A transformation called "twist"

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The transformations found in secondary mathematics curriculum are typically limited to stretches and translations (e.g., ACARA, 2010). Advanced students may find the transformation, twist, to be of further interest. As most available resources are written for professional-level readers, this article is intended to be an introduction accessible to advanced secondary students.

Definition and explication

Define *twist* as anticlockwise rotation about the origin, of each point on a graph, by $t(r)$ radians, where $t(r)$ is a function of the point's distance r from the origin. Transformations of this nature are widely used in computer graphics and engineering (see e.g., Krishnamurthy, 2002). How should an equation $f(x, y) = 0$ be changed so that its graph is twisted? Since twist involves rotation, the polar coordinate system will prove to be useful. Begin with polar axis L. Twist it, and call the twisted polar axis L' (see Figure 1).

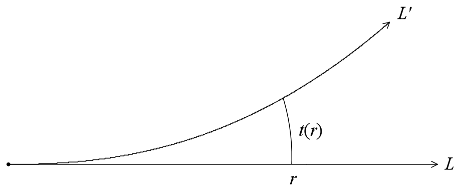


Figure 1. Twisting the axis, L.

For any point P, call its coordinates on L (r, θ), and its coordinates on L' (r', θ') as shown in Figure 2. Note that $r' = r$, and $\theta' = \theta - t(r)$.

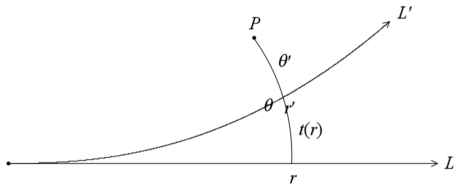


Figure 2. Transformation of a point, P.

Now consider the graph of any equation $g(r, \theta) = 0$.

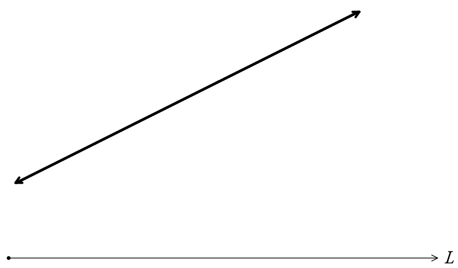


Figure 3. The graph of $g(r, \theta) = 0$.

Twist the graph of $g(r, \theta) = 0$ and L together, as shown in Figure 4.

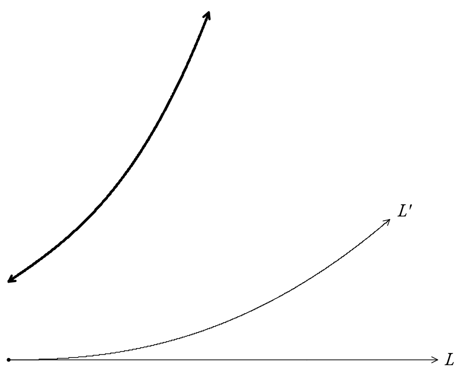


Figure 4. The graph of g after a twist.

The twisted graph has equation $g(r', \theta') = 0$, which is the same as $g(r, \theta - t(r)) = 0$. So to twist the graph of $g(r, \theta) = 0$, substitute $\theta \to \theta - t(r)$.

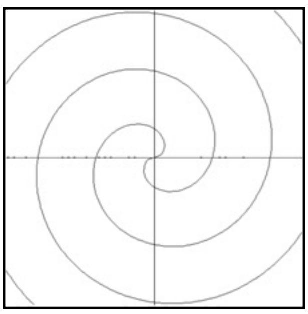
To twist the graph of $f(x, y) = 0$, convert the equation into a polar equation ($x \to r\cos\theta$, $y \to r\sin\theta$), then substitute $\theta \to \theta - t(r)$. That is, substitute

$$x \to r\cos(\theta - t(r))$$
$$y \to r\sin(\theta - t(r))$$

Examples

The following examples show that the twist transformation can yield aesthetically pleasing graphs. In each example, an equation and $t(r)$ are given. The equation of the twisted graph is found, and the equations are plotted.

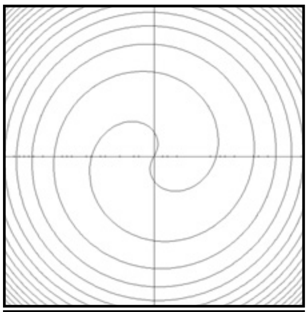
Example 1.
$y = 0,\ t(r) = r$
$\sin(\theta - r) = 0$



Example 2.

$y = 0,\ t(r) = 2^r$

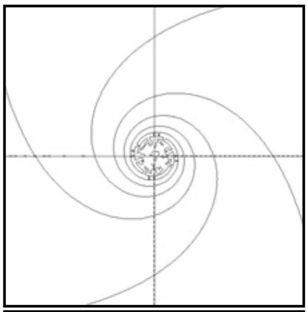
$\sin(\theta - 2^r) = 0$



Example 3.

$xy = 0,\ t(r) = \dfrac{1}{r}$

$$\sin\left(2\theta - \frac{2}{r}\right) = 0$$

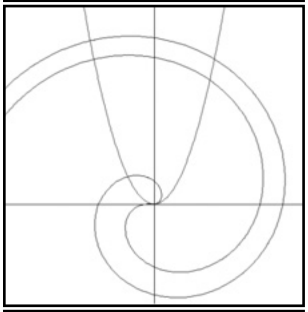


Example 4.

$y = x^2,\ t(r) = r$

$\tan(\theta - r)$

$= r\cos(\theta - r)$

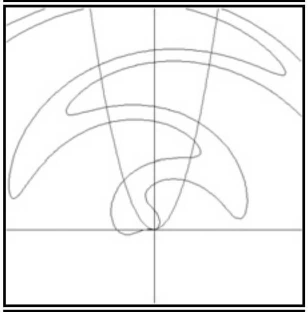


Example 5.

$y = x^2,\ t(r) = \sin r$

$\tan(\theta - \sin r)$

$= r\cos(\theta - \sin r))$

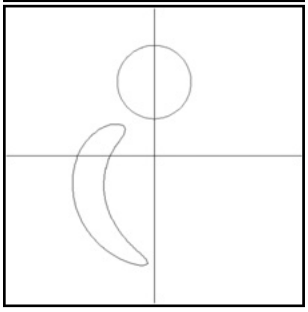


Example 6.

$x^2 + (y - 2)^2 = 1,$

$t(r) = r$

$r^2 - 4r\sin(\theta - r) = -3$



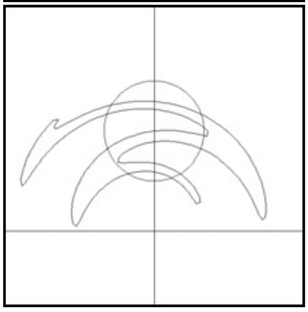
Example 7.

$x^2 + (y - 2)^2 = 1,$

$t(r) = \sin 5r$

$r^2 - 4r\sin(\theta - \sin 5r)$

$= -3$



Suggested classroom tasks

Students with access to suitable graphing software (e.g., Illustrator, see http://diganimation.info/pages/resources/illustrator/05_illu_transform_ex.pdf) may:

1. Given various combinations of $f(x, y) = 0$ and $t(r)$, use paper and pencil to predict what the twisted graph will look like; then use the software to check.
2. Produce a gallery of aesthetically pleasing twisted graphs.
3. Investigate how the unit square changes under various twist transformations.

References

Australian Curriculum Assessment and Reporting Authority (2010). http://www.australiancurriculum.edu.au/SeniorYears/Mathematics/Mathematical%20Methods

Krishnamurthy, N. (2002). *Introduction to computer graphics.* New Delhi: Tata McGraw-Hill.

Further reading

http://en.wikipedia.org/wiki/Screw_theory