Finding the general term for an arithmetic progression: *Alternatives to the formula*

Joseph B. W. Yeo
Nanyang Technological University, Singapore
<josephbw.yeo@nie.edu.sg>

Secondary school students in Singapore are expected to find an expression for the general or nth term of an arithmetic progression (AP) *without* using the AP formula $T_n = a + (n - 1)d$, where a is the first term, n is the number of terms and d is the common difference between successive terms. This type of question is worth only one mark in the GCE O-Level Examination but the usual method taught in school is far too long and many students feel that it is not worth trying so hard for one mark. This led me, when I was a teacher, to look for other shorter methods. In this article, I will discuss five methods to find an expression for the nth term of an AP, including arguably the fastest method that I have invented. I believe teachers will not only find the "fastest" method useful for their students but they can also challenge their students to be innovative and creative: find as many methods as possible to find the general term of an AP.

Method 1: Usual method taught in school

Let us consider the AP: 1, 4, 7, 10, … The usual method taught in Singapore secondary schools is to find a pattern by rewriting the terms as follow (actually, this is the gist of the derivation of the AP formula):

$$\begin{array}{rll}
1 & = 1 & = 1 + 0 \times 3 \\
4 & = 1 + 3 & = 1 + 1 \times 3 \\
7 & = 1 + 3 + 3 & = 1 + 2 \times 3 \\
10 & = 1 + 3 + 3 + 3 & = 1 + 3 \times 3 \\
 & & \vdots \\
T_n & & = 1 + (n - 1) \times 3
\end{array}$$

Most students will not see the need to express 1 as $1 + 0 \times 3$, or $4 = 1 + 3$ as $1 + 1 \times 3$, so the trick is to start by simplifying $7 = 1 + 3 + 3$ as $1 + 2 \times 3$, and $10 = 1 + 3 + 3 + 3$ as $1 + 3 \times 3$, and then use this pattern to guide students to express the first two terms in the same way. The difficulty lies

in finding an expression for the nth term: some students think that $T_n = 1 + n \times 3$. Teachers are probably familiar with using the following method to guide students to see why T_n should be $1 + (n - 1) \times 3$:

$$\begin{array}{llll}
n & T_n & & \\
1 & 1 & = 1 & = 1 + 0 \times 3 \\
2 & 4 & = 1 + 3 & = 1 + 1 \times 3 \\
3 & 7 & = 1 + 3 + 3 & = 1 + 2 \times 3 \\
4 & 10 & = 1 + 3 + 3 + 3 & = 1 + 3 \times 3 \\
 & & \vdots & \\
n & & T_n & = 1 + (n - 1) \times 3
\end{array}$$

Some of my students still found this difficult and concluded wrongly that $T_n = 1 + n \times 3$. I have even seen students who argued that $T_n = 1 + (n + 1) \times 3$ as follows:

$$\begin{array}{llll}
n & T_n & & \\
1 & 1 & = 1 & = 1 + 0 \times 3 \\
2 & 4 & = 1 + 3 & = 1 + 1 \times 3 \\
3 & 7 & = 1 + 3 + 3 & = 1 + 2 \times 3 \\
4 & 10 & = 1 + 3 + 3 + 3 & = 1 + 3 \times 3 \\
 & & \vdots & \\
n & & T_n & = 1 + (n + 1) \times 3
\end{array}$$

Therefore, this method is not only long but it poses a problem to some students.

Method 2: A variation of Method 1

My first attempt to solve the above problem creates another problem. But it is worthwhile discussing it here because it is related to the "fastest" method (see Method 4 later):

$$\begin{array}{rll}
1 & = -2 + 3 & = -2 + 1 \times 3 \\
4 & = -2 + 3 + 3 & = -2 + 2 \times 3 \\
7 & = -2 + 3 + 3 + 3 & = -2 + 3 \times 3 \\
10 & = -2 + 3 + 3 + 3 + 3 & = -2 + 4 \times 3 \\
 & & \vdots \\
T_n & & = -2 + n \times 3
\end{array}$$

Most students are able to see that the pattern for T_n is $n \times 3$, and not $(n - 1) \times 3$ or $(n + 1) \times 3$. But it creates another problem. Why must you express 1 as $-2 + 3$ and where does -2 come from? Although the teacher can explain how to get -2 from $1 - 3$ (i.e., first term a minus common difference d), most students will not remember what they do not understand (there is actually a basis for this which will be discussed in Method 4). Moreover, this method is still too long to be worth only one mark in the examination.

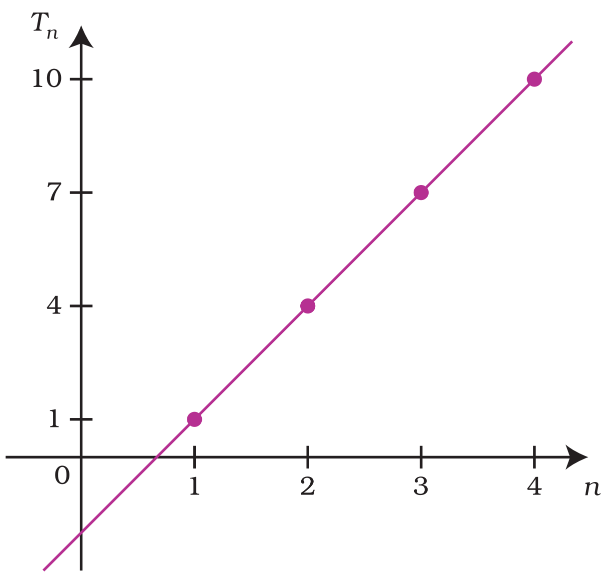
Method 3: Number line

My next attempt to find a shorter method involves the use of a number line:

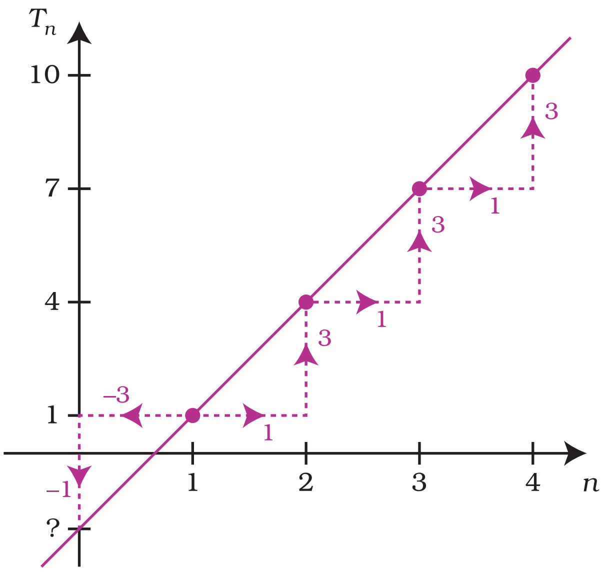
Guide the students to observe that there are two gaps between the first and the third term, three gaps between the first and the fourth term, etc. The teacher can also use the analogy of four lampposts on a straight road and three gaps between the four lampposts. Then there are $n - 1$ gaps between the first and the nth term. Since the length of each gap is 3 units and the first term is 1, then the nth term is $1 + (n - 1) \times 3$. Compare this with Method 1 and you may see the similarity between the two methods: the difference is that Method 1 is algebraic while Method 3 is the pictorial version. In fact, Method 3 is how you can interpret the AP formula $T_n = a + (n - 1)d$ from a more visual perspective. Although my students prefer Method 3 to Method 1, some of them still have problem obtaining the $n - 1$ gaps. Moreover, it is still too long for just one mark.

Method 4: Arguably the fastest method

The concept for Method 4 will take quite a while to explain but once students have understood it, the shortcut will be faster than using the AP formula. Just like Method 3, this method also makes use of a straight line:



The equation of the straight line is $T_n = mn + c$ (in the form of $y = mx + c$). The gradient m can be found using the concept of “rise over run”. From the graph below, the run is always 1 and the rise is always the common difference d. So $m = d = 3$.



The T_n-intercept is c. From the graph above, using the concept of the gradient, $c = 1 - 3 = -2$. Just like in Method 2, most students will have a problem remembering that c is equal to the first term a minus the common difference d. But there is another way to look at c: it is the value of T_n when $n = 0$, i.e., $c = T_0$. I will call this the zero-th term — what else to call it? Most students know how to find a missing term in an AP, whether it is the next or the previous term. So the missing term T_0 is just $1 - 3 = -2$.

$$\underset{\xleftarrow{-3}}{?\quad 1,}\quad \underset{\xrightarrow{+3}}{\quad 4,}\quad \underset{\xrightarrow{+3}}{\quad 7,}\quad \underset{\xrightarrow{+3}}{\quad 10,}\quad \ldots$$

If students have understood from the graph why c is equal to the zero-th term, then they will remember it more easily than memorising the formula $c = a - d$ without understanding. Compare this with Method 2 and you may see the similarity, especially $c = 1 - 3 = -2$. However, there are two main differences: the first is that Method 2 is more algebraic while Method 4 is more pictorial; the second is that students can see more clearly the reason why there is a need to find -2 in Method 4, which is not so obvious in Method 2.

To illustrate how the shortcut for this method works, I will use another AP: 3, 7, 11, 15, … Since the common difference is 4, then $T_n = 4n + ?$ The zero-th term T_0 is $3 - 4 = -1$. So $T_n = 4n - 1$. If you use the AP formula, then $T_n = 3 + (n - 1) \times 4 = 3 + 4n - 4 = 4n - 1$. But simplifying $3 + (n - 1) \times 4$ will usually take a longer time than finding the zero-th term which can be found easily using mental calculation. I will leave the readers with two APs to try using Method 4 to find an expression for the nth term, and to decide for themselves whether it is faster than using the AP formula: 2, 5, 8, 11, … and 1, 5, 9, 13, …

The problem with Method 4 is that students need to know the concept of graphs (if you teach the shortcut without the concept, most students will not remember it). But in Singapore, students learn number patterns (including AP but without the formulae) in Secondary One before they learn graphs. Moreover, they only learn the concept of gradient in Secondary One but the concept of the y-intercept only in Secondary Two. So there is a need for teachers to revisit number patterns after they have taught graphs. In the meantime, are there other shorter methods that are suitable for students who have not learnt graphs?

What about primary school students going for Maths Olympiad? One

way is to teach the formula without understanding. Another way is to use Method 3 which only some primary school students can understand (Methods 1 and 2 are too algebraic for most of them). So there is a need to develop a method that is suitable for primary school students going for Maths Olympiad, and for secondary school students who have not learnt about graphs.

Method 5: Transform to another sequence

This method involves transforming the AP to another sequence which is easier to find the general term. Since the common difference of the above AP (1, 4, 7, 10, …) is 3, the ideal choice of another sequence is the multiples of 3 (starting from 3) because each term of the AP must be transformed by the same amount and the general term for the multiples of 3 can be found easily:

$$\begin{array}{cccccc} 1, & 4, & 7, & 10, & \ldots & 3n-2 \\ {\scriptstyle +2}\big\downarrow & {\scriptstyle +2}\big\downarrow & {\scriptstyle +2}\big\downarrow & {\scriptstyle +2}\big\downarrow & & \big\uparrow{\scriptstyle -2} \\ 3, & 6, & 9, & 12, & & 3n \end{array}$$

Then you can obtain the general term of the original AP by transforming $3n$ back by the same amount to get $T_n = 3n - 2$. The main difficulty for the average primary school students is deducing that the general term for the multiples of 3 is $3n$ because they are usually weak in algebra and the concept of the nth term probably eludes them. But high-ability primary school students, who need this for their Maths Olympiad, have found this method useful. Furthermore, secondary school students, who have not learnt graphs, generally have no difficulty with using this method which is easier to apply than Methods 1 to 3.

Conclusion

There is usually more than one way to approach a problem. Teachers can get their students to find as many methods as possible to solve the problem. But we should not stop there. We can go one step further and ask our students to evaluate which method(s) is/are more efficient or easier (more efficient does not mean it is easier for the students), or more elegant (whatever that may mean to the readers). This article outlines five methods to find the general term of an AP. Some methods (e.g., Methods 1 and 2) are too long and troublesome but they can be useful, e.g., Method 1 is the precursor for the algebraic derivation of the AP formula. Some methods (e.g., Methods 3 and 5) are more visual and they are usually easier for students to understand than algebraic methods (e.g. Methods 1 and 2). Method 4 is the most efficient but the concept is not easy for primary school students to understand. Which methods, do you think, are more elegant? I would think Methods 4 and 5 are more elegant although readers may have a different opinion.