Investment return calculations and senior school mathematics

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Businesses and individuals both like to make money from their investments. Individuals may focus on the return they earn in their at call savings accounts, in their term deposits or from their share portfolios. Businesses have similar interests to individuals in this way but may also be interested in the return they can expect to earn from a new business venture. Why, from a financial perspective, would a business proceed with a venture that can only promise a return on investment which is lower than it could earn in a risk-free savings account? The methods for calculating returns on investments are taught to undergraduate level business students. In this paper we demonstrate how such calculations are within the scope of senior school students of mathematics. In providing this demonstration we hope to give teachers and students alike an illustration of the power and the utility of the mathematics they are working with in the senior school classroom which in turn can be a source of motivation for them.

Everyone and every business wants to earn a return on their investments. Suppose a school is considering putting on a production by the students in the drama class. Before such a production can go ahead, there will be a range of expenditures relating to hire of the venue, costume hire and employment of staff to work with the student actors. All these costs will need to be met before a single dollar of revenue can be earned from the sale of tickets to parents and fellow students. Suppose the school can reliably estimate the initial costs and the revenues to be earned from ticket sales. It can then calculate the return earned on its investment. Of course in reality there will be some unknowns in this exercise—an obvious one is whether the ticket sales will match that anticipated. We will assume that the school has done sufficient research to be able to make reliable estimates of its costs and revenues. Schools may consider making available to students information on revenues and expenses related to a particular drama production for the purposes of this example. Indeed, some students may elect to run their next production on a commercial basis after assessing their expected return on investment.

This paper bridges the gap between calculations of rates of return and the senior mathematics syllabus. It is organised as follows. Following this introduction, the next section defines precisely what we mean by a rate of return on an investment. The subsequent section describes how a business might determine the rate of return it requires before making an investment or before undertaking a new venture. Next we provide illustrations of the method of calculating returns on investments. Links between the senior school mathematics syllabi and these calculations are made to enable teachers to see how this application can be incorporated into many aspects of their teaching. The practical problem that some projected cash flows may have multiple rates of return and the issues this can cause for businesses are then considered. We also provide a new proof that a particular characterisation of the projected cash flows from a business venture will lead to a unique positive rate of return. This will provide a thought-provoking challenge to capable senior students. We also provide a practical example in that section to illustrate the result proven there.

Rate of return for a financial project

Students from Year 9 onwards (Board of Studies, 2007) are familiar with the concept of a compound interest rate. In order to establish our notation, we repeat here a formula that will be familiar to all,

$$A = P(1 + i)^n \tag{1}$$

where A is the accumulated value (or final value) at time n years when P dollars is invested at time 0 at rate of interest i per annum.[1]

Equation (1) can be solved for P. We obtain a unique solution which works as a formula for determining the value today, at time 0, of money available in the future. The expression is

$$A = P(1 + i)^{-n} \tag{2}$$

where P is commonly called the *present value* (PV) of A dollars available at time n years, where interest earns at i per annum. Clearly, for positive interest rates, i, we have $P < A$.

The annual (effective) rate of return for a project is defined to be the rate of interest, i, such that the present value (at time 0) of all cash inflows or receipts (revenue from ticket sales in the example given in the introduction) matches the present value of all cash outflows or *payments* (costs of hiring the venue and costumes and costs linked to employment of the training staff in our example given in the introduction).

To illustrate, suppose a particular business venture has anticipated cash receipts, r_j at time j for (say) j = 3, 4 and 5. Suppose the set up costs or payments linked to this project are denoted p_j at time j for (say) j = 0, 1. Note that the payments occur before the receipts. This is common, although as we shall see later, not universal, in new business ventures. The rate of return, often called the *internal rate of return* (IRR), for this project, is the rate of inter-

1. This formula can be easily used to show, using logarithm laws, that money invested today with interest applying at 10% per annum will take about 7.3 years to double in value.

est i which solves the equation

PV of cash receipts = PV of cash payments

$$\frac{r_3}{(1+i)^3}+\frac{r_4}{(1+i)^4}+\frac{r_5}{(1+i)^5}=p_0+\frac{p_1}{1+i} \qquad (3)$$

This equation is often written with all the terms on the left hand side. When written this way, the IRR calculation can be thought of as determining the rate of interest i which equates to zero the present value of cash receipts minus the present value of cash payments. This interpretation is encapsulated in the following equation

$$-p_0-\frac{p_1}{1+i}+\frac{r_3}{(1+i)^3}+\frac{r_4}{(1+i)^4}+\frac{r_5}{(1+i)^5}=0 \qquad (4)$$

The IRR is just one of many different rates of return that can be considered by businesses. We describe one other rate of return, the *continuously compounded rate of return*, denoted δ.

You may reasonably ask what we mean by a continuously compounded rate of return. A useful way to start thinking about this concept is to think about your bank account. You may see that interest is credited to your account monthly. This is not continuous compounding but rather it is compounding twelve times per annum. Another example is if you check out your home loan statement. You may find that interest is levied on the amount you owe *every* day! That is still not continuous compounding, rather it is compounding 365 times per year or daily compounding. Daily compounding is so close to compounding occurring at every instant in time that it is often approximated by continuous compounding. The analogue to Equation (1) for continuous compounding is shown in the equation below

$$A = Pe^{\delta n} \qquad (5)$$

where A is the accumulated (or final value) of an investment at time n years when P dollars is invested at time 0 at rate of interest δ per annum with continuous compounding.[2,3] We note that this form of compound interest is identical to the exponential growth model considered for population sizes, amongst other applications, in Australian senior mathematics syllabi (e.g., Board of Studies, 2000; VCAA, 2005). As above Equation (5) can be solved for P to give the present value of A dollars at time n where the rate of interest is δ per annum with continuous compounding:

$$P = Ae^{-\delta n} \qquad (6)$$

Similarly, the continuously compounded rate of return on our investment described just before Equation (6) can be found by solving for δ in

$$-p_0 - p_1e^{-\delta} + r_3e^{3\delta} + r_4e^{4\delta} + r_5e^{5\delta} = 0 \qquad (7)$$

2. This formula can be easily used to show, using logarithm laws, that money invested today with interest applying at 10% per annum compounding continuously will take about 6.9 years to double in value. Note that money accumulates faster in value with a continuously compounding rate of interest than with a standard per annum interest rate.

3. By noting that the accumulation under equation (1) and equation (5) must be equal, we can derive a relationship between the annual rate of interest, i, and the continuously compounded rate of interest, δ: $(1 + i)^n = e^{\delta n} \Rightarrow \delta = \ln(1 + i)$.

How to determine your required rate of return?

Suppose we are running a business and that we are considering a new venture as a means of expanding our operations. As part of our decision-making process with respect to this new venture, we should assess the benefits and the costs to our business of this new venture. Having quantified these anticipated cash payments and receipts, the business will determine the implied IRR. If this IRR is less than what the business can earn, risk free, then the new business venture is not worth pursuing. Thinking just about financial returns, why would a business expose itself to risk for a return no greater than that it could achieve by investing money in a bank?

The business will often require an IRR in excess of that which can be earned risk free. This will be particularly relevant if the new business venture involves significant risk. The higher IRR that the business requires can be thought of as a reward for the additional risk that the business now faces.

Calculation of the rate of return for a given set of cash flows and links to the mathematics syllabus

In this section we give a series of examples of IRR and continuously compounded rate of return calculations and how these can be included as applications of diverse areas of the senior mathematics syllabi. An overview is provided in Table 1. Elaborations of some of the cases are then presented.

To illustrate in more detail on the use of the examples provided in Table 1, we expand on two of the cases there: *equations reducible to quadratics* and *integration involving exponential functions.*

Example 1: Equations reducible to quadratics

Suppose that a school canteen launches a small advertising campaign. The costs connected to the advertising amount to $1000. The increased revenue for the next year, assumed for simplicity to be earned at the end of that first year, is $700. The assumed additional revenue in the second year, again assumed to be all earned at the end of that year is $600. After two years the effect of the advertising will have worn off—students have short memories! If the school normally earns 5% per annum on its investments, determine whether the advertising campaign was worthwhile. To answer this question, calculate the IRR for the investment and see if this exceeds the school's normal, risk free, annual effective return of 5%.

Solution to Example 1

Using (4), we need to solve

$$-1000+\frac{700}{1+i}+\frac{600}{\left(1+i\right)^2}=0$$

where i is the annual effective rate of return. To solve this, we make the substi-

Table 1. Calculating the rate of return and links to senior mathematics syllabus.

Mathematics Syllabus Topic	IRR Calculation
Solving linear equations	Initial investment at time 0 followed by a (larger) cash receipt exactly one year later.
Equations reducible to quadratics by substitution	Initial investment at time 0 followed by cash receipts after one year and after two years. The sum of the cash receipts should exceed the initial investment in order to ensure a positive solution for the IRR.
Polynomials	Initial investment at time 0 followed by cash receipts after one year, two years and three years.
Problem solving ("Guess-check-refine") or for solving with a graphics calculator	An initial investment followed by a series of cash receipts at a range of times.
Exponentials and logarithms	Calculation of the continuously compounded rate of return requires the use of exponentials and logarithms. Consider an initial investment followed by a single (larger) cash receipt. The continuously compounded return can be calculated using exponential and logarithmic functions.
Integration involving exponential functions	Suppose an initial investment of \$1000 will yield a continuous stream[4] of funds payable for t years at the rate of \$$r$ per annum. The continuous rate of return can be calculated as the solution, in δ, of $1000 = r\int_0^t e^{-\delta s}\,ds$ Other rates of cash receipt, possibly dependent on time t could be used as a way of applying other integration techniques such as integration by parts.

tution, $u = 1 + i$ to obtain

$$-1000 + \frac{700}{u} + \frac{600}{u^2} = 0 \Rightarrow 10u^2 - 7u - 6 = 0$$

Using the "quadratic formula", the solutions for u are

$$u = \frac{7 \pm \sqrt{49 + 240}}{20}$$

or $u = 1.2$ or -0.5. Disregarding the negative solution, and substituting back for $u = 1 + i$, we obtain an annual effective rate of return of 20%. This far exceeds the 5% that the school can earn from its risk free investments. So, assuming that expansion of the school's canteen sales is agreeable to the school community (healthy foods, etc.!), this is certainly a viable advertising campaign.

Note that investments involving cash flows at more than three times (only three separate cash flows were considered in the quadratic equation example above), will produce IRR equations of the form (4), which, if payments are at

4. In reality we can only receive cash flows at discrete times rather than continuously. In practice, however it is often convenient to model cash flows that occur very regularly as continuous cash flows. An example is the toll receipts for cars using a busy tollway. The funds are being received every minute and possibly even more frequently. Such cash receipts could accurately be modelled as a continuous cash flow.

positive integer times, polynomials of higher degree than 2. Polynomials of degrees 3 and 4 are treated in some depth in the New South Wales and Victorian senior mathematics syllabi, amongst others (e.g., Board of Studies, 2000; VCAA, 2005). Often however the solutions of such equations can be difficult to locate and for higher order polynomials the difficulties are compounded. Such equations offer a suitable application of either the Goal-Seek function (or the Solver function) in Excel or the solve function on a CAS calculator, such as the TI-NSpire. (For details, see for example http://mathbits.com/MathBits/TInSection/Opennpage.htm.)

Example 2: Integration involving exponential functions

Suppose that an initial investment of $1000 produces returns at rate $300 per annum, continuously, for five years. Find the IRR related to this investment.

Solution to Example 2

Students will need guidance in relation to the concept of a continuous cash flow in this example. "$300 per annum, continuously" can usefully be thought about using a bathtub analogy. Suppose the tap is turned on at a constant rate of flow. Instead of water, the tap yields money! After one year of continuous and constant flow of money, the bathtub will contain $300 worth of money.

Using a continuous analogue of (7), we need to solve

$$-1000 + 300\int_0^5 e^{-\delta t}\,dt = 0$$

where δ is the continuously compounding IRR. Note here that the integral replaces the summation of individual terms in (7). Integrals, as discussed in the calculation of areas under graphs of functions bounded by the co-ordinate axes in Year 11 and 12 calculus, are the continuous analogue of summation.

Solving, we have

$$-1000 = \frac{300}{\delta}\left[e^{-\delta t}\right]_0^5 \text{ or } 1000 = \frac{300}{\delta}\left(1 - e^{-5\delta}\right)$$

The solution to this equation in δ can be obtained using the solve function on a CAS calculator or the Goal-Seek (or Solver) function in Excel. We find, using Excel, a continuously compounding rate of return of 17.48%, correct to two decimal places. This is a very high IRR and, based on our calculations and assumptions, we would support going ahead with this investment.

Multiple IRRs and conditions on cash flows to ensure a unique IRR

One well-documented problem with the IRR calculation is that it can, in some situations, lead to multiple solutions—that is more than one possible rate of return can be calculated. This should not come as a complete surprise. Table

1 shows that IRR calculations can be found by solving quadratic and higher order polynomial equations. Such equations are well known to have multiple solutions and indeed these multiple solutions can even be complex rather than real-valued.

A well-known example of multiple IRRs occurs when you have an initial investment of \$10 (at time 0), followed by a return of \$21 after one year (at time 1) and a final payment of \$11 after two years (at time 2). Such payment patterns are common in mining projects where there are initial exploratory expenses and final clean-up related expenses. We set up our IRR equation as follows

$$-10+\frac{21}{1+i}-\frac{11}{\left(1+i\right)^{2}}=0 \tag{8}$$

When we solve (8) for the IRR, i, we obtain internal rates of return of 0 and 10%! Which one is correct? The answer is that both are correct. This situation however does create real problems for financial decision makers who are interested in finding the IRR rather than a set of IRRs.

To see the intuition behind the two IRRs possible from this example, consider the difference between earning 0% and earning a positive return on investment in this example. The cash flow at time zero, \$10 in our example, has present value of \$10 regardless of the interest rate earned. The present value of the positive, \$21 in our example, cash flow at time 1 is reduced when interest rates are positive compared to when interest rates are zero. This is clear by calculating the present values at 0% and at some positive investment rate using (6). Consider now the negative cash flow, –\$11 in our example, at time 2. This will have present value of –11 if interest rates are zero. However, if interest rates are positive, the present value of this negative \$11 cash flow at time 2 will be less negative. So positive interest rates, compared to interest rates of zero, reduce the present value of positive cash flows and increase the present value of negative cash flows. In our example, when the interest rate is 10%, these two effects on the cash flows at times 1 and 2 exactly offset each other. Hence the two possible IRRs.

It is of interest then to determine which patterns of cash flows (payments and revenues) will ensure that there is a unique IRR. One such result, first proven, to our knowledge, by Norstrom (1972) is given below.

Result

If the cumulative cash flows under an investment change sign only once, then there is a unique positive solution for the IRR.

Proof

Norstrom's proof involved a fairly complicated argument which ultimately establishes a contradiction if there are multiple IRRs when the cumulative cash flows change sign only once. In addition his proof only applied in the case of discrete cash flows. Here we provide a new and alternative proof for this result which is accessible to senior students of mathematics. We assume that cash flows are payable continuously.

Denote the net rate of cash flow at time t under the project as $c(t)$. Note that $c(t) = r(t) - p(t)$ where $r(t)$ and $p(t)$ are the continuous rates of cash receipts and cash payments respectively under the project at time t.

Denote the accumulated value (without interest) of the net cash flows received up to time t as $f(t)$. It may help to think of $f(t)$ as the amount of money held at time t if all receipts are kept in a secure box and all payments are made from this box. Mathematically,

$$f(t) = \int_0^t c(r)dr \tag{9}$$

To help understand (9), note that an integral is a continuous version of summation. We see then that $f(t)$ is simply the total of all cash receipts minus all cash payments up to time t.

The IRR (expressed as a continuously compounded rate of return), is the solution, in δ, to the equation

$$\int_0^\infty c(t)e^{-\delta t}\,dt = 0 \tag{10}$$

To see that (10) is true, we write $c(t) = r(t) - p(t)$. Then (10) becomes

$$\int_0^\infty r(t)e^{-\delta t}\,dt - \int_0^\infty p(t)e^{-\delta t}\,dt = 0$$

Note that $r(t)e^{-\delta t}$ is the present value of a cash receipt of $r(t)$ at time t, using (6). So in (10), we are equating the total present value of receipts to the total present value of payments. The value(s) of δ that solve this equation is/are the IRR(s).

Equivalently our IRR is the solution, in δ, to

$$\int_0^\infty \frac{d}{dt}\big(f(t)\big)e^{-\delta t}\,dt = 0 \tag{11}$$

We now define

$$F(\delta) = \int_0^\infty f(t)e^{-\delta t}dt$$

Using integration by parts, taught for example in the NSW Year 12 Mathematics Extension 2 course (Board of Studies, 2000) or a well-known property of Laplace transforms, we see that

$$\int_0^\infty \frac{d}{dt}\big(f(t)\big)e^{-\delta t}\,dt = \delta F(\delta) - f(0) = \delta F(\delta) \tag{12}$$

where the final equality holds since the accumulated cash flow prior to any cash receipts or payments, $f(0)$ is 0. Using Equation (12), we see that the solution(s) to the IRR equation are the same as the solution(s) to $F(\delta) = 0$.

We now consider the function $F(\delta)$. As the continuously compounding rate of interest, δ, tends towards zero, we have

$$\lim_{\delta \to 0} F(\delta) = \lim_{\delta \to 0} \int_0^{\infty} f(t) e^{-\delta t} dt \tag{13}$$

which is clearly positive for investments where ultimately cash receipts exceed cash payments.

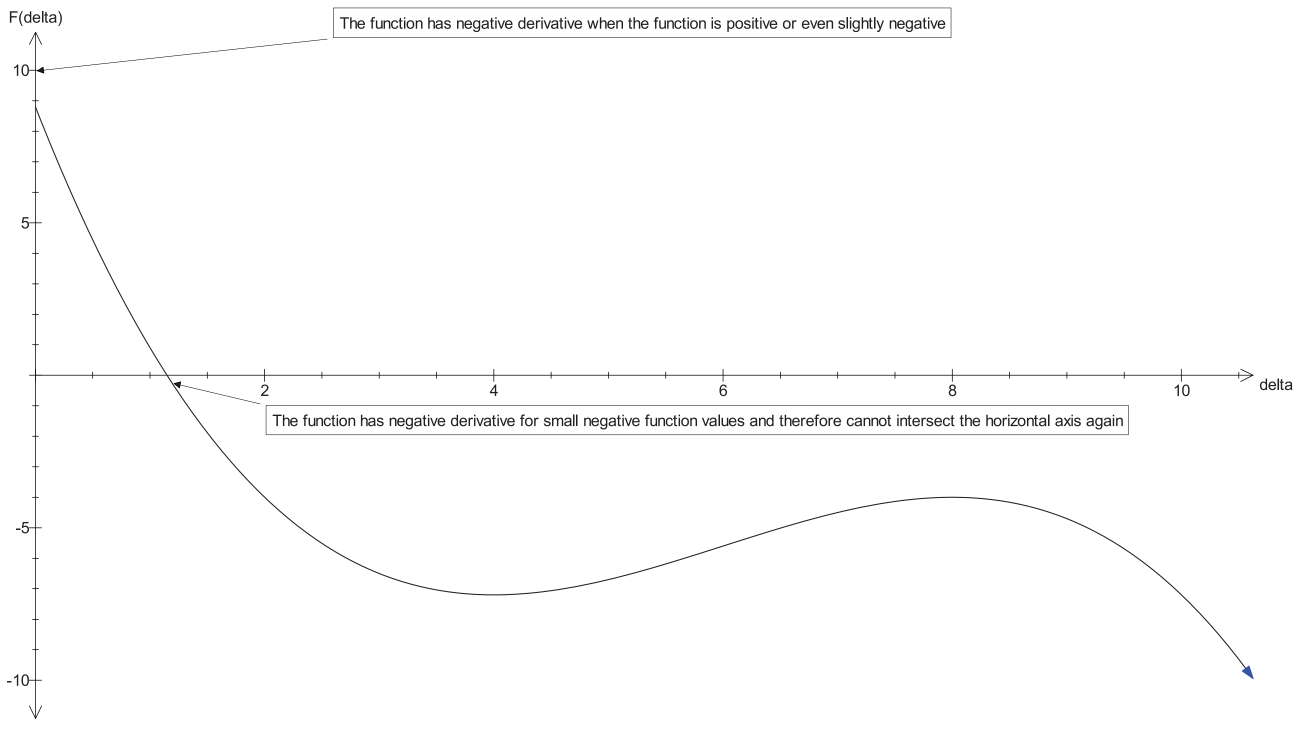
We next find the first derivative of $F(\delta)$ with respect to δ. Assuming (10) is a convergent integral, which would be the case for the very vast majority of practical applications of IRR calculations, we obtain

$$F'(\delta) = -\int_0^{\infty} t f(t) e^{-\delta t} dt \tag{14}$$

It is clear then that if $F(\delta) \geq 0$ then $F'(\delta) < 0$. In fact, for $F(\delta) = -\varepsilon$ for ε sufficiently small, it will also be the case that $F'(\delta) < 0$. This is true since if the integral represented by $F(\delta)$ is multiplied by a linear and negative factor, $-t$, it will become positive if it were originally negative by a sufficiently small amount under the condition that $f(t)$ changes sign, from negative to positive, only once.

So the function $F(\delta)$ is positive as δ tends to zero. As δ increases, the function values reduce until they pass through the horizontal axis. For small negative values of the function $F(\delta)$, the function is still decreasing. Since the function $F(\delta)$ is continuous, and for small negative values of the function the function is a decreasing function in δ, it is therefore impossible for the function to pass through the horizontal axis at a value of δ greater than the first positive value for which it passes through this axis. An example of how $F(\delta)$ could appear is given in Figure 1. Hence there is a unique positive solution to the IRR.

Figure 1. The IRR equation can only have one positive root.



Example 3: Illustrating the application of the result proven in this section

We consider two investments opportunities. Both opportunities require an initial investment of \$10. This initial investment occurs at what we will call time 0. The first opportunity promises a positive cash payment of \$31 at time 1 and a subsequent investment at time 2 of \$22. The second opportunity promises a positive cash payment of \$21 at time 1 and a subsequent investment at time 2 of \$10. Note that for this example time is measured in years. The cash flows under each of the two opportunities and the cumulative cash flows are shown below in Table 2.

Table 2. Cash flows and cumulative cash flows for two investment opportunities.

Time (years)	**0**	**1**	**2**
Investment one cash flow	–10	31	–22
Investment one cumulative cash flow	–10	21	–1
Investment two cash flow	–10	21	–10
Investment two cumulative cash flow	–10	11	1

From Table 2 it is clear that for investment one the cumulative cash flows, calculated as the running total of the actual investment cash flows, change sign twice. For investment two, the cumulative cash flows only change sign, from negative to positive, once. From the result proven in this section, investment two will have a unique positive IRR. Investment one, also as an application of the result proven in this section however, *may* have multiple IRRs.

In order to find the IRR(s) under investment opportunities 1 and 2, we use (7). We plot the value of the left hand side of (7) as a function of the continuously compounding interest rate δ. The IRR(s) occur when the left hand side of (7) is zero. Graphically, this corresponds to a horizontal axis intercept in the graph shown in Figure 2.

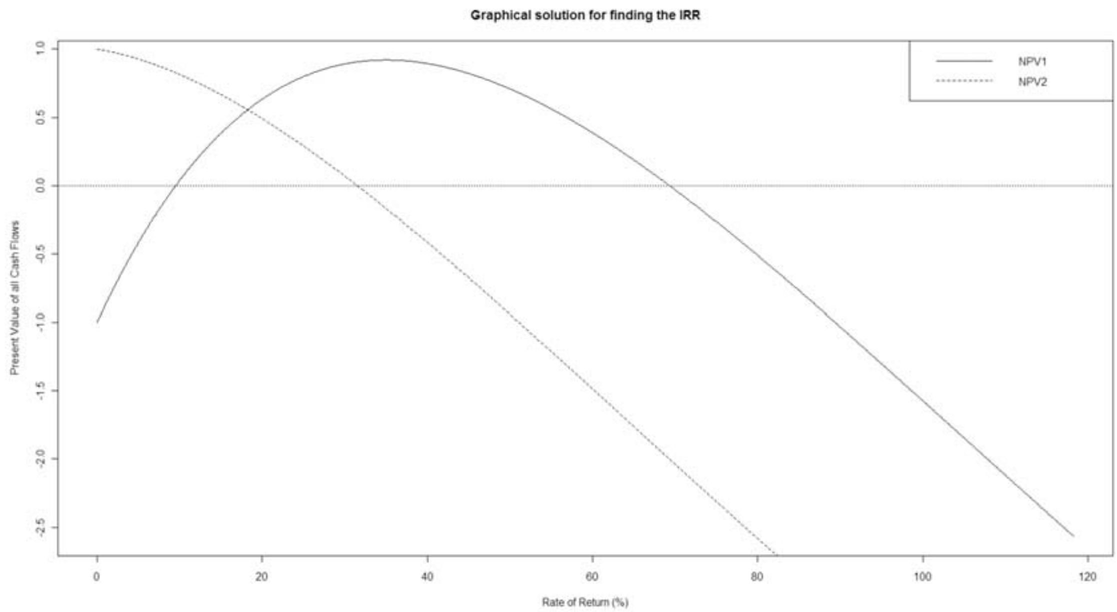


Figure 2. Graphical solution for finding the internal rate of return.

We see from Figure 2 that the first investment opportunity, indicated by the line marked NPV1, has two IRRs between 0 and 120%. The second investment opportunity only has a single positive IRR over this range, consistent with the result proven in this section.

This example illustrates the graphical approach to solving equations. Throughout the paper we have illustrated how IRRs can be found by solving equations by hand, for example linear or quadratic type equations, by graphical methods as in Figure 2 or by the use of technology such as a CAS calculator or Excel. It is valuable for students to understand that all three methods of solving equations: analytical, graphical and numerical (technology) will yield the same results. The IRR calculations provide a useful application for demonstrating these important ideas that go a long way towards unifying the senior mathematics syllabi.

Conclusion

The process for calculating rates of return on investments is generally only taught to undergraduates in business and actuarial science. This paper aims to show how these calculations can be conveniently woven into the mathematics education of senior school level students. In addition to this we have provided a new proof to a result about uniqueness of IRRs. Again the proof of this result has been written so as to be within the grasp of an enthusiastic and able student of senior school mathematics. Examples throughout illustrate the methods for solving IRR equations: analytical approach, graphical approach and numerical approach through technology. The selection of an appropriate method is determined by the complexity of the problem. Importantly the business application presented here will help to illustrate to students the equivalence of different methods of solving equations while at the same time introducing them to some applications of mathematics to the financial world.

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