Mathematics Education Research Journal 2010, Vol. 22, No. 2, 108-120

(Fish) Food for Thought: Authority Shifts in the Interaction between Mathematics and Reality

Irit Peled
University of Haifa

This theoretical paper explores the decision-making process involved in modelling and mathematizing situations during problem solving. Specifically, it focuses on the authority behind these choices (i.e., what or who determines the chosen mathematical models). We show that different types of situations involve different sources of authority, thereby creating different degrees of freedom for the problem solver engaged in the modelling process. It also means that mathematics plays different roles in these problems and situations. This epistemological analysis on the meaning of modelling implies that we should reconsider the mathematical status of realistic solutions and raises questions on the validity of some traditional choices of mathematical models and their use in diagnosing children's conceptions. It also suggests constructing modelling tasks by choosing a certain variety of situations that might lead to a better understanding of the roles of mathematics.

Recent reviews (Niss, Blum, & Galbraith, 2007; Stillman, Brown, & Galbraith, 2008) indicate that the community of researchers that investigate issues of modelling and applications is involved in many research perspectives. Many of the works deal with the effect of modelling tasks on student learning of mathematical concepts, student developing of modelling skills, teacher attitudes towards modelling tasks and teacher learning from observing students' modelling processes.

Within modelling issues, this article touches on several perspectives. From the Epistemological perspective, which is less investigated (Stillman et al., 2008), it aims at analyzing the nature and meaning of choosing mathematical models in a given situation. From the Authenticity and Goals perspective it investigates the roles of mathematics in problem solving, and views the development of this knowledge as a curriculum goal that can have an impact on the choice of modelling tasks. Our goals are also motivated by the need to promote a radical change in teacher beliefs about the roles of everyday knowledge and the roles of mathematics in problem solving and modelling (Bonotto, 2007).

For these purposes, we will compare different types of problems in an effort to find what determines the problem solver's modelling choices. We use the term *authority* in the sense of a source of knowledge or power that either suggests or imposes the choice of mathematical structures.

We believe that teachers and students who engage in modelling tasks or any other problem solving activities should be conscious of their reasons for making choices in applying mathematical concepts. We also believe that this awareness can change teacher and student beliefs about the roles of mathematics.

The Authoritative Power of Reality over Mathematics

Issues that focus on authority are not new and have always been a trigger for fascinating philosophical debates. Some of the main issues involve the relationship between mathematics and the real world. While we look briefly into the role of real-world phenomena in the construction of mathematical concepts, the focus of this paper is in the other direction; that is investigating the role of mathematics in the analysis of situations during problem solving.

Scholars and thinkers have long pondered the question that is at the heart of the field of mathematics: What is the authority that creates mathematical objects and determines mathematical rules and "truths"? More simply, what is the relationship between mathematics and reality? As Sfard (2007) put it in her discussion of the emergence of negative numbers: "Back then in the 17th century the real, albeit unspoken, question was about the rules of mathematical game: Who is the one to decide what counts as mathematically acceptable - the reality itself or the participant of the mathematical discourse" (p. 582)?

For a long time the prevailing view was that observed phenomena determine the construction of mathematical structures, and that mathematical truths could only emerge from observed reality. Acceptance of the view that mathematical structures might be born in the human mind without an explicit real-life source did not come easily, especially since most examples supported the opposite view. Today, of course, it is relatively easy to support the authority of the human mind. Plenty of mathematical ideas that are not grounded in any real-world phenomenon, and that were therefore once rejected, have turned out to be powerful tools with a legitimate existence in the mathematical world.

As suggested by the quotation from Sfard (2007) above, one of the earliest examples is negative numbers, which originated from an algebraic need for closure (Hefendehl-Hebeker, 1991), and which were not accepted by some mathematicians when they were introduced. Other mathematical objects and theorems that were developed without a basis in reality later triggered ingenious inventions and applications. One of the most famous of these is the RSA coding method, named after its inventors (Rivest, Shamir, & Adleman, 1978), which is based on the near impossibility of deducing two very large prime numbers from their product. The world financial system depends upon this and related coding systems.

The mere existence of these cases has caused a shift that has freed mathematics from the need to obey the authority of reality even in cases that have not yet proven to be "useful". In the wake of this shift, mathematical theory could develop without having to lean on objective, real-world "truths".

Mathematics Power in Modelling and Mathematizing Situations

Whatever their source, be it some realistic phenomenon or imaginative inspiration, the field of mathematics now offers us a wide range of

structures and models that can be used in different situations. The choice of mathematical models is what interests us in this paper.

A brief note on terminology is warranted. When problem solving involves a process of organizing a given situation and fitting a mathematical structure or representation, the term *modelling* is used for the entire process, including the decision-making element. The act of representing a situation using the terms and symbols of the mathematical concepts that seem to fit it is termed *mathematizing*. Our use of the terms modelling and mathematizing will be relaxed in the sense that we will discuss fitting mathematics to situations in cases that involve different degrees of decision making, and might also involve relatively little organization.

What is the Nature of the Modelling Process?

When a problem solver makes a decision about what mathematical concept or concepts to apply in a certain case, the question that we ask is: What is the source of information or the authority on which the solver can make this choice? Other questions follow in its wake. How much freedom does the problem solver have in choosing mathematical concepts? Does the context influence these choices, and in what way? To answer some of these questions, we analyze a few examples, compare their solution processes, and make some generalizations.

Examples of Problem Solutions

We start by looking at a relatively standard problem, the *Lottery Problem*, together with its common solutions. Standard problems traditionally require very few decisions, and often the location of the problem in the textbook reveals the mathematics that the problem composer sees as fit to describe or solve the given situation. Still, the fitting of mathematical concepts in such problems might contribute to our understanding of this process.

> Two friends, Anne and John, buy a lottery ticket together. The price of the ticket is $5. Anne pays $3 and John pays $2. They win $40. How should they split their winnings?

Some version of this problem often appears in textbooks following instruction on ratio and proportion. It is expected that students will identify the problem as a ratio and proportion problem and that their solution will involve splitting the $40 according to the investment ratio, 3:2. Using this ratio Anne would receive $\frac{3}{5}$ of $40 (i.e., $24), and John would receive $\frac{2}{5}$ of $40 (i.e., $16).

Why do we automatically employ proportion in this case? The obvious response is that we use proportion because this is a proportion problem; that is, the problem appears in the chapter of the textbook dealing with proportion, so this model is an obvious fit. However, for the purpose of our current analysis, we are interested in going beyond this sort of tautological answer ("It's a proportion problem because it's a proportion problem") to figure out why this example belongs to the collection of conventional ratio and proportion problems.

In fact, an implicit assumption is made by the textbook writer that a constant ratio should be maintained – in other words, that the ratio between John and Anne's investments should equal the ratio between their profits. What is the rationale for this ratio equivalence? Most likely, behind it stands an implicit assumption that each invested dollar should earn the same amount of money. In calculating the profit shares in the Lottery Problem, since the total investment is $5, the profit per dollar is a fifth of the total winnings. The rest of the calculation then follows by multiplying this intensive quantity by the invested amount. Using Schwartz's (1988) terms, this intensive quantity can also be viewed as a scalar because we get dollars divided by dollars, and we can then refer to a given individual lottery drawing situation as having some profit scalar showing by how many times one's investment has grown.

Is a similar implicit assumption used in other proportion problems?

We will take a look at another problem considered by its presenter as a proportion problem. This problem is more complex than the standard example given above, and calls for some situation analysis before fitting a mathematical model.

In working with pre-service teachers, Koirala (1999), a teacher educator, asked them to solve the following *Shoe-Sale Problem*:

> Two friends are shopping together when they encounter a special "3 for 2" shoe sale. If they purchase two pairs of shoes at the regular price, a third pair (of lower or equal value) will be free. Neither friend wants three pairs of shoes, but Pat would like to buy a $56 and a $39 pair while Chris is interested in a $45 pair. If they buy the shoes together to take advantage of the sale, what is the fairest share for each to pay?

In offering his solution, Koirala explained that it could be reached in two different but equivalent ways, one involving the concept of percentage and the other the concept of ratio. The first suggests that the two friends should get a same-percent discount. This calls for calculating the total discount in percentage terms and then using this to figure out the individual discounts. The second route suggests that the ratio between the new costs should equal the ratio between the original costs. Either approach leads to Pat having to pay $68.54 and Chris, $32.46. According to Koirala, this is the only appropriate solution to the problem, and he evaluated any alternative solution by how far it was from it.

Why should the problem be solved by using proportion? As in the *Lottery Problem*, there is probably an implicit assumption here that would lead to Koirala's claim that the ratio between the new (reduced) costs should be the same as the ratio between the original costs. This assumption might say that each buyer should pay a constant amount for each dollar of the original cost.

Phrased in this way, the implicit assumptions in the two problems are similar. In the *Lottery Problem* one is expected to earn the same profit for each dollar invested, and in the *Shoe-Sale Problem* one is expected to get the same discount for each dollar of the original price. In both cases this results in a scalar operator that linearly determines the amount earned or saved. Does a problem solver have to use these assumptions? What is the status of a solution that does not use them?

Initial Reservations: Alternative Solutions

While the two examples offered here appear to have "correct" solutions, in both cases alternative solutions have been suggested. We will examine these solutions and consider their mathematical status.

Alternative solutions to the first example: The Lottery Problem. Ron, a sixth-grader who solved this problem, suggested three different solutions, given here with his calculations and comments:

> Solution 1: 40:2 = 20. Each child gets $20.
>
> Solution 2: One child (the one who paid $2) will get $19 and the other (the one who paid $3) will get $21 although the difference is $2 [while the difference in the amount paid is $1].
>
> Solution 3: 40:5 = 8, 3×8 =24, so $24 to the child who paid $3, 2×8 = 16, so $16 to the child who paid $2.

After writing these solutions Ron concludes: "In my opinion, the first solution is the most fair, but the third is most right because of the ratio". Ron, aware of classroom norms, knew that the teacher expected him to give the third solution, even if it did not feel quite right to him. Is proportional sharing really the "right" (and unique) model? Do we have to use the implicit assumptions discussed earlier?

Alternative solutions to the second example: The Shoe-Sale Problem. Koirala (1999) was unhappy with the performance of his pre-service teachers, since none of his 32 teachers (not even his best students) suggested a solution that he considered appropriate. Instead of using the equivalent same-percent or same-ratio solutions, they used different price or savings split criteria. The most frequent answers were:

1. Split the price according to the ratio of shoes bought (rather than the price ratio). Thus, the friend who buys 2 of the 3 pairs pays $\frac{2}{3}$ of the total price. (This answer was given by 8 of the 32 pre-service teachers.)
2. Split the savings (the price of the cheapest pair) evenly. (This answer was given by 7 of the 32 pre-service teachers.)
3. Split the savings according to the ratio of shoes bought. Thus, the friend who buys 2 pairs gets $\frac{2}{3}$ of the total discount. (This answer was given by 5 of the 32 pre-service teachers.)

Koirala described a dialogue with his best student, Ayaz, who suggested dividing the savings evenly. In describing his effort to convince Ayaz that his solution was inappropriate, Koirala wrote that he "provided a counter-argument saying that the sharing between Pat and Chris can be considered mathematically fair only if their savings are proportional to their original costs" (p. 166).

Is this really an argument, or is it an assumption? In either case, is it justified? While there is some sense in the same-cost-ratio criterion, it also has flaws. Specifically, it does not take into account the fact Pat (who bought

two pairs of shoes) contributed more to the friends' ability to jointly take advantage of the sale (though she also benefited from going home with two pairs). From the opposite perspective, perhaps Chris (who bought the $45 shoes) was not really thinking of buying anything, and joined the action to make her friend happy. In that case, Pat, who might have been willing to buy two pairs with no discount, should be happy with any discount at all.

What is the status of these alternative solutions? Should we accept them as valid because in a real-world situation, considerations besides those of pure mathematical parity come into play? What does accepting them as valid solutions then mean? Do they have the same status as a mathematical solution to the problem, or do we believe that there is a unique mathematical solution to each problem, and the alternative solutions are useful for the real world but, as Koirala suggests, "mathematically inappropriate"?

Often, it is easier to analyze an example by looking at other examples and comparing them. We will continue our search for answers by looking at yet another traditional problem and identifying similarities and differences in the features and solutions of the two problems.

The Fish Food and Eel Food Problems

In their thorough and detailed description of the development of ratio and proportion conceptions in a fifth grader, Lo and Watanabe (1997) present several fish-feeding tasks adapted from Piaget, Grize, Szeminska, and Bang (1977). The following Fish Food Problem is one of these tasks.

> Fish A is 18 centimeters long and Fish B is 12 centimeters long. If Fish B needs 60 pieces of food every day, how many pieces of food will Fish A need every day? (p. 225)

The *Fish Food Problem* (in this or similar versions) is a common problem often used in class or in intelligence tests to diagnose proportional and analogical thinking. The problem solver is expected to say that the mathematical relationship between the fish lengths should also hold for the fish food; that is, a fish that is k times longer should also eat k times as much food as the other fish.

Why should we accept the assumption that a fish that is, for example, twice as long as another fish, eats twice as much? Come to think of it, its volume might be around 8 times the volume of the first, and by the same rationale perhaps this fish should eat 8 times as much as its smaller cousin. Figure 1 demonstrates what happens when we double the length of a fish, keeping the new fish proportionally similar to the original one. In spite of the fact that only 2 of the 3 dimensions are represented, one can imagine fitting the smaller fish into the bigger one more than twice.

In fact, if anything, those who use a linear relationship have fallen into the linearity trap discussed by DeBock, Dooren, Janssens, and Verschaffel (2002) in their extensive investigation of the sources of children's incorrect application of proportion (or a linear connection) in cases that involve more than one dimension. So, on the one hand we are unhappy when children use linearity where it should not apply, and on the other hand we expect them

to use it in a similar situation and identify them as having a poor understanding of proportionality if they do not!

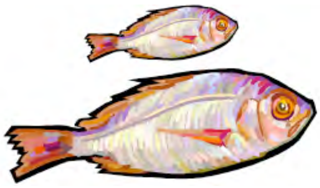

Figure 1. Fish of different lengths.

Being aware of this issue, Piaget et al. (1977) eliminated additional dimensions by using fish that looked like eels. Following their examples, Lo and Watanabe (1997) use a picture of two lines to symbolize the length of the two fish. They indicate that "these pictures helped to establish a linear relationship" (p. 222). Piaget et al.'s (1977) task was also adapted by Hart (1981) and by many others. Some tasks use fish and some use eels, with some variations on the type of food as well. Figure 2 demonstrates a typical picture of eels that accompanies an *Eel Problem* (detailed below).

Figure 2. Eels of different lengths.

The following version appears in Misailidou and Williams (2003) as adapted from the *Concepts in Secondary Mathemetics* project (Hart, 1981).

> There are 3 eels, A, B and C, in the tank at the Zoo.
> A: 15 cm long B: 10 cm long C: 5 cm long
> The eels are fed sprats, the number depending on their length. If C is fed 2 sprats, how many sprats should B be fed to match?

Similar versions can be found in other works or tests. For example, the teaching resources site of the Department of Education and Early Childhood Development (2007) in Victoria, Australia, presents a similar item for testing multiplicative reasoning. The teacher is directed to show the child a card with a picture of 3 eels (as depicted in Figure 2) while saying:

> This card shows three eels. The blue eel is twice as long as the red eel and the orange eel is three times as long as the red eel. The eels are fed food pellets according to their length. If the red eel gets two food pellets, how many pellets would be fed to the other two eels?

Not knowing how eels feed, we cannot reject the intended and implicit assumption that eels are fed according to a linear function that depends on their length; but from where would the problem solver deduce this knowledge? As can be seen, in these two eel problems the problem solver is told that the eels are fed *depending on their length* or *according to their length*. This tells us that there is some relation between the length of the eel and the amount of food that it eats, but we are not even told that the relation is positive and that the longer the eel, the more it eats. We only assume this is the case. If the relation is multiplicative, and the problem solver is expected to deduce it, the solver should be given data on eels and their feeding habits. The common short versions of the earlier original Piagetian tasks do not provide such data. We will discuss the value and diagnostic power of these versions in the concluding remarks.

A Mysterious Solution: The Case of the Three Widows

In the previous examples we raised some questions about the legitimacy of composing problems based on implicit assumptions and the use of proportional reasoning. In the following case we find ourselves puzzled for the opposite reason: in a problem that looks like a proportion problem, the mathematical model seems to be something else. By investigating the rationale for the solution of this problem, we hope to learn something about modelling and mathematizing in other problems.

This next example is taken from a Mishna in the Babylonian Talmud and deals with the hypothetical case of the death of a man who was married to three women, each of whom had a different marriage contract (called *Kethuba*, plural *Kethubot*). The widows had been promised 100, 200, and 300 gold coins, but the estate consists of an amount less than 600. How should the estate be divided among the widows? If we were to ask a similar problem in a mathematics textbook, we would probably suggest dividing the estate proportionally; that is, the widows would receive $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{3}{6}$ of whatever the estate is, so that the ratio between what they receive is the same as the ratio between the marriage contracts. However, the Mishna that presents the case did not make life that easy for scholars who tried to interpret it. Table 1 presents the three examples given in the Mishna for demonstrating how the estate should be divided (the representation is adapted from Aumann and Maschler, 1985). According to Table 1, for an estate value of 100, the Mishna's instructions are to split the money evenly. With 200, the first widow receives half of her Kethuba, and the two others split the rest evenly. With 300 each receives half of her Kethuba.

On the face of it, each case employs a different mathematical model for sharing, where only the third case uses proportion. For a long time this seeming inconsistency puzzled scholars who looked at the problem from moral, mathematical and pragmatic perspectives. A practical reason for confusion was the fact that without a better understanding of the reasoning behind the three decisions, it was not possible to generate a generalized sharing procedure for additional cases.

Table 1
Three estate sharing cases (from the Mishna Kethubot)

		Marriage contracts (Kethubot) (in coins)		
		100	200	300
	100	33⅓	33⅓	33⅓
Estate	200	50	75	75
(in coins)	300	50	100	150

A possible solution to this puzzle was offered by Aumann and Maschler (1985). Being mathematicians familiar with game theory, Aumann and Maschler identified a similarity between the demonstrated sharing values and the values that would appear in a matrix of a coalitional game situation. This led them to identify a general model that when applied generates values equivalent to those given in the three Mishna cases. The mysterious seemingly inconsistency was solved. Now all that was left was to figure out why the Mishna had chosen this solution.

Aumann, the first author of the above mentioned work and winner of the Nobel Prize in Economics for 2005, explains the rationale in another paper (1999). In short, the idea is that in order to allow each widow to satisfy at least some basic needs, a widow should not get less than some minimal amount. Only when these basic needs are satisfied can a proportion model be used. This is a social rationale that leads to certain mathematical constraints that happen to be well fit by the coalitional game model.

As it turns out, this is not the only case where a problem that looks like a classical case of proportion is handled by using a different model. The widows' case is a special example of a more general situation of sharing where there are several creditors, and where the sum of the debts is larger than the available estate (as is the case in bankruptcy). In these situations, most countries have enacted laws and rules for resolving debts that do not simply involve sharing the assets according to the proportion between the debts. Other cases that are handled differently involve situations such as sharing the cost of an elevator between tenants or the price of an airport runway among users who have different needs. These cases involve alternative sharing algorithms such as the use of Shapli's Value, the Nash Equilibrium, or a Coalitional algorithm as in the widows' case.

Comparisons and Generalizations

We have observed a situation where a proportion model could be applied, but due to a social rationale, another solution was decided upon. Is the same sort of freedom available in any of the previously discussed examples? That is, can a non-proportional model be used in the lottery, shoe-sale, and fish cases? Here is where we need to generalize and categorize.

The *Lottery Problem* and the *Shoe-Sale Problem* are similar to the *Widows' Case* in being based on some social-moral decision. Since the decision is a matter of a social agreement, the friends in either of these two stories can

decide what division criterion they want to use. The authority or the source for deciding what model to use lies in their own hands. Thinking that one *has* to use a proportion model would mean giving mathematics a role and a power that it does not and should not have.

Can the same be said about the *Fish Problem*? For this analysis we look at an additional example: The *Paint Mixing Problem*.

> Tom mixed 3 cans of yellow paint with 6 cans of blue paint and got a nice green for his garden fence. The next day he wanted to get some more of the same shade of green by mixing blue paint with the 5 cans of yellow that he had left. How many cans of blue should he use?

This, too, is a traditional proportion problem. Why is a proportion model used here? What would happen if we tried using another model? For example, we might suggest that since Tom is using 2 more cans of the yellow paint, he should also use 2 more cans of the blue and mix the 5 yellow cans with 8 blue cans. If we tried this hypothetical alternative answer, we would either obtain the same green and affirm our hypothesis about the mathematical model that fits this context, or obtain a different green and know that our hypothesis was refuted and we should try a better fit. Having some knowledge about the phenomenon of mixing colours, we know that the additive model will fail to satisfy the prediction.

What is this a case of? The mixing colors situation is a scientific phenomenon in which the behavior of the paints is dictated by the relevant science. Figuring out what model to use requires familiarity with this phenomenon. Indeed, Tourniaire (1986) found that only 37% of the elementary school children he tested on a *Paint Problem* succeeded in solving it, in comparison with 60% who succeeded in solving an *Orange Juice Problem*, in spite of the fact that both problems dealt with mixtures. He concludes that perhaps it is not simply a matter of familiar context, but rather a matter of *familiarity with the use of ratios in the context*.

The *Eel Problem* should also fall in the same category of scientific phenomena. There are biological rules that model the connection between the length of eels and their feeding habits. Assuming that the proportion model is not a wild invention of the problem composer, further observations can support or reject the problem solver's predictions. It should be noted that this process is not always simple. Sometimes a given list of observations might fit several different models, and the model needs refinement by more observations. Sometimes a model might fit for a range of conditions but fail in others. Greer (1993), for example, suggests that in baking recipes, even though in normal daily use a proportion model helps calculate ingredients for different sizes of cake, this model might fail when very large quantities are involved.

Concluding Remarks

With the goal of promoting modelling as an integral part of the school curriculum, children and teachers need to know more about the meaning of modelling and the roles of mathematics. Our theoretical analysis can be relevant for working with teachers on changing their conceptions and for

curriculum writers, who can design a sequence of modelling tasks that takes this analysis into account.

The analysis has led us to identify different sources for deciding what mathematics can be used in modelling and mathematizing a situation, showing that these sources depend on the problem context. Specifically, in problems that involve social or moral behavior, the problem solver has the freedom to decide upon criteria or norms for applying a mathematical model, and then choose a model that fits these criteria. The role of mathematics lies solely in suggesting a repertoire of models and thereby giving problem solvers tools to come up with possible and reasonable criteria. Then it provides the solvers with calculation tools to realize their decisions.

In cases such as the *Lottery* or *Shoe-Sale Problems*, problem solvers can suggest a wide range of money-splitting strategies. From a mathematical point of view all these solutions have the same mathematical status. The solutions offered by Koirala are good, but cannot be said to be better than all the so-called alternative solutions offered by his students. Moreover, this is true not because in reality these alternative solutions would be acceptable, but because from a modelling perspective these solutions are not inferior mathematically.

In problems that involve a scientific situation (e.g., mixing paint), the source for the mathematical model lies in the phenomenon itself. The problem solver is constrained by the relations between the involved variables and by observations of how the phenomenon is realized. Sometimes there are several possible models that might fit the situation, but this is where the solver's freedom ends. The good news is that fitting a model to a given situation, at least in real life, can serve as a prediction and can be tested.

This means that in cases such as the *Fish Food*, *Eel Food* or *Paint Mixture Problems*, the source of the model is in the phenomenon itself; therefore, if we want children to apply a mathematical model in a situation of this kind they should either be familiar with the behaviour of the phenomenon (as is true for some children with regard to mixing paints), or get enough information from which they can deduce it. If such relevant data is missing, the use of problems such as the *Eel Food Problem* in diagnosing children's knowledge might lead to "Ill diagnosis"…

The use of relatively simple and traditional problems in this article allows us to make some remarks on the issue of realistic considerations and instructional goals. Often the avoidance of realistic considerations in traditional problems is legitimized by classroom goals and their corresponding norms. Traditional problems are viewed as an opportunity to teach specific mathematical concepts and therefore children are expected to use certain mathematical schemes without too many deliberations. In contrast, modelling problems focus on developing modelling skills encouraging a realistic attitude towards problem situations.

Hopefully, our analysis has shown that this dichotomy between traditional problems and modelling problems does not hold even if the two types of problems have different goals. Through the divergent examples we have tried to demonstrate that the avoidance of realistic considerations is problematic not because things work differently in real life, but because of

the meaning of fitting a mathematical model to a given situation and the roles that mathematics plays in different situations. These meanings and roles hold for traditional problems as well as modelling problems and should not depend on their goals.

Mathematics plays important roles in problem solving, but its roles are different and the limitations and nature of its authority should be understood. Following this realisation, our next goal is to design instruction that will promote teacher and student understanding of these roles and what they mean in terms of constructing and solving problems. While the examples in this article were mostly simple word problems, the designed tasks can combine the goal for developing modelling skills with the goal for understanding the roles of mathematics leading to the construction of modelling tasks *beyond the low hanging fruit*, to use Galbraith's (2007) expression.

Acknowledgement

This research was supported by the Israeli Science Foundation (ISF) grant No. 59/06. I wish to thank Alan Bishop for the stimulating conversations, and thank Peter Sullivan, Doug Clark, Anna Sfard, and Roza Leikin for their helpful comments.

References

Aumann, R. (1999). On the matter of the man with three wives. *Moriah, 22*(3/4), 98-107 (in Hebrew).

Aumann, R., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, 36, 195-213.

Bonotto, C. (2007). How to replace word problems with activities of realistic mathematical modelling. In W. Blum, P. Galbraith, M. Niss, & H-W Henn (Eds.), *Modelling and applications in mathematics education. The 14th ICMI Study (New ICMI Studies Series)* (Vol. 10, pp. 185-192). New York: Springer.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*(3), 311-334.

Department of Education and Early Childhood Development, Victoria (2007). *Mathematics Developmental Continuum P-10*. Retrieved April 16, 2009 from www.education.VIC.gov.au/studentlearning/studentresources/maths/mathscontinuum/number/n55004p.htm

Galbraith, P. (2007). Beyond the low hanging fruit. In W. Blum, P. Galbraith, M. Niss, & H-W Henn (Eds.), *Modelling and applications in mathematics education: The 14th ICMI Study (New ICMI Studies Series)* (Vol. 10, pp. 79-88). New York: Springer.

Greer, B. (1993). The mathematical modelling perspective on wor(l)d problems. *Journal of Mathematical Behavior, 12*(2), 239-250.

Hart, K. (1981). *Children's understanding of mathematics: 11-16*. London: John Murray.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics*, *11*(1), 26-32.

Koirala, H. P. (1999). Teaching mathematics using everyday context: What if academic mathematics is lost? In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 161-168). Haifa, Israel: PME.

Lo, J., & Watanabe, T. (1997). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education, 28*(2), 216-236.

Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *Journal of Mathematical Behavior*, *22*(3), 335-368.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, M. Niss, & H-W Henn (Eds.) *Modelling and applications in mathematics education: The 14th ICMI Study (New ICMI Studies Series)* (Vol. 10, pp. 3-33). New York: Springer.

Piaget, J., Grize, J. B., Szeminska, A., & Bang, V. (1977). *Epistemology and psychology of functions*. Dordrecht, The Netherlands: Reidel.

Rivest, R. L., Shamir, A., & Adleman, L. (1978). A method for obtaining digital signatures and public-key cryptosystems. *Communications of the ACM, 21*(2), 120–126.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale, NJ: Erlbaum.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences*, *16*(4), 565 – 613.

Stillman, G., Brown, J., & Galbraith, P, (2008). Research into the teaching and learning of applications and modelling in Australia. In H. Forgasz, A. Barkatas, & A. Bishop (Eds.), Research in mathematics education in Australia 2004-2007 (pp. 141-164). Rotterdam, The Netherlands: Sense Publishers.

Tourniaire, F. (1986). Proportions in elementary school. *Educational Studies in Mathematics, 17*(4), 401-412.

Author

Irit Peled, Department of Mathematics Education, Faculty of Education, University of Haifa, Israel. Email: <ipeled@construct.haifa.ac.il>