Technology Focus:
Enhancing Learning of Limits of Functions with Dynamic Sketches

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After students have constructed a conceptual understanding of a mathematical idea, applying mathematical symbolism to represent and define the idea concisely becomes an easier task. In fact, Moore (1994) found that students "often needed to develop their concept images through examples, diagrams, graphs, and other means before they could understand formal verbal or symbolic definitions."

Case in Point: The Formal Definition of the Limit of a Function

A good example is the formal ε-δ definition of the limit of a function. Without a coherent pictorial model, students struggle to make sense of this definition, and it is no wonder when one considers its complexity: "A function f(x) has a limit L as x approaches c if for every $\varepsilon > 0$, there exists an integer $\delta > 0$ such that for every x, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$" (adapted from Bartle & Sherbert, 1992, p. 113). Beginning students are often unable to connect their naïve, motion-oriented visual models of limit with the static absolute value inequalities in the verbal definition (Cottrill et al., 1996).

Furthermore, they may not understand why the universal quantifier ("for every") and the existential quantifier ("there exists") are necessary, and either attach them to the wrong variable or overlook them entirely. Even after instruction, if students' pictorial mental models are not yet fully developed and integrated with the verbal representation, they may have trouble explaining or even remembering this definition (Cory, 2005). Some give up trying to understand altogether and resort to rote memorization, frequently confusing the formal definition with other limit notions.

While the formal definition takes effort to master, with appropriate guidance, manipulating an interactive, mathematically-consistent visual representation of the formal limit concept may help learners enhance their understanding of the definition. In this way, interactive sketches may make the formal definition less formidable, thus providing a more positive experience for students during one of their first introductions to advanced mathematical thought.

The Dynamic Sketches

Using *The Geometer's Sketchpad* (Jackiw, 2002), we designed a collection of interactive dynamic sketches that could be utilized to help students overcome hurdles involved in understanding the formal definition of the limit of a function. Although beginning calculus students intuitively connect the limit concept to motion along the curve, a first hurdle is to coordinate the movement of x toward c (the domain process) with the simultaneous movement of f(x) toward L (the range process; Cottrill et al., 1996).

In our sketches, after clicking on the "Show trace function" button, students can manipulate a visual representation of these coordinated processes by dragging a lavender-colored x-value along the x-axis toward c while simultaneously viewing the movement of the corresponding f(x)-values along the y-axis toward L (see Fig. 1). The x-value and

Figure 1. Coordinating the domain and range processes

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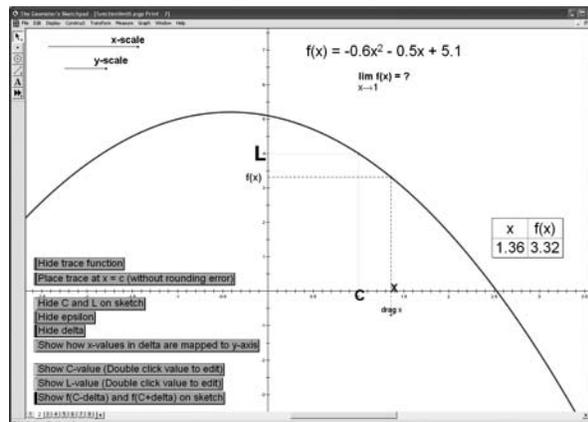
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corresponding f(x)-value also determine a point that moves simultaneously along the curve toward (*c*, f(*L*)). (Note that the buttons on the sketches toggle between "Show" and "Hide.")

Once this coordination is understood, the next step is to reconstruct the coordinated processes in terms of the absolute-value inequalities in the definition (Cottrill et al., 1996). In each dynamic sketch, the student can display a δ-interval along the *x*-axis around *c* by clicking on the "Show delta" button and an ϵ-interval along the *y*-axis around the limit, *L*, by clicking on the "Show epsilon" button (see Fig. 2).
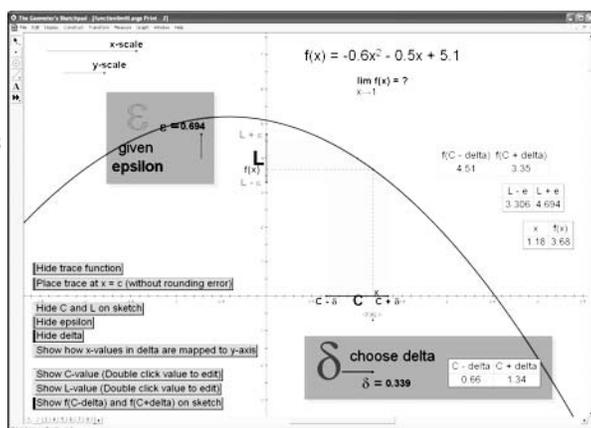

Figure 2. Applying the formal definition dynamically

After the instructor adjusts the red ϵ-slider to display an appropriately large ϵ-interval, students can drag the lavender-colored *x*-value back and forth within the δ-interval to see that all corresponding f(x)-values fall within the ϵ-interval. This gives a visual representation of "for every *x*, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$." To see this more clearly, the "Show how x-values are mapped to *y*-axis" button colors a light-blue band initiating from the δ-interval, traveling up to the curve, then over to *y*-axis to fall within the ϵ-interval.

After students understand the absolute-value inequalities, they are ready to tackle the universal and existential quantifiers. One method (the "ϵ-δ game") is for the instructor to set the red ϵ-slider to a particular value. Then a student volunteer is asked to find one δ-value that "works," i.e., for which all *x*-values (except *x* = *c*) within the δ-interval correspond to f(x)-values within the ϵ-interval. This is done by either dragging the δ-slider or adjusting the endpoints on the δ-interval.

If the student can find an appropriate δ for the given ϵ, he or she wins the first round of the game. Note that students can use the *x*- and *y*-scale sliders to zoom in around *x* = *c* and *y* = *L* as needed. After playing two more rounds using two increasingly smaller values of ϵ, the students are asked, "For any value of ϵ, could you always find a δ that 'works'?" If the limit exists, our students have responded with explanations such as, "Yes, because you can stretch out your *x*-scale and see the tiniest, tiniest little number near *c*?that would make you fall into whatever ϵ you wanted me to hit," or "Yes, because this is a continuous function." Finally, to connect the visual to the verbal, we have students carefully compare their interactions with the sketch to each portion of the formal definition.

Going through similar processes with sketches of various functions can help students further connect the visual dynamic representation with the verbal representation. Our collection includes two quadratic functions, a constant function, a rational function, and a piecewise-defined function. A nice option for each sketch is to give students a table of ten or more ϵ-values for which to find an appropriate δ. In this way, the concept of the universal quantifier is further reinforced.

Accessing the Dynamic Sketches

We encourage you to use our collection of dynamic sketches in your own classes. We also welcome your suggestions for improving this tool. The sketches are available by subscribing to NCTM's *ON-Math* online journal. To open the sketches, The Geometer's Sketchpad version 4 or higher must be installed on your computer.