A Lakatosian encounter with Probability

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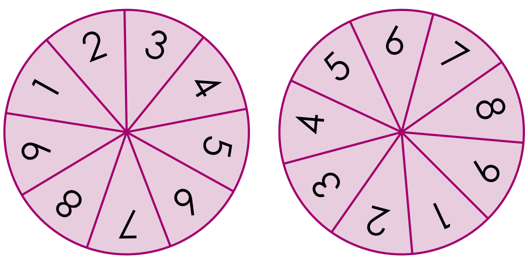
There is much to be learned and pondered by reading *Proofs and Refutations* by Imre Lakatos (Lakatos, 1976). It highlights the importance of mathematical definitions, and how definitions evolve to capture the essence of the object they are defining. It also provides an exhilarating encounter with the ups and downs of the mathematical reasoning process, where conjectures are proposed, countered, and justified. The main part of the book has its mathematical home in the domain of three-dimensional geometry, and Lakatos uses the literary device of presenting, in script form, the dialogue of a cast of Greek-letter-named characters as they debate definitions, proofs, and counterexamples in a classroom.

My educational background included some formal geometry, so there was something quite familiar about the domain of 3D solids that Lakatos explores. Despite the tumult that Lakatos puts us through—as the definitions and proofs are examined and refuted—the reasoning and logic applied had a comforting feel because of its traditional domain.

Recently, however, I observed a probability game (Feely, 2003a,b) that raised a number of interesting issues about how probability is learned, understood, and reasoned with. The game is a simple one, for two players. There are two spinners, each divided into nine equal sections numbered 1 to 9. Both are spun, and the results added. If the total is odd, then player 1 wins a point; if the total is even, player 2 wins a point. The winner of a game is the first person to get 10 points.



I wondered what would happen if Lakatos's class encountered the game, and what reasoning might arise in this rather different area of mathematics. This is my attempt to imagine such a debate. (Those of you familiar with Lakatos's work please note that although I have borrowed the names of some of his characters, their personalities here don't match those of their original counterparts.)

The class begins with our Lakatosian characters, in pairs, playing the game as described, and recording the scores from their "first to ten" contest. They end up with the following table, showing the final totals achieved for the odd player and the even player in each of the pairs.

Student Pair	Odd's Score (player 1)	Even's Score (player 2)
1	8	10
2	10	2
3	8	10
4	6	10
5	7	10
6	10	6
7	10	4
8	9	10
9	5	10
10	10	7

TEACHER: Well, what do you think?

THETA: Hah! It's clearly biased. Even won more times. See: six times out of the ten.

GAMMA: That could just be a fluke. You know, luck of the draw and all that.

OMICRON: No, no. It's biased all right. I've worked out why. It's easy. Since even plus even is even, and odd plus odd is even and, finally, even plus odd is odd, then that's two out of three evens. Therefore Even has a much better chance of winning.

KAPPA: Quick thinking, dude.

BETA: Yeah, but it is it correct thinking? I'm not sure I'm convinced.

GAMMA: Why not? Even won more games. Looks good to me.

BETA: Well, it seems…

ALPHA: You said it was two-thirds biased in favour of Even?

OMICRON: Yep.

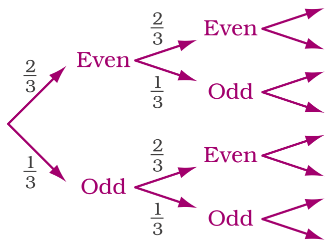
THETA: I see what you're thinking, Alpha. You're saying that the two-thirds bias means that Even should have won 67% of the time, and it only won 60% of the time.

ALPHA: Well, no… Worse than that, I think. Remember a game is made up of lots of spins, because we played "first to ten". But it's the spins that have the two-thirds bias, not the whole game. Each spin has a two-thirds chance of coming up in favour of Even, if Omicron's argument is right.

BETA: Ah, yes, now I see what Alpha's getting at. As you play the first to ten game, Even must surely have a really good chance of winning, because each spin has a two-thirds chance of coming up Even, and so after lots of spins it's really, really likely that Even is going to be ahead at the end of a whole game. In fact, I reckon Even must almost be nearly guaranteed to win.

SIGMA: How can you be so sure?

ALPHA: Well you could do one of those tree diagram things, with the first spin showing a split, with even on one branch with a two-thirds chance, and odd on the other branch with a one-third chance; and then each of those splits for the next spin, and so on, and so on. [Alpha starts to sketch this on the board.]



KAPPA: That's a lot of "and so ons"!

ALPHA: Yep.

SIGMA: And how do you know when to stop? You could have someone win after only ten spins, so long as that person won ten in a row.

Of course, the other games might go on for longer. And then you have to do all that multiplying and adding stuff for combining the probabilities ... and not everyone in the class knows about that anyway.

ALPHA: Yeah, it's a bit messy. But you could do it, I reckon.

BETA: Hang on, what if we did it experimentally?

SIGMA: What do you mean?

BETA: We could simulate it. You know: do an experiment lots of times and see what happens.

ALPHA: That's not a mathematical proof. It doesn't guarantee that it's true. It will come out differently each time.

BETA: Well, no it's not a proof ... and yeah, I agree it could come out differently each time ... but it might provide evidence.

SIGMA: But how are you going to simulate it?

OMICRON: What if we use a die? Say we roll it, and if we get a 1 or a 2, then we'll say that means Odd wins, and if we get 3, 4, 5 or 6 then we'll say that means Even wins. So, rolling the die is like spinning once, with Even having a two-thirds chance of winning.

SIGMA: Oh, yeah, I get it.

TEACHER: Okay, here are some dice. Rolling once is like a spin, and a whole game is playing first to ten points again. Don't forget to keep score so we can make a record like we did before. [Students roll a die in pairs, and record a table of results.]

Student Pair	"Odd's" Score (player 1, who gets 1 or 2)	"Even's" Score (player 2,who gets 3, 4, 5 or 6)
1	4	10
2	10	7
3	3	10
4	2	10
5	5	10
6	5	10
7	4	10
8	6	10
9	3	10
10	2	10

GAMMA: Oh, wow. I'm convinced. Look how often Even won, and Odd usually got pretty low scores, too. If it really was two-thirds in favour of Even, then Even should have blitzed Odd, that's for sure.

ALPHA: I'm glad the empirical evidence supports my argument, but, of course, it's still not a proof.

KAPPA: Okay, okay. Just because you like your proofs! But you've gotta admit that's pretty strong evidence.

SIGMA: So, hang on a tick, what have we just done?

BETA: Omicron used a nice argument about odds and evens to suggest that in the spinner game spinning favoured Even with a two-thirds probability. However, Alpha said that if this was true then Even should actually thrash Odd in a game, and especially in a whole bunch of games.

ALPHA: And our two-thirds dice simulation supports that—but isn't a proof, mind you!

BETA: Yes, yes. But the real point is that when we actually played the original spinner game, we didn't get such a strong result. This makes me think that Omicron's "even plus even is even, odd plus odd is even, and even plus odd is odd" argument is not right.

OMICRON: Bother. It sounded so good!

BETA: Ah! Coin tossing!

KAPPA: What?

ALPHA: Not another simulation! That's not real maths.

BETA: No, no. Coin tossing. Remember when we were tossing two coins to see what the outcomes were? And we listed the possibilities…

GAMMA: And… and we thought at first that there were only three possibilities: heads-heads, tails-tails and heads-tails.

OMICRON: Oh, yeah… I remember now, too. We were really surprised that we got lots of heads-tails instead of all three outcomes being roughly the same.

GAMMA: Because we'd forgotten about tails-heads.

SIGMA: Huh? What do you mean?

THETA: Hey, wait up. What if you look at the overall totals in our original results? Umm, if I do some quick addition, I reckon there's a total of… 83 odds, and… 79 evens. Therefore Odd wins.

KAPPA: But not by much.

SIGMA: Theta, you're confusing me. Don't change the topic. I'm still wondering what they're on about with the coins. I can't see what the difference is with heads-tails and tails-heads. And then I can't see what it's got to do with our spinners.

BETA: When you toss two coins, you could get heads on one and heads on the other. Right?

SIGMA: Yep. I'm with you so far.

BETA: And you could get tails on one and tails on the other.

SIGMA: Obviously.

BETA: And now for the tricky bit. You could get heads on the first coin and tails on the second, or you could have tails on the first coin and heads on the second.

SIGMA: Ummm…

ALPHA: Imagine that one coin is painted green and one is red. Then when you toss the coins you could have green heads-red heads, green tails-red tails, green heads-red tails, and green tails-red heads.

SIGMA: Ah, yes, now I see it. There are two ways of getting the heads-tails combination … and so that's why it ended up occurring much more often than each of the others. Got it!

BETA: In fact, it just so happens that the probability of the heads-tails combination is the same as the total of heads-heads and tails-tails.

SIGMA: I'm not sure how that relates to the spinners, though.

THETA: Oh, I do now. I see what you're on about, Beta. The spinners are like the coins. Omicron forgot to allow for the fact that we can get an odd total in two ways: odd on one spinner and even on the other, and the other way around.

GAMMA: Oh, neat. So that must mean that there are four possibilities: odd-plus-odd-is-even and even-plus-even-is-even, together with even-plus-odd-is-odd and odd-plus-even-is-odd.

OMICRON: Ahhh. So that means that it's not two-thirds at all, but 50-50: half the time you'll get odd and half the time you'll get even.

THETA: And that matches what I was saying before about the totals. I mean, 83 odds to 79 evens is pretty close to half-and-half.

KAPPA: It's not 81-81 though!

BETA: Oh, come on, Kappa! You know that when the probability is a half, it means that when you do an experiment the experimental probability will approximate the theoretical probability. 83/162 is close to a half. Remember when we tossed a single coin 100 times? The number of tails was around 50, and rarely equal to it; in fact, we got even got the occasional value as low as 40 or as high as 60.

KAPPA: Yeah, I knew that. I was just trying to wind you up! It worked.

BETA: Okay, you got me.

SIGMA: But hang on a moment. We've forgotten something.

KAPPA: Here we go again. Now what?

SIGMA: Well, the spinners have got more odd numbers on them than even numbers.

GAMMA: So what? There's still only the two alternative types of numbers, and so only four sets of combinations.

SIGMA: I'm not sure it's that simple. It just doesn't feel balanced somehow.

ALPHA: Oooh, no… You're right, Sigma, there really is a problem.

GAMMA: But how?

ALPHA: Well, suppose you spin one of the spinners only. What are you more likely to get: odd or even?

GAMMA: Odd, I guess. Sigma said there were more odds than evens on each spinner.

ALPHA: Well, that must mean that when you spin two spinners you're more likely to get an odd and an odd in comparison to getting an even and an even.

SIGMA: But both those combinations give even totals anyway, so maybe it doesn't matter.

OMICRON: But what if the other two combinations give more odd totals?

SIGMA: Ummm…

THETA: Hah! What if the spinners only had odd numbers on them? Then you could only get an even total. Odd could never win then.

KAPPA: Now you're being silly.

ALPHA: No, that's pretty clever.

KAPPA: All right. It's clever silly.

THETA: So, I think the number of odds and evens on each spinner will make a difference and we need to think about it.

[There is a general rush for pens and paper, with a few working on their own and some muted discussion between pairs. The scrawl becomes more hasty as they realise everyone seems to be working towards an answer.]

BETA: Got it.

OMICRON: Me too.

THETA: And me

SIGMA: Yep. Done.

GAMMA: I'm there, I just haven't finished the counting yet. … 38, 39, 40. Finished.

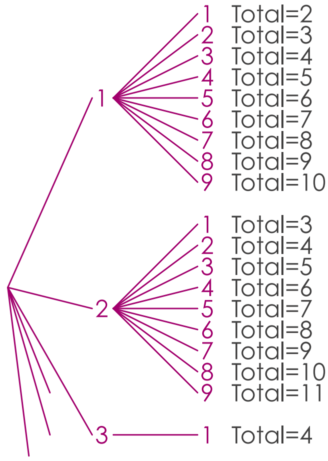
TEACHER: Okay, Gamma, what did you do?

GAMMA: Well, I just listed absolutely everything. 1 + 1, 1 + 2, 1 + 3, up to 1 + 9, and then 2 + 1, 2 + 2, 2 + 3, and on and on. That got me 41 even totals and 40 odd totals.

SIGMA: I think what I did was similar but different looking. I did a gigantic tree diagram.

KAPPA: That's massive.

SIGMA: That's only part of it. Anyway, I've got branches numbered 1 to 9 for all the different outcomes on the first spinner, and then each of them has branches numbered 1 to 9 coming off, giving all the different possibilities for the second spinner. Gamma's 2 + 3, for example, is down the 2 branch followed by the 3 branch. Then I just counted everything. I got 41 evens to 40 odds as well.



KAPPA: We'd be in trouble if they were different!

THETA: Well, mine looks totally different, but I think it ends up the same too. I did a table. I listed the numbers 1 to 9 for Spinner 1 across the top, and 1 to 9 for the second spinner down the side, and then filled in the table. Only I couldn't be bothered adding the actual totals, so I just filled in the table with odd or even, so for 3 + 4 I just wrote O for odd. And, guess what? 41 evens and 40 odds for me too.

	1	2	3	4	5	6	7	8	9
1	E	O	E	O	E	O	E	O	E
2	O	E	O	E	O	E	O	E	O
3	E	O	E	O	E	O	E	O	E
4	O	E	O	E	O	E	O	E	O
5	E	O	E	O	E	O	E	O	E
6	O	E	O	E	O	E	O	E	O
7	E	O	E	O	E	O	E	O	E
8	O	E	O	E	O	E	O	E	O
9	E	O	E	O	E	O	E	O	E

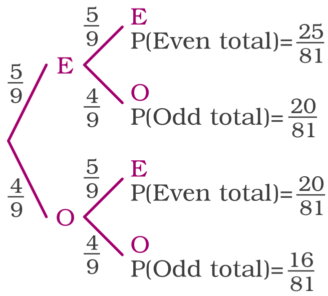
BETA: It's pretty neat how there's so many different ways of coming up with same thing.

OMICRON: I think mine's a simplified version of yours, Theta. I did a table too, but instead of going 1 to 9 across the top and down the side, I did 1, 3, 5, 7, 9 together, followed by 2, 4, 6, 8. This meant all the odds were together and all the evens were together. Since odd plus odd is even, and I had five odds across the top and five odds down the side, then I got 25 evens all together from that. Even plus even, with four for each even, gave me 16 more even totals, and then odd at the top and even on the side gave me 20 odd totals, and vice versa gave another 20 odds. So, I got 41 evens and 40 odds all together.

	1, 3, 5, 7, 9	2, 4, 6, 8
1, 3, 5, 7, 9	25 evens	20 odds
2, 4, 6, 8	20 odds	16 evens

KAPPA: Oooh. What a surprise! 41 and 40 again!

BETA: I did a simple tree diagram and combined the probabilities. On the first spinner there's a 5/9 chance of getting odd, and a 4/9 chance of getting even. From each of these, you branch again for the probabilities on the second spinner. Then you combine using multiplication: odd followed by odd is 5/9 times 5/9, which means a 25/81 chance of an even total. Even followed by even is 4/9 times 4/9, giving an extra 16/81 chance of an even total. Then we have 5/9 times 4/9 and 4/9 times 5/9 for odd-even and even-odd, and so we have 40/81 for an odd total. Oh, and we had 41/81 from those two even calculations. I think this is the same as Omicron's cunning table, but I just used the fractional probabilities right from the start.



KAPPA: And I, good people, did it all in my head. Nine possibilities on each spinner means 81 possibilities all together. Five odds with five odds, and four evens with four evens gives 25 + 16 even totals; five odds with four evens and four evens with five odds gives 20 + 20 odd totals. Lo and behold: 41 evens to 40 odds.

THETA: A smart solution from Kappa, as well as the usual smart-alec comments!

KAPPA: Ha! We are such a bunch of legends.

GAMMA: All right!

SIGMA: Yeah… but what exactly does it mean?

OMICRON: It's biased in favour of Even.

BETA: Yeah, but only just. I mean, 41/81 to 40/81 is really close to half and half. No wonder it was possible for Odd to win when Theta added up the totals. In fact, I wouldn't be surprised that if we all played first to ten again, it would be possible to have Odd winning more times than Even. Sometimes, anyway.

OMICRON: But in the long run, it favours Even.

BETA: It'll have to be a humongous long run, I reckon.

KAPPA: Hey, Alph! You've been pretty quiet. Wotcha need a calculator for? We did it all in our heads … apart from the diagrams!

ALPHA: I'd worked out the 41 to 40 pretty quickly.

KAPPA: Typical!

ALPHA: Of course, you all know that's just for one spin. Then I wanted to go back to the whole first-to-ten game.

BETA: You mean, spin once and then spin again, and then keep going?

ALPHA: Yeah. I wanted to work it out with that tree diagram I started doing ages ago before you all got sidetracked on the simulation.

OMICRON: Oh, wow. That was going to be huge.

KAPPA: No wonder you wanted a calculator.

ALPHA: It was a bit of a brain strain, and that was even before I started mucking around with the actual probability calculations on the calculator. I actually had to work out what all the branches on the tree looked like … and then I stopped thinking about it as a tree and just tried to imagine strings of Odd and Even.

BETA: Go on.

KAPPA: I'm not sure I want to know!

ALPHA: Well, I'm not going to give you all the gory details…

KAPPA: Hallelujah!

ALPHA: But you've got to keep track of the number of different ways that Even can win the first-to-ten. So, Even could win ten in a row.

THETA: Which is unlikely.

ALPHA: But still possible. To get the probability you just work out 41/81 to the power of 10, because you need Even and Even and Even and Even and so on, ten times, and each Even has probability of 41/81 and we multiply the probabilities together.

THETA: And that 41/81 to the power of 10 is not very big, so it's unlikely, like I said.

BETA: But that's not the only way Even can win.

ALPHA: Which is why it gets complicated.

BETA: For example, they could play 11 games, with Even winning 10 of them, which means you have to count the number of different ways that Even could win 10 games from 11.

ALPHA: But you don't want to count the possibility that Odd won the eleventh game, because Even would already have reached 10 by then. And you have to work out the product of the probabilities for ten evens and one odd.

SIGMA: Okay, I'm starting to lose it a bit here.

ALPHA: Well, it gets worse. Because they could have had to play 12 games before Even won 10…

GAMMA: Or 13 games…

BETA: Or anything up to 19. Whoah! That's hard to keep track of!

ALPHA: Yeah. So to win from 15 games, for example, Even has to have 10 wins out of the 15. If you use combinations, there are "15 choose 10" ways for Even to win 10 out of 15 but you have to not count the "14 choose 10" ways that would have had Odd winning the fifteenth game. You don't want that, because Even has already got to 10 by then. Actually, you can sort this out easily. It just turns out to be the same as having 9 evens in the first 14 slots, keeping Even's tenth win for the fifteenth game. So in fact we just have to do "14 choose 9".

GAMMA: I've lost it, too, I'm afraid.

OMICRON: I'm struggling.

BETA: I think I'm still with you, Alpha, but I'll be happy to trust you for the fine details.

THETA: But did you get an answer?!

ALPHA: Well, if you add up all the probabilities for all the different ways that Even can win first-to-ten, you get…

KAPPA: Drum roll…

ALPHA: That Even has a 52% chance of winning the first-to-ten game.[1]

GAMMA: Woo-hoo! Great effort, Alpha.

THETA: I'm impressed… but… well… all that effort to discover that it's not much different from just tossing a coin anyway.

ALPHA: So it would seem. Of course, the really cool thing is that we've now got the theoretically proved value of the probability. It's not some simulation.

KAPPA: Go Alpha, the proof-obsessive!

BETA: Oh, I don't know. I wouldn't mind seeing a simulation as well.

TEACHER: As it happens, I do have a little Excel spreadsheet that simulates the game, and also does a best of 1000 competition as well.[2]

SIGMA: It's a bit of a boring game, now. I mean, if it's not much different from coin tossing, then it would take ages before you were ahead.

ALPHA: Yeah, but it's that little bit that casinos rely on to make their money, I'll bet.

SIGMA: I s'pose, but I'd like a game with a little more bias, so I can beat my friends without having to keep playing until half-way through next year.

KAPPA: You big cheat!

BETA: Well, why not redesign the game then? Put some different numbers on the spinners so it's more biased.

OMICRON: Only you have to make it so it doesn't look like it's biased.

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1. What Alpha worked out was

$$\sum_{i=0}^{9} \binom{9+i}{9} \left(\frac{41}{81}\right)^{10} \left(\frac{40}{81}\right)^{i} \text{ where } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

2. The teacher's spreadsheet can be downloaded from http://staff.edfac.unimelb.edu.au/~chick/homepage_files/Downloads/TwoSpinnerOddEven.xls

GAMMA: I think we can do that.

BETA: And then work out the probabilities for the different outcomes after spinning.

THETA: Yeah, that can be done, too.

ALPHA: And then work out the first-to-ten probabilities!

KAPPA: Not likely, mate! But it'll be fun to design a new game, though.

BETA: Oh, I don't know. I might give the first-to-ten a go this time. But I might program a simulation too.

Could this kind of reasoning happen in a classroom? Omicron's initial argument about evens and odds came up very naturally in two Grade 5 classrooms that I observed, and at least one student at that age was also able to come up with a nearly complete enumeration of the sample space. With teacher guidance most students in one of those Grade 5 classes could list all 81 possibilities. Pre-service primary teachers, with high school maths backgrounds, were able to construct nearly all of the different methods of listing all the possibilities.

In this classroom I have allowed the students to do all the reasoning and counter-arguing. In a real classroom perhaps some of this might need to be prompted by suitable questions and doubts raised by the teacher, but the capacity to produce the arguments is within the reach of secondary students and can also be developed in primary students. This kind of reasoning and justification about situations is a fundamental part of learning to think and work mathematically. This essential characteristic is achievable in classrooms where it is expected, supported, and encouraged. The only possible exception to accessibility is Alpha's computation of the first-to-ten probability. I've included this, because Alpha made me finally get around to doing it for myself. I don't mind that Sigma, Gamma, Kappa, Theta, and Omicron didn't get it this time around; they've already come a long way.

References

Feely, J. (2003a). *Nelson maths for Victoria: Teacher's resource Year 5*. Melbourne: Thomson Nelson.

Feely, J. (2003b). *Nelson maths for Victoria: Year 5 student book*. Melbourne: Thomson Nelson.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Notes

This article first appeared in the May 2010 issue of *Mathematics Teaching*, published in the UK by the Association of Teachers of Mathematics. It is reprinted here with their kind permission. Helen Chick is a Senior Lecturer in Mathematics Education at the University of Melbourne. This article was prompted by a day of studying Lakatos' Proofs and Refutations while on study leave at the University of Oxford.