical worlds in ways that emphasise processes that are related but different. Modelling activities require students to develop a mathematical model by mathematising a real world situation, whereas application problems require students to apply a previously learned mathematical model to a real world context (see Figure 1).

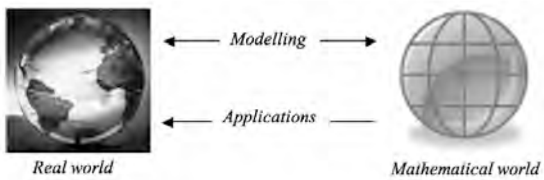

Figure 1. The difference between modelling and applications, adapted from Lesh & Doerr (2003, p. 4).

A group of researchers has developed a class of modelling activities called Model-Eliciting Activities (MEAs) that are designed to mimic the kinds of real world problems encountered in business, engineering, science, and other mathematics-heavy fields (Lesh, Hoover, Hole, Kelly, & Post, 2000). The researchers claim that MEAs are more productive instructional activities than application problems, citing the enhanced conceptual understandings students develop while *mathematising* real world situations (Lesh & Doerr, 2003). However, they warn that the effectiveness of MEAs largely depends on the timing of their implementation (Lesh, Yoon, & Zawojewski, 2007). When MEAs are implemented before any direct instruction on the topic, they serve their intended role of encouraging students to develop their own understandings through the process of mathematising. In contrast, they argue that when MEAs are implemented at the end of an instructional unit, they resemble application problems, in which students can apply what they have already been taught. Does it necessarily follow that students will not engage in mathematising in a MEA that is administered after an instructional unit?

In this article, we investigate the claim that students develop deep conceptual understandings by mathematising real world contexts after an instructional unit, using data from the implementation of a calculus MEA. The MEA is set within the context of tramping, and was implemented after participants received direct instruction on differentiation and integration. Therefore, the participants were able to approach the MEA in two ways: (a) as a modelling activity in which they mathematised the tramping context, or (b) as an application problem in which they applied their previously learned knowledge of calculus to the tramping context. In this article, we analyse the mathematical understandings of four participants who approached the MEA in both ways. While their mathematisations led to primitive understandings of how a gradient graph reveals the value of the anti-derivative as a measure of height, their attempts to apply their prior knowledge of integration were unsuccessful and revealed their limited conceptual understanding of the topic. We suggest that there are important educational benefits in implementing MEAs even at the end of an instructional unit, as they give students an opportunity to deepen their understanding of the mathematical topics by mathematising, as well as an opportunity to apply their knowledge.

We begin this article by describing the theoretical framework in which we use the terms *applying* and *mathematising* in the context of MEAs. Next, we describe the participants in our study, and the methods used to collect and analyse the data. We then present and analyse the mathematical thinking of the four participants, and compare their applied knowledge with their mathematised knowledge. In the final section, we discuss our results and call for calculus instruction to adopt more modelling activities, which emphasise the process of mathematising over the application of a particular technique. This article is an extension of Yoon, Dreyfus, and Thomas (2009) where initial findings were reported; we have incorporated an additional case study (of two teachers) and included a more detailed theoretical framework and discussion section.

Theoretical Framework: Mathematising and applying in MEAs

The framework we use to analyse our participants' mathematical thinking is based on the two processes of mathematising and applying that can occur within MEAs. In this section, we clarify what we mean when we use the terms *mathematising* and *applying* by considering them in the context of modelling activities and application problems respectively. Then we justify how we propose to identify both mathematising and applying within MEAs. Our framework draws on the emerging "Models and Modeling Perspective" (Lesh & Doerr, 2003), which supports the use of MEAs.

Applying and Applications Problems

The majority of "real world problems" that are found in calculus textbooks can be classed as application problems, which require students to apply some knowledge they have previously learned. The following is a typical example of an application problem taken from "Delta Mathematics", a calculus textbook that is popular in New Zealand:

> The top and sides of the curtains for the stage in a theatre are decorated with a string of small electric lights 24 m long. Calculate the height of the top above the floor of the stage if the area of the curtains is as large as possible. (Barton & Laird, 2002, p. 121)

This problem comes from a chapter called "Applications of differentiation", which follows chapters on differentiating various types of functions, using the product and quotient rule, and identifying turning points and points of inflection. Thus, if students follow the progression of chapters in the textbook, they will have learned the calculus techniques that are needed to solve the problem. Furthermore, the textbook provides students with a list of seven steps that detail how to solve problems like the one described in which they are required to "set up a function in terms of one variable only before differentiating to find a maximum or minimum" (Barton & Laird, 2002, p. 118). Thus, students encountering the above problem have already learned how to solve problems of its type, and need only see through the disguise of curtains, electric lights, and theatre stage to recognise what it calls for. Researchers consider use of context in these kinds of application problems as merely "dressing up" a mathematics problem that the student has already learned how to solve (Blum & Niss, 1991). Consequently, the main thinking process involved in these problems is "undressing" the real-world context to decipher which procedure to apply.

Application problems can be useful because they give students the chance to see the utility of the mathematics they have learned by applying it in an extra-mathematical context. However, because the goal of such problems is to demonstrate the application of a particular procedure, the context is often sparse and contrived, and needs to be interpreted in a non-realistic way. The overuse of such activities can give the impression of mathematics only being useful in unrealistic contexts (Lesh et al., 2000). In contrast, modelling activities tend to be context-rich problems that do not assume the student has already learned a procedure for solving the problem;

instead they require students to create their own models by mathematising the context and constructing new understandings in the process.

Modelling and Mathematising

Mathematical modelling involves the complex coordination of a variety of processes that can be depicted around the modelling cycle as in Figure 2. The modelling cycle begins in the real world, where one determines which pieces of information in the real context are relevant to the problem. Next, one interprets the relevant information in the real world mathematically to create a mathematical model. This model is then used to find a mathematical result, which is in turn interpreted back into the real world context. The fitness of the model is then assessed, and if necessary, the cycle begins again in pursuit of a model with a better fit (for a more comprehensive discussion of the modelling cycle, see Stillman, Galbraith, Brown, & Edwards, 2007).

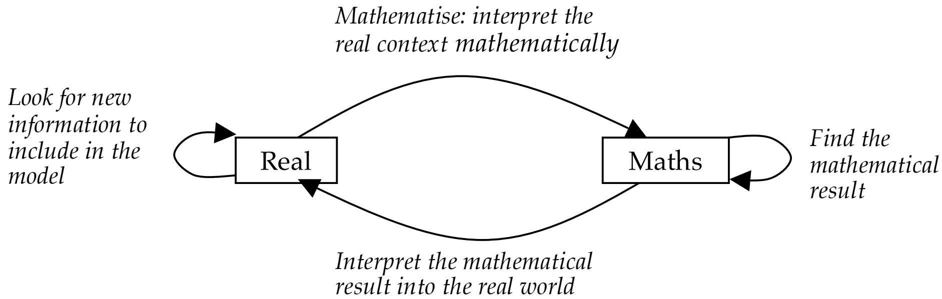

Figure 2. Elements of the modelling cycle.

Although we acknowledge that modelling involves many processes, in this article we focus particularly on the process of mathematising. Lesh and Doerr (2003) define this process of mathematising in MEAs in the following way:

> Model-eliciting activities usually involve mathematizing – by quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities. (p. 5)

The "objects, relationships, actions, patterns, and regularities" in the above definition are echoed in Niss, Blum and Galbraith's (2007) description of a mathematical model:

> A mathematical model consists of the extra-mathematical domain, D, of interest, some mathematical domain M, and a mapping from the extra-mathematical to the mathematical domain. Objects, relations, phenomena, assumptions, questions, etc. in D are identified and selected as relevant for the purpose and situation and are then mapped – translated – into objects, relations, phenomena, assumptions, questions, etc. pertaining to M. (p. 4)

We interpret the "objects, relations, phenomena, assumptions, questions, etc." in the extra-mathematical domain as comprising a system in the real world, whereas the "objects, relations, phenomena, assumptions,

questions, etc." pertaining to the mathematical domain corresponds to the mathematical entities relevant for the solution of the mathematised real world problem. Thus, we characterise the process of mathematising as interpreting the structural aspects (i.e., the objects, relations, actions, patterns, regularities, assumptions, etc.) in a real world system, and expressing this structure in a mathematical model using mathematical representations such as symbols, text, graphs, diagrams, and so forth.

Mathematising and Applying in Model-Eliciting Activities

As mentioned at the beginning of this article, MEAs were designed to provide students with authentic experiences of modelling through the mathematisation of a rich context. In our study, we created a MEA called *The Tramping Problem* in close adherence to the six principles for designing these types of activities (for a detailed explanation of the six principles, see Lesh et al., 2000). In accordance with the *reality principle*, the problem is set in the context of tramping, and begins with a newspaper article that discusses the inadequacy of difficulty ratings for tramping tracks in New Zealand. After reading the newspaper article, students work on warm-up activities that ask them to find the gradient graph (i.e., derivative) of a distance-height graph of a tramping track.

The problem statement then presents the gradient graph of a tramping track shown in Figure 3, and asks students to develop a method for finding the distance-height graph of the original track, and generalising their method so that it works for any gradient graph. By asking students to develop a method instead of merely providing their solution, the activity fulfils the *model construction principle*, and by asking them to generalise their method, it also fulfils the *model generalisation principle*. In mathematical terms, the problem amounts to creating a method for finding an anti-derivative of the given gradient graph, and the embedding of this mathematical task in the tramping context satisfies the *simple prototype principle*.

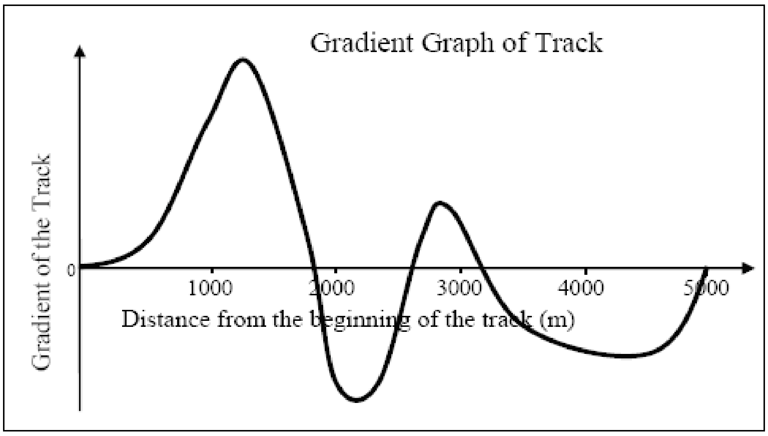

Figure 3. The gradient graph given in the modelling task.

The problem statement asks students to write their method in the form of a letter to clients who wish to determine whether the track is suitable for their purposes, thereby fulfilling the *model documentation principle*. Finally, students are instructed to use their method to draw the distance-height graph, which gives them a chance to test and revise their method, thereby satisfying the *self-assessment principle*.

Lesh, Yoon, and Zawojewski (2007) acknowledge that even though MEAs are designed to encourage authentic modelling experiences, their ultimate success in this endeavour is subject to the timing of the MEA implementation. They describe two possible types of implementation scenarios, which they call (1) Making mathematics practical, and (2) Making practice mathematical. In the first scenario, the MEA is implemented at the end of an instructional unit, which "guides students along (necessarily narrow) conceptual trajectories towards a textbook's (or teacher's) cleaned-up version of the meaning of the relevant concepts or abilities" (Lesh et al., 2007, pp. 316-317). This approach turns the MEA into an opportunity for students to apply what they have been taught, thereby seeing how the mathematics they learned is practical.

In the second scenario, the MEA is implemented before any instruction in the relevant mathematical concepts or abilities. Students are encouraged to mathematise the realistic context and "express, test, and revise their own relevant ways of thinking" (Lesh et al., 2007, p. 316) before being formally introduced to the mathematical concepts or abilities. In the instructional unit following the implementation of the MEA, teachers focus on helping students to "clean up" their primitive ways of thinking, and "endow them with more elegance, power, sharability, and reusability" (p. 316). Thus, students working on a MEA in this approach are encouraged to develop mathematical understandings in response to a practical need.

In our study, we chose to implement the MEA *after* students had received traditional instruction on the relevant topics of differentiation, anti-differentiation, and integration. This implementation can therefore be construed as adhering to the "Making mathematics practical" scenario, which emphasises the process of *applying*. However, we set the tramping problem within a purely graphical representation, which meant that a straight-forward application of integration was not obvious. Although the students had some previous experience in finding graphical derivatives without recourse to algebraic representations, they had not been exposed to the inverse problem (i.e., finding a graphical representation of an anti-derivative). Thus, although we implemented the MEA at the end of the instructional unit, the design of the problem statement meant that students were encouraged to engage in the process of *mathematising* at least as much as the process of applying. Thus, we use the theoretical framework of mathematising and applying in MEAs to analyse the students' modelling during the *Tramping* MEA.

Methodology

The data presented in this article were collected within a larger study that investigated the calculus knowledge of undergraduate mathematics students. Eighteen participants overall were involved in this study: Sixteen

participants were undergraduate students who were enrolled in a calculus course at a large university in New Zealand, and two were secondary school mathematics teachers who participated in the pilot test. The participants worked in pairs on four 1-hour long activities that focused on graphical representations of anti-derivatives. They worked in the presence of an interviewer in a quiet room, and were audiotaped and videotaped. The undergraduate students completed these tasks outside of class time, and were given $25 per hour each, whereas the secondary school mathematics teachers completed the activities for professional development. The interviewer presented the tasks in each activity, but refrained from giving advice on how to solve the problems.

In this paper, we report on the work from four of the participants from this study: Cam and Sid, who were two male undergraduate students, and Ava and Noa, the two female secondary school mathematics teachers from the pilot study. Cam and Sid were enrolled in a bridging course that covered similar levels of calculus content to that found in Year 13 (the last year in New Zealand secondary schools). By the time they started work on the activities in this study, they had already been taught differentation, anti-differentiation, and integration in their course. Ava had seven years of teaching experience and Noa had two years of teaching experience. Both Ava and Noa had taught up to Year 12 mathematics, but neither had taught Year 13 calculus. Nevertheless, they had encountered the concepts of differentiation, antidifferentiation, and integration as secondary and tertiary students. Thus, all four participants had the mathematical preparation to make sense of *The Tramping Problem*. We report our analyses of these four participants' work during the first activity – *The Tramping Problem* MEA, which was described in the previous section. We choose to focus on these participants because they were the only ones who engaged fully in *both* processes of mathematising and applying when trying to determine the value of the anti-derivative as a measure of height.

We analysed a variety of sources in an attempt to infer the participants' mathematical thinking. First, transcriptions of the participants' audiotapes were annotated with descriptions, diagrams, and photos of their gestures, inscriptions, and interactions that were captured in the video footage. These annotated transcripts were then coded to identify the mathematical concepts the participants considered throughout the activity, and the participants' mathematising and applying. The participants' written work was analysed in conjunction with these coded, annotated transcripts to create narratives of the participants' attempts to use mathematising and applying to determine the value of the anti-derivative as a measure of height. We present these narratives in the next section.

Results and Analysis

We present seven narratives that describe the engagement of Cam and Sid, and Ava and Noa in the processes of mathematising and applying while they tried to determine the value of the anti-derivative as a measure of height. The problem of finding the height of the tramping track was only one aspect of *The Tramping Problem*, and the four participants engaged in many more instances of mathematising and applying when investigating

other aspects of the problem. They also considered, for example, the tramping track's summits, valleys, uphills, downhills, relative steepness and points of inflection. However, in this article, we restrict our analysis to those instances of mathematising and applying that relate directly to the problem of finding the height of the tramping track.

Narrative 1: Cam and Sid's First Mathematising Attempt

Cam and Sid began by correctly determining that positive portions of the graph (coloured light and marked as A and E in Figure 4) correspond to uphill portions of the track, whereas negative portions of the graph (coloured dark and marked as C and G in Figure 4) correspond to downhill portions of the track. They also correctly ascertained that the x-axis intercepts on the graph indicate summits (B and F in Figure 4) and a valley (D in Figure 4) on the track.

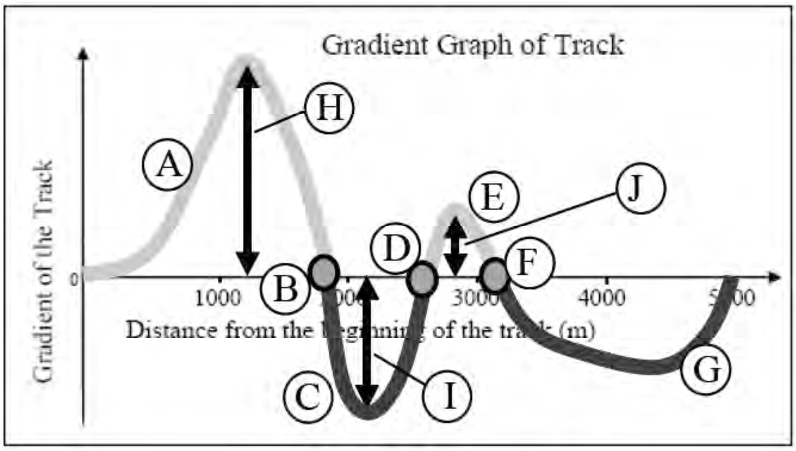

Figure 4. Features (A-J) of gradient graph referred to by Cam, Sid, Ava, and Noa.

When they drew these features in their first distance-height graph of the track (shown in Figure 5), they assumed that the bottom of the valley is at sea level. They then encountered the problem of the height of the track for the first time when they tried to determine the height of the second summit in relation to the first summit.

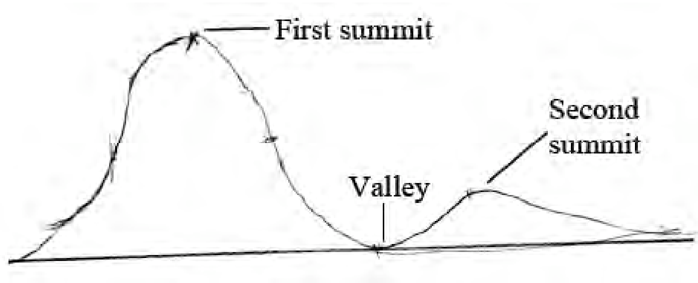

Figure 5. Cam and Sid's first drawing of the distance-height graph of the track (with labels added to indicate the summits and the valley).

Cam initially suggested that the second summit is one third of the height of the first summit, since the height of the second maximum on the gradient graph (indicated as J in Figure 4) is one third the height of the first maximum (indicated as I in Figure 4). Thus, in his first attempt to mathematise the height of the track, he inferred a proportional relationship between the heights of the local maxima on the gradient graph, and the heights of the summits of the tramping track:

$$\frac{\text{maximum}_2 \text{ (of gradient graph)}}{\text{maximum}_1 \text{ (of gradient graph)}} = \frac{\text{height of summit}_2 \text{ (of the track)}}{\text{height of summit}_1 \text{ (of the track)}}$$

However, he soon realised that this mathematisation was incorrect, and corrected himself, saying "Oh no it just gets a third as steep, that's got nothing to do with distance (points to J in Figure 4), that's not as steep, so it's just flatter" (note that in this excerpt, he appears to use the word "distance" to refer to the *height* of track). He explained that the height of the gradient graph indicates the steepness of the track, not the track's height.

Narrative 2: Cam and Sid's Attempt to Apply Integration

Cam then wondered what features of the graph would help him find the height of the track, and brought up an idea he remembered from class. Note that in the following excerpt, he again uses the word "distance" when referring to the *height* of the track:

Cam: What does the area under the gradient graph mean? Doesn't it mean something as well?

Sid: No because well you can't have negative areas so…

Cam: No but the absolute value of an area, isn't that distance? (Directs question to interviewer) Are we allowed to use our textbook?

Cam opened his textbook to the section describing "area under a curve", but dismissed it after finding it written in the context of speed. He decided that he could not apply his knowledge about the area under a curve to determine the height of the track at the second summit, because his

textbook-based knowledge was situated in the context of speed, which he could not relate to the tramping context of the problem. Thus, Cam's attempt to apply his knowledge about integration was unsuccessful, and the question of how to find the height of the antiderivative remained.

Narrative 3: Cam and Sid's Second Mathematising Attempt

Later, Sid drew a "good copy" of the graph they drew in Figure 5, and again drew the bottom of the valley at sea level (see Figure 6). While redrawing the graph, Cam and Sid revisited the question of the second summit's height in relation to the first. This time, Cam suggested that the second summit is not as high as the first summit because although sections E and A (in Figure 4) are "fairly similar shapes", E occurs over a shorter horizontal length than A. Thus, he correctly recognised that the horizontal distance is an important factor in determining the height of the antiderivative. He remarked that a summit that is reached by travelling at a shallow gradient over a short horizontal distance is not as high as the summit reached by travelling at a steep gradient over a long horizontal distance. Thus, Cam incorporated the horizontal distance into his second mathematisation of the relationship between the gradient graph and the height of the antiderivative. However, this mathematisation was not formalised until the third drawing of the graph.

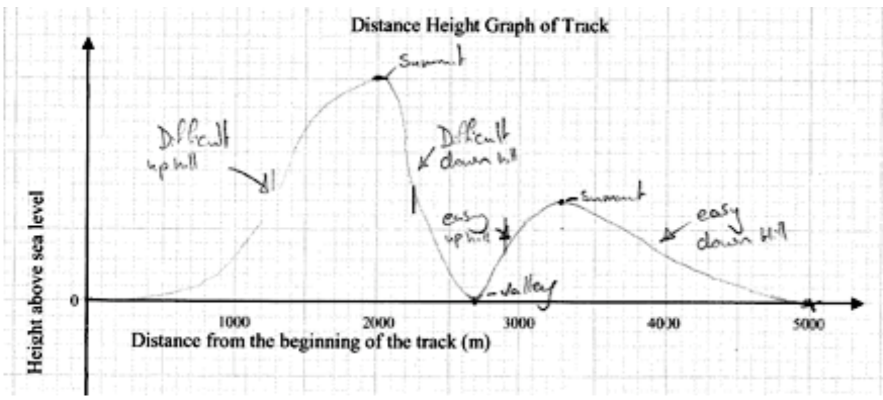

Figure 6. Cam and Sid's second drawing of distance-height graph of track.

Narrative 4: Cam and Sid's Third Mathematising Attempt

After writing down their method in a letter, Cam observed that the steepest downhill portion of the track should only be "50% as steep" as the steepest uphill portion of the track, since the (absolute value of the) height of the first minimum on the gradient graph (I in Figure 4) is half that of the first maximum (H in Figure 4). Thus, he inferred a proportional relationship between the heights of the first relative maximum and minimum on the gradient graph, and the steepness of the corresponding parts of the track.

Cam then redrew the track (see Figure 7) to reflect the differences in steepness, and in doing so also corrected the height of the bottom of the valley in the track with the comment: "I think what misled us was that this (points to the first downhill portion of the track in Figure 7) was over a shorter distance, so maybe it (points to the valley in Figure 7) doesn't go all the way down to the ground."

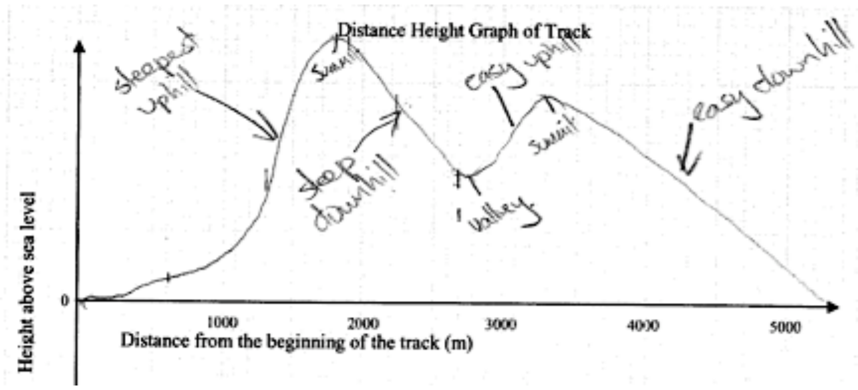

Figure 7. Cam and Sid's third drawing of distance-height graph of track.

Next, Cam explained to Sid that the horizontal distance in the gradient graph is as important as the amplitude when determining the height of the track:

Cam: I think that the distance covered (referring to the horizontal distance) on the gradient graph does indicate the height climbed or descended in the…

Sid: Yup. I'd be more that the amplitude for this section here (points to C in Figure 4) is not as big as the amplitude here (points to A in Figure 4), so therefore it doesn't reach ground level again.

Cam: Yeah, do you think it's maybe a combination of the two, 'cos if that was a lesser amplitude (points to C in Figure 4) but over the whole graph (points along the x-axis in Figure 4) then it would be going downhill the whole way?

Sid: Yeah but then we would see that as it goes the whole way, but just for that part…

Cam: Yeah, but for any graph. Just general graphs. The amplitude, it's kind of that the amplitude times how far it's gone is going to give you an indication of how high you're going to go, how far you're going to drop?"

Sid: Yeah.

Cam summarised this insight in a postscript to their letter: "you should take into account both the amplitude and horizontal distance as an indicator of the change in elevation for each slope". Although Cam did not talk in

terms of area under the curve, he commented in the excerpt above that the change in elevation for the track is determined by "the amplitude times how far it's gone", or in other words:

$$\text{Change in elevation} = \text{amplitude of gradient graph} \times \text{horizontal distance}.$$

This third mathematisation is essentially a crude approximation of the area under the curve using one Riemann rectangle for each area (see Figure 8). Thus, Cam initially dismissed the application of the "area under the curve" idea he had previously learned because it was embedded in a speed context in the textbook. However, he ended up developing a mathematical model that incorporates a primitive version of the very same idea in his third mathematisation of the relationship between the gradient graph and the height of the antiderivative.

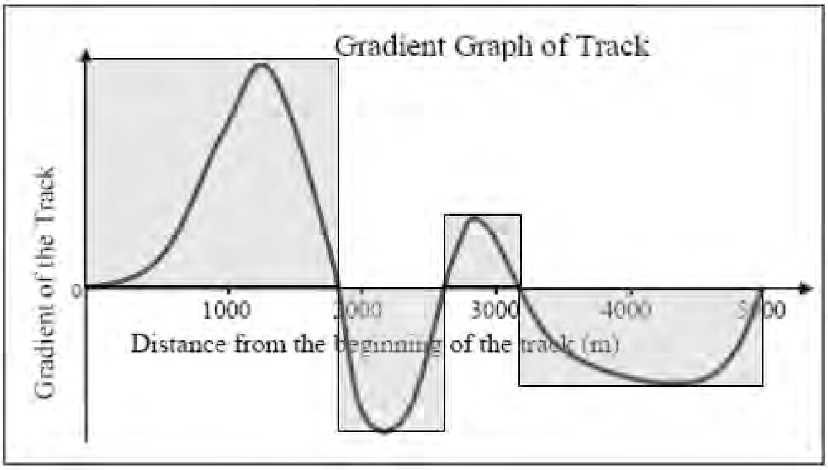

Figure 8. A graphical representation of Cam's insight into the relationship between amplitude of the gradient graph and the horizontal distance.

Narrative 5: Ava and Noa's First Mathematising Attempt

Ava and Noa began by using gestures to ascertain the general shape of the tramping track, based on the gradients given in the gradient graph. They first became concerned with the issue of the height of the antiderivative when they came to draw the graph of the tramping track on paper. Ava correctly drew the initial ascent up to the summit in the graph, which corresponds to section A in Figure 4. However, she expressed doubt as to how far down the subsequent valley (which corresponds to section C in Figure 4) needed to descend. She initially drew the valley as descending all the way down to sea level, then paused with her pen at sea level saying, "Am I here, should I be here?" (see Figure 9a).

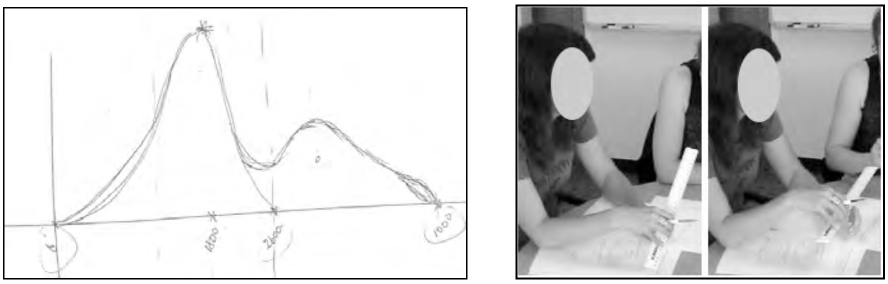

Figure 9. (a) Ava's drawing of tramping track (initial drawing of valley descends to sea level), (b) Ava using a ruler to represent steep and shallow gradients.

In trying to resolve the issue of how far down the valley went, Ava suggested that the size of the "humps" (A and C in Figure 4) indicated how high the track ascended. "If this hump (points to A in Figure 4) is bigger than this hump (points to C in Figure 4), what does that say in terms of the gradient? That says you've gone steeper so you've gone higher, surely?" When Noa disagreed, Ava used a ruler to emphasise her conjecture: "If you've gone steeper, you must have gone higher" (see Figure 9b).

Eventually, Noa agreed, stating: "Steeper for longer must be higher." Thus, their first mathematisation of the height of the antiderivative consisted of relating the steepness of the gradient with the duration of the steepness to the height of the antiderivative.

Narrative 6: Ava and Noa's Attempt to Apply Integration

Later on, while writing their method up, Ava again questioned how to find the height of the track from the gradient graph, and suggested they apply integration to solve the problem.

> Ava: If you've got the gradient function how do you get back to the initial function? You need to integrate. Right, so we could integrate this function to provide a value? Yes we could (shades in the area underneath A in Figure 4), couldn't we?
>
> Noa: Yeah. That's clever now isn't it? Yes, that's right. The area under the curve.

Ava and Noa were enthusiastic about the application of integration, but as it had been a few years since they had learned integration, they first tried to recall what they remembered about the topic:

> Noa: So what do we know about integration? We know yeah – distance, time graph – that's right. Distance, time, so *ds* by *dt* is the gradient function which is the speed…

This application of integration led to them incorrectly interpreting the gradient graph of the tramping track as representing the speed of the tramper, rather than the steepness of the slope. They subsequently

abandoned this approach while expressing low confidence in their ability to apply integration to the problem – "my brain's going… it's amazing how quickly it goes out of your head isn't it?"

Narrative 7: Ava and Noa's Second Mathematising Attempt

After abandoning the application of integration, Ava and Noa returned to their first mathematisation, and discussed the extent to which the gradient was responsible for determining the height of the antiderivative. Noa introduced a hypothetical scenario in which a shallow gradient could lead to a high hill. She stated: "you could have one that's just got a shallow gradient but just goes on for ages. And then… you could end up being really high, but not really steep to get there". Ava agreed, and suggested that they adjust their letter to mention portions of the track that corresponded to a "long hard slog" like the one Noa described. Thus, Ava and Noa refined their initial mathematisation of how gradient and horizontal distance interact to give the height of the antiderivative, to acknowledge that a shallow gradient over a long horizontal distance could give rise to a high hill. This second mathematisation reflects a more advanced understanding of gradient and the height of the antiderivative than the first mathematisation, which only allowed that steeper gradients could give rise to a higher antiderivative.

Discussion and Conclusions

The MEA in this study was implemented after the participants had received direct instruction on integration. Hence the participants had the necessary mathematical tools to make sense of, and progress with, the problem, since they could approach it as an application problem that required them to integrate the given function to find the height of its antiderivative. However, all four participants solved this problem by mathematising the tramping context, while their attempts to apply their knowledge of integration were unsuccessful. This reveals two limitations in the participants' prior knowledge. First, their knowledge of integration was closely linked to the context of speed, which did not carry over to the context of tramping. Cam rejected the applicability of his speed-bound knowledge of integration to the tramping context outright, whereas Ava and Noa tried unsuccessfully to convert the tramping context into a speed context. This result resonates with Gravemeijer and Doorman's (1999) observation that the teaching community's predominant use of the context of speed to illustrate integration concepts may be limiting students' ability to apply their knowledge of integration to other contexts.

A second interesting feature is their apparent lack of representational versatility (Thomas, 2008) in their prior knowledge of integration. Since the function was given only in the graphical form, an application of integration would have required the participants to use graphical and numerical representations of integration. For example, they could have used triangles to approximate the area under the curve, and thereby approximate the difference in height between each summit and valley. The fact that none of the participants chose to apply a graphical and numerical representation of integration to the problem echoes Thomas and Hong's (1996) finding that

students' knowledge of integration is largely confined to algebraic representations. This finding is strengthened further by the following observation: Cam and Sid reconstructed (a primitive version of) integration by mathematising in Narrative 4, and Ava and Noa reconstructed (a primitive verbal version of) integration by mathematising in Narrative 7. However, both participant pairs failed to recognise their reconstructions as cases of integration.

Although the students' work on the modelling activity revealed some limitations in their previously learned knowledge of integration, it also gave them an opportunity to build up their conceptual understandings through the iterative process of mathematising. Cam went through at least three cycles in the modelling diagram (Figure 2), and eventually mathematised the height of the track as the product of the amplitude of the gradient graph and the horizontal distance in the track. The knowledge Cam developed through the process of mathematising was more primitive than the sophisticated integration ideas he had previously rejected. However, his preference for this primitive understanding suggested that it was a deeper conceptual understanding of integration than his textbook knowledge of the topic. Similarly, Ava and Noa went through two modelling cycles in which they refined their understanding of how gradient and horizontal distance interact to indicate the height of the antiderivative.

Our research suggests that even when MEAs are implemented after direct instruction in the topic, students may still approach them as modelling activities, particularly if the direct instruction did not lead to deep conceptual understandings of the topic in the first place. In the case of *The Tramping Problem*, the combination of the unfamiliar context of tramping and the unfamiliar graphical representation meant that a direct application of integration is only possible for those possessing a deep understanding of the topic. A wealth of research suggests that although most students of calculus are reasonably proficient in performing various calculus techniques, they often lack a conceptual understanding of the core ideas (Eisenberg & Dreyfus, 1991; Thomas & Hong, 1996; Thompson, 1994). This article suggests that real world problems that emphasise the process of mathematising, such as *The Tramping Problem* MEA, can help students develop, express, test, and revise their own conceptual understandings of calculus concepts.

In this article, we have discussed the recommendation for MEAs to be implemented at the beginning of instructional units, rather than at the end (Lesh, Yoon, & Zawojewski, 2007). However, in practice, MEAs are often seen as standalone activities that are more likely to be used on a rainy day, than to be integrated into an instructional unit, whether at the beginning or the end. One reason for this common perception is the lack of available follow-on activities to most MEAs. We have developed three follow-on activities (to which we alluded in the methodology section) that are designed to help students strengthen and consolidate the primitive understandings they initially develop in *The Tramping Problem*. An important goal for future modelling research is to identify which design principles are most useful for creating productive follow-on activities to MEAs. Such research would increase the likelihood that MEAs are implemented at the beginning of instructional units, thereby enabling the

understandings that students develop through mathematising to be most effectively leveraged.

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