LEANING ON SOCRATES TO DERIVE THE PYTHAGOREAN THEOREM

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"Knowledge needs to be presented to, or accessed by, students through a variety of means, enabling them to construct the knowledge and make sense of it, and then transform it" (Perso, 2007).

Introduction

The one theorem just about every student remembers from school is the theorem about the side lengths of a right angled triangle which Euclid attributed to Pythagoras when writing Proposition 47 of *The Elements*. Usually first met in middle school,[1] the student will be continually exposed throughout their mathematical education to the formula $b^2 + c^2 = a^2$, where a is the length of the hypotenuse. It is used to determine the length of a triangle's sides, to measure the distance between points in the plane, in trigonometry, and in tertiary study as the Euclidean metric in n-dimensional space. Possibly the second most familiar equation, that of a circle $x^2 + y^2 = r^2$, is another incarnation of the Pythagorean theorem. The ratio of diameter to circumference was also of particular interest to the students of Pythagoras.

1. Level 6, Years 9 and 10, *Victorian Essential Learning Standards.*

It is so important in school mathematics and in mathematical thinking that the student deserves to be able to derive the Pythagorean theorem with an appropriate degree of rigour. Our aim, in this article, is to provide a repertoire of derivations that range from the visual and geometrical to the algebraic and, in doing so, expose the interconnectedness of many parts of the school curriculum. We begin, following the advice of Cavanagh (2008), by "allowing students to explore concrete examples". The simple case for a right isosceles triangle is easily seen to be true by construction. The construction can then generalised to any right-angled triangle. The student performs "dissections and recombinations of shapes", (de Mestre, 2003; Coad, 2006) requiring no algebra or non-geometric manipulation.

The simple visual demonstration, used initially, can be revisited for two rigorous geometrical proofs, one reinforcing the concepts of supplementary and complementary angles and the other providing a useful application for the properties of angles of a traversal of parallel lines. Further developing the geometry, at this stage, leads to a geometric proof of the algebraic squaring of sums formula, $(b + c)^2 = b^2 + 2bc + c^2$. We conclude by using the "squaring of sums" formula to give the simple and common ancient Hindu proof of the Pythagorean theorem favoured by Gough (2001).

The history and methods of proof of the Pythagorean theorem are, to this day, of great interest to the mathematical mind. A new aspect of the story that we bring here is a demonstration of the proof of the theorem using a hinged wooden toy, that can be used as a "prop" to dramatically transform two squares of area b^2 and c^2 to form a square of area a^2.

A Socratic proof

Let us begin with a little more history to enrich the learning environment. In *Meno*, one of Plato's famous "Socratic dialogues" (Hamilton & Cairns, 1961), Socrates leads a slave boy to construct a square with double the area of a given square. Three other squares of the same size are placed together with the first and each is divided along a diagonal into two isosceles triangles. Two of the isosceles triangles make up the original square and four make up the shaded square of double the area (Figure 1).

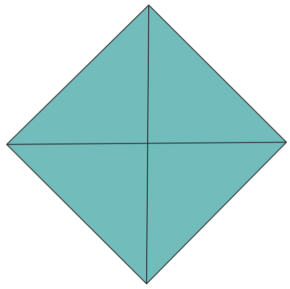

Figure 1

It is mathematical folklore that Socrates adapted his original demonstration of doubling the area of a square into a proof of the Pythagorean Theorem for the case of a right isosceles triangle. Any proof of this particular instance of the Pythagorean Theorem using eight isosceles triangles has loosely been called the "Socratic proof". The rearrangement may be done in many different ways, so there is no unique Socratic proof. Heath (1956, p. 352) and Loomis (1940, p. 102) name several authors of this style of proof.

The version that we give below can be constructed as a wooden "toy" used to demonstrate the derivation. Similar hinged "toys", such as the Dudeney dissection, transforming an equilateral triangle into a square, are charming in their appeal to all ages (de Mestre 2003, Frederickson 2008). The student can perform the proof for themselves using two paper squares cut into four isosceles triangles. Draw a line on a piece of paper and arrange the squares so they have a diagonal on the line and they form two sides of triangle ABC which can be drawn on the piece of paper.

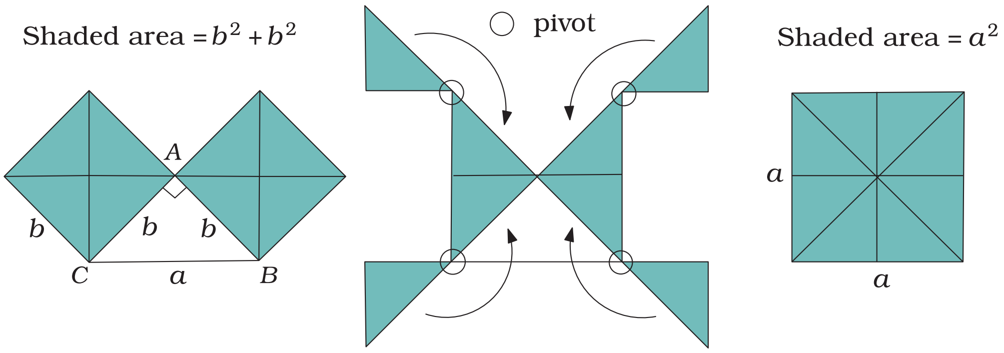

Figure 2

A leaning Socratic proof

The method of the Socratic proof can be generalised to design a wooden "toy" that will demonstrate the derivation for any right angled triangle, not necessarily isosceles. This general construction is, in a sense, the same as the Socratic proof with some vertical components pushed into a leaning, non-vertical orientation, so we call this dissection a "leaning Socratic proof". There are many other approaches to teaching the proof of the Pythagorean Theorem (for a selection, see Chambers, 1999). The advantage of a two-step approach is that the Socratic proof is simple; an accessible starting point which, when leaned on, leads directly to the general proof. Interest is stimulated by a dramatic practical model. It is not necessary in the first instance to understand the "proof without words" (*a la* Nelsen, 1993; 2000) given in the diagram. However, the "toy" can be revisited to revise the concepts of complementary and supplementary angles and their application in the proof. The fact that α is the angle between two pairs of parallel lines is the only other geometry needed.

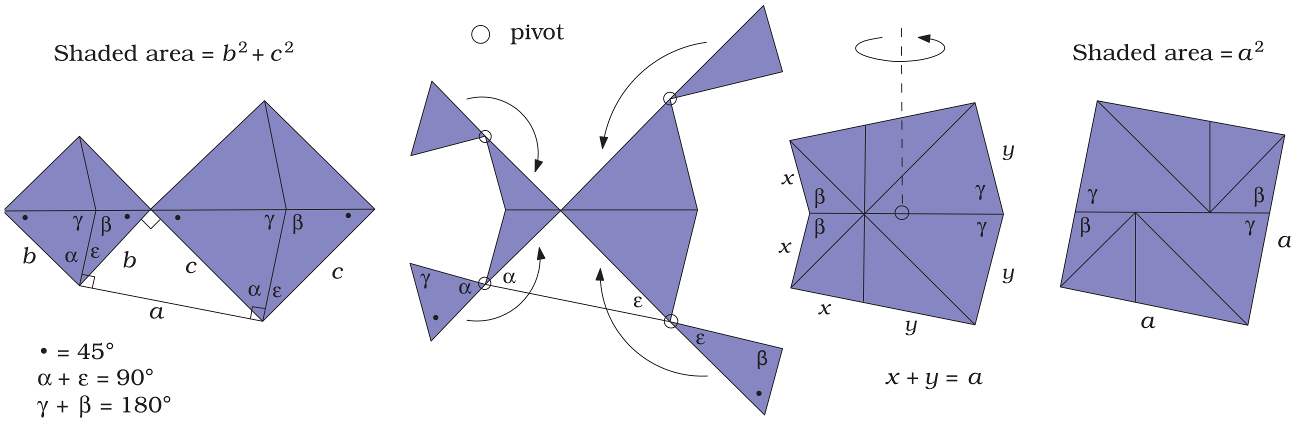

Figure 3

Unlike the Socratic dissection, two of the "cuts" are not along a diagonal, yet they retain the property of being at right angles to the hypotenuse. The hypotenuse 'leans' away from horizontal and so the cuts lean away from vertical when compared to the Socratic dissection. Similar dissections are recorded by Loomis (1940, p. 112, p. 114) but the dissected pieces are not rearranged the same way and are not suitable for a pivoting demonstration aide.

Rotating the triangular pieces as before gives a six-sided polygon. Turning over, or swivelling, the "top half" of that polygon, gives a square of side length a, which completes the demonstration. Socrates might also have swivelled the 'top half ' of his square but we would not notice any difference on his diagram.

Another proof with a twist

In the "leaning Socratic proof ", the "outer" squares were dissected, the triangles rotated and finally, the upper half of the construction was rotated. Rotating the upper half first, allows two copies of the original right triangle to be dissected and these can be positioned to make up a square of side length *a* (also see Nelson, 2000, p. 6). The indicated proof provides a useful application of the congruent angles created when a traversal crosses parallel lines. This elegant visual derivation of the general theorem is more suitable than the "leaning Socratic proof" for a Year 9 student to perform with paper. Unfortunately a swivelling wooden toy demonstrating this proof would fall apart!

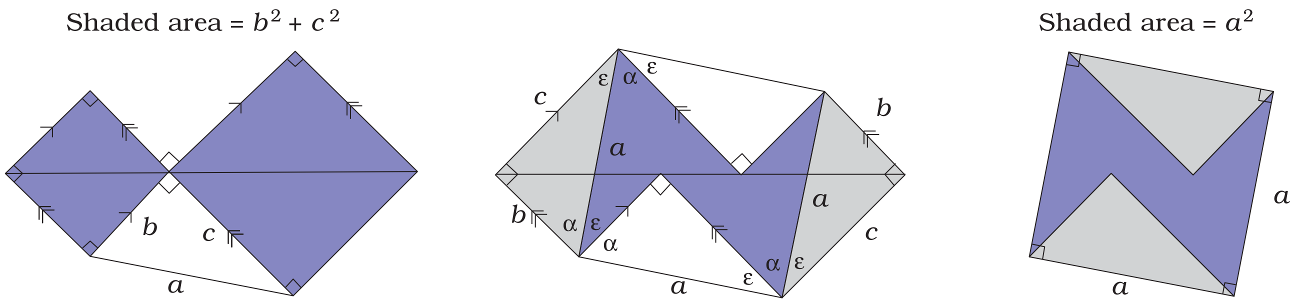

Figure 4

Geometry to algebra and back

When looking at the "proof with a twist" it could be worthwhile to take a short detour and see some interplay between geometry and algebra. Taking the shaded section from Figure 4, which has an area of $b^2 + c^2$ and adding two rectangles of area bc we get a square of area $(b + c)^2$ demonstrating the algebraic "square of sums" formula $(b + c)^2 = b^2 + 2bc + c^2$ (Figure 5).

The formula $(b + c)^2 = b^2 + 2bc + c^2$ can be rearranged to give $(b + c)^2 - 2bc = b^2 + c^2$ which we can see from the Hindu diagram to the right (Figure 6) leaves the inner square having the area $b^2 + c^2 = a^2$ since the outer square, taking away the four triangles, has area $(b + c)^2 - 2bc$. Despite its simplicity, this is an indirect proof since it involves the area $(b + c)^2$ which is not a part of the Pythagorean formula.

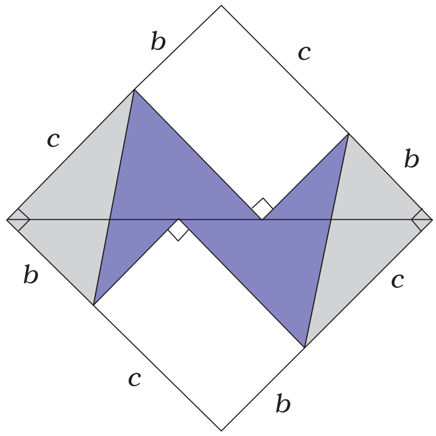

Figure 5

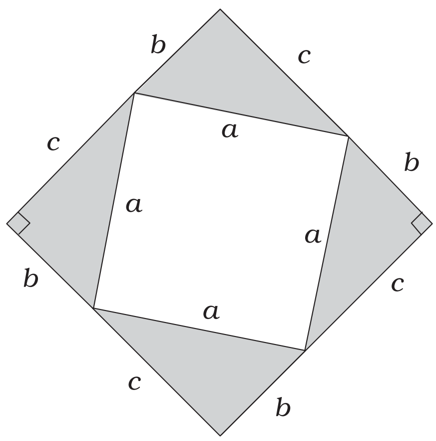

Figure 6

Conclusion

The Pythagorean theorem is a foundation stone of mathematics. It is possibly the simplest important mathematical theorem and as such it deserves to be understood by all students of mathematics. We have suggested a sequence to make the derivation of this theorem attainable, beginning with a demonstration of the isosceles case leading to an arbitrary case which is performed by dissecting and repositioning two squares of area b^2 and c^2 to form a square of area a^2. This demonstration can be abstracted as a general geometrical proof of the theorem using either the "leaning Socratic proof" or the similar "proof with a twist".

The interplay between the algebraic formula $b^2 + c^2 = a^2$ and geometric areas of shapes can be pushed further to show that $(b + c)^2 = b^2 + 2bc + c^2$ which we can then use to wrap up the exploration with a favoured proof of the central concept: the Pythagorean theorem.

By following the suggested sequence, students will have constructed the knowledge, made sense of it and transformed it; geometry and algebra interplaying through a common theme. At the very least, students will have experienced it happening!

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From Helen Prochazka's Scrapbook

Prefixes for SI Units

Prefix	Symbol	Factor
yotta	Y	10^{24}
zetta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deka	da	10^{1}

Prefix	Symbol	Factor
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}