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Structuring Numbers 1 to 20: Developing Facile Addition and Subtraction

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The Numeracy Intervention Research Project (NIRP) aims to develop assessment and instructional tools for use with low-attaining 3rd- and 4th-graders. The NIRP approach to instruction in addition and subtraction in the range 1 to 20 is described. The approach is based on a notion of structuring numbers, which draws on the work of Freudenthal and the Realistic Mathematics Education program. NIRP involved 25 teachers and 300 students, 200 of whom participated in an intervention program of approximately thirty 25-minute lessons over 10 weeks. Data is drawn from case studies of two intervention students who made significant progress toward facile addition and subtraction. Pre- and post-assessment interviews and five lesson episodes are described, and data drawn from the activity of the students during the episodes are analysed. The discussion develops a detailed account of the progression of students' learning of structuring numbers, and how this can result in significant level-raising of students' arithmetical knowledge as it becomes more formalised and less context-dependent.

In early addition and subtraction in the range 1 to 20, students can progress from using strategies involving counting by ones to using more facile strategies that do not involve counting. Researchers recognise this progression to facile addition and subtraction as critical mathematical learning, yet many low-attaining students do not make the progression successfully. There is a pressing need to understand how low-attaining students can progress to facile addition and subtraction, and to design instruction that facilitates such progress.

As part of a design research project investigating intervention in number learning in 3rd and 4th grade, we have been developing instruction in addition and subtraction based on Dutch approaches to *structuring numbers* (Freudenthal, 1991). This article comprises one iteration in our design cycle, as we analyse student learning in the context of our experimental intervention instruction. The purpose of this paper is to formulate students' development toward facile addition and subtraction as an activity of structuring numbers. We aim to articulate the activity of structuring numbers, and how it can result in significant advancement in students' arithmetical knowledge. Such an analysis can in turn inform our refinement of the instructional design.

In this article we first review research on early addition and subtraction, and the need for intervention. We then present the notion of structuring numbers, and our structuring numbers approach to instruction, drawing on the work of Freudenthal and his successors, which serve as the theoretical framework for our analysis of students' learning. We then describe the larger research project from which the data presented in this article are drawn. Learning episodes from case studies of two students in intensive intervention, who made significant progress toward facile addition and subtraction, are presented. In the data analysis and discussion, we formulate the students' activity during the episodes as structuring numbers.

Background

Facile Addition and Subtraction

Young children's learning of addition and subtraction was the subject of considerable research in the 1980s and early 1990s (e.g., Carpenter & Moser, 1984; Fuson, 1988; Resnick, 1983; Riley, Greeno, & Heller, 1983; Steffe & Cobb, 1988). A broad consensus picture emerged of a progression of key developments in children's numerical thinking, as summarised in Fuson's research review (Fuson, 1992). In early learning of numbers, children use strategies involving counting by ones, and will rely on visible objects to count. Later, children can count visualised objects, fingers, and their own recited counting words (Steffe & Cobb, 1988). An example of a relatively sophisticated use of counting is when a child solves 6 + ? = 13 by counting on from 6 to 13, while keeping track of the seven counts using fingers. Children make a qualitative change in number thinking when they can solve additive tasks without counting by ones (Fuson, 1992; Riley et al., 1983; Steffe & Cobb, 1988). The task 6 + ? = 13 might be solved as "6 (makes 12) and 1 more—7". From research literature characterising this more facile number thinking, we identify four significant aspects.

First, conceptual analysis reveals that to use the strategy just described, the child must regard both the 6 and the missing addend as units, and simultaneously conceive of their sum 13 as a unit. In contrast to children who use counting by ones, in this solution none of the numbers need to be counted out to have meaning. This can be described as a part-whole construction of number (Resnick, 1983; Hunting, 2003; Young-Loveridge, 2002) — the ability to partition a whole number into number parts. Such part-whole thinking indicates a construction of the number sequence as a "bidirectional chain" (Fuson, 1992); or as an Explicitly Nested Number Sequence (Steffe & Cobb, 1988). This thinking also constitutes a more formal construction of the operations of addition and subtraction—the child can begin to use addition and subtraction as inverses (Steffe & Cobb, 1988).

Second, facile additive thinking involves solving tasks without counting by ones. Thus, facile students use a range of informal non-counting strategies, such as near doubles (as in the example above), adding through 10, and compensation (Thompson, 1995; Thornton, 1978; van de Walle, 2004), and they can use these skilfully.

Third, the non-counting strategies require the child to have automated knowledge of some number combinations such as double 6 is 12. Informal non-counting strategies commonly build on knowledge of doubles, combinations with 5 and with 10 (such as 5 and 3, 10 and 6), and the partitions of 10 (1 and 9, 2 and 8, 3 and 7, 4 and 6, and 5 and 5). These combinations are generally the most familiar to children, and probably arise from reflection on finger patterns (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Treffers, 1991).

Fourth, using non-counting strategies requires relating different number combinations to each other. In the example discussed above, the task 6 + ? = 13 has been related to the known combination 6 + 6 = 12. Noticing and using such relationships requires a form of number sense or numerical reasoning (Threlfall, 2002)

In summary, facility with addition and subtraction can be described in terms of four interrelated aspects of children's developing number knowledge: (a) the part-whole construction of number; (b) the use of non-counting additive strategies; (c) automated knowledge of key number combinations and partitions; and (d) a relational number sense, relating unknown combinations to known combinations.

Intervention Instruction for Facile Addition and Subtraction

Development from counting strategies to facile non-counting strategies for addition and subtraction in the range 1 to 20 is regarded as an important accomplishment of early childhood mathematics (Resnick, 1983; Wright, 1994; Wright, Martland, Stafford, & Stanger, 2006; Young-Loveridge, 2002). As well as facilitating calculation in the range 1 to 20, the non-counting strategies and part-whole thinking are required to calculate in higher decades (Heirdsfield, 2001; Treffers, 1991), and to understand multiplication and fractions (Olive, 2001; Resnick, 1983). Further, relational thinking and knowledge of number combinations are important aspects of number sense (Bobis, 1996; McIntosh, Reys, & Reys, 1992; Treffers, 1991). In short, facility in adding and subtracting without counting is a critical goal in achieving children's numeracy.

Some students do not achieve this facility. Instead, they persist with strategies involving counting by ones for addition and subtraction in the range 1 to 20, and in turn use counting strategies in the higher decades. Persistent counting is characteristic of students who are low-attaining in number learning (Denvir & Brown, 1986; Gervasoni, Hadden, & Turkenburg, 2007; Gray, 1991; Treffers, 1991; Wright, Ellemor-Collins, & Lewis, 2007). Low-attaining 3rd and 4th grade students might typically solve the subtraction task 17 − 15, for example, by counting back 15 counts from 17. They often show little knowledge of number combinations, for example, finding 8 + 8 by counting rather than using a known doubles fact. Further, they typically do not relate unknown number combinations to known combinations: for example, knowing that 6 + 6 is 12, but finding 6 + 7 by counting. Such persistent counting strategies result in inefficiency and error (Ellemor-Collins, Wright, & Lewis, 2007), and disable further generalisation of arithmetic strategies. Persistent counting is a mathematical dead end (Gray, 1991).

Numeracy is a principal goal of mathematics education (The national numeracy project, 1998; Australian Government, 2008; Principles and standards for school mathematics, 2000). Hence, there are calls for intervention in the learning of low-attaining students to enhance numeracy outcomes (Bryant, Bryant, & Hammill, 2000; *Mapping the territory*, 2000; Pearn, 1998; Rivera, 1998). For example, the recent National Numeracy Review in Australia recommended increased resources for intervention for students at risk, particularly in the early years of schooling, with a focus on "enabling every student to develop the in-depth conceptual knowledge needed to become a proficient and sustained learner and user of mathematics" (Australian Government, 2008, p. xiii). If we are to develop numeracy intervention, there is a pressing need to understand how low-attaining students can progress from strategies based on counting to facile addition and subtraction, and to design instructional procedures which support this learning.

This need motivates our design research work, which includes an aim to study low-attaining students' progress with addition and subtraction, and to design intervention instruction for facile addition and subtraction. In designing instruction, we have drawn on the Realistic Mathematics Education (RME) approach to calculation up to 20 (Treffers, 2001), which involves a notion of structuring numbers. Our design goal is to further develop the structuring numbers approach of RME as an instructional approach applicable to intensive intervention. This paper contributes to that design goal by pursuing a detailed analysis of how students' learning in the context of the instructional approach can be formulated as an activity of structuring numbers. We describe below the notion of structuring numbers, and the structuring numbers approach to instruction, which together serve as the theoretical framework for our analysis of students' learning.

Theoretical Framework

Structuring Numbers

Our use of the term *structuring* is informed by Freudenthal and his successors. Freudenthal recognised that doing mathematics consists, in part, of organising phenomena into increasingly formal or abstract structures (e.g. Freudenthal, 1991, pp.11, 15; Treffers, 1987, p.59). He proposed that students learn mathematics, in part, by doing this organising, which he often termed structuring. "By structuring rather than forming concepts we get a grip on reality" (1991, p. 26). He used structuring as a relatively general term, meaning "emphasising form" (p. 10). *Structuring numbers*, in turn, means organising numbers more formally: establishing regularities in numbers, relating numbers to other numbers, and constructing symmetries and patterns in numbers. For example, consider a student adding 5 and 8 who first makes 10 from 5 + 5 and then uses a known fact that 10 and 3 more is 13. The student is structuring the numbers around 10 as a reference point: organising the numbers and the operation by realising that two fives make 10, and by using the formal decimal regularities of teen numbers to add 3 to 10.

Additive structuring of whole numbers

In his *Didactical Phenomenology of Mathematical Structures* (1983), Freudenthal laid out in some detail the sorts of phenomena students might try to organise and the structures that are valuable for students to develop. In discussing the learning of the natural numbers, he introduced the additive structure of the natural numbers to be "as it were, the whole complex of relations a + b = c" (p. 104) and then gives the following as an example.

> $a + b = c$ can be structured by prescribing c and asking for the totality of solutions (a, b), the list of splittings
> 8 = {8 7 6 5 4 3 2 1 0
> +{0 1 2 3 4 5 6 7 8
> which exhibits a striking structure of increasing and decreasing sequences and a central symmetry. Of course, splittings are also useful for the algorithm of passing over the tens when adding, but there is more to it. (p. 105)

There are also many other ways of structuring the numbers, into doubles, multiples, sequences, and so on. Treating numbers as commutative or associative, or using the equivalence of a + b = c and c − b = a (p. 105) also involves structuring. Freudenthal emphasised structuring by bundling into tens, or decimalising, as critical to learning numbers (p. 90). Thus, structuring numbers involves developing a coherent, richly networked knowledge which organises number combinations, relations, and operations.

Level-raising

Structuring has an important quality of level-raising, of vertical reorganisation. In the example above from Freudenthal (1983), we structure a + b = c by finding the list of splittings. But then we structure the list of splittings by recognising sequence and symmetry. On a larger scale, "the relation between addition and subtraction arises as a matter of content before it is formally applied, in order to become once again subject matter and content in the context of algebraic structures" (Freudenthal, 1991, p.12). Each structure becomes content to be organised by new structures, in "a never ending cyclic process" (p. 10). Treffers emphasises ever-progressing level-raising as "essential for mathematical activity" (Treffers, 1987, p. 53). Such recursive level-raising is familiar in many characterisations of doing mathematics, for example Sfard's reification (1991), and Pirie and Kieren's folding back (1994).

Mental object versus concept attainment.

In our view it is important to distinguish structuring from learning about structures. Structuring is an activity that begins with content, experienced as realistic or common sense, and organises it into more formal structures. On the other hand, formal structures can simply be imitated: "schemes of thought can be imposed, algorithms can be taught as rigidly as computers are programmed" (Freudenthal, 1991, p. 11). Freudenthal was concerned that the latter is a superficial, impoverished and problematic approach to teaching mathematics. He was adamant that learning mathematics consists of an active interplay of content and form—structuring content into form, which in turn becomes content at a new level—it cannot consist of imitating structured form alone (e.g., pp. 11, 27). To help draw attention to this issue, he made an important distinction between constituting mental objects and attaining concepts (p. 18). For example, we can define the concept of whole number, perhaps using set theory, and try to teach this concept, which Freudenthal suggests was the approach of New Math in the 1960s (Moon, 1986). Such an approach equates learning with concept attainment. However, for a learner, whole number is generated through structuring the process of counting, rather than through learning a succinct definition. When first generated, whole number becomes a mental object, "a matter of common sense rather than a concept" (Freudenthal, 1991, p. 19). Recognising this, we ought not teach superficially for concept attainment. Rather we should teach to support students' constitution of mental objects. To clarify the distinction between a concept and a mental object, we could say that a genuine learning of a concept arises as a reification of a mental object. For example, a student first constitutes whole number as a mental object, and only later—as an adult even—might reflect on this object, to make sense of the *concept* of whole number. The distinction between *mental object* and concept is observed in the history of mathematics too: "the mental object of group preceded the *group concept* by about half a century" (Freudenthal, 1991, p. 19).

In our approach to the development of students' facility in addition and subtraction, our approach is not to aim for students to know what the structure of numbers is; rather, we aim for students' mental object of number to be constituted with structure—we want the students to structure their numbers. For example, the first aim is not for a student to be able to answer the question "What are the partitions of 10?" Rather, it is for the student, when faced with 7 and 3, to know "there are 10 altogether", and again later when faced with 4 and 6; that is, an aspect of the students' construction of the numbers 1 to 9 is as parts of 10. The aim is for the student to structure the numbers, but they need not be aware of how their understanding has changed, and may not know the term "partitions of 10" yet. Similarly, we want the student to constitute the mental operation of facile addition, rather than to learn about addition.

Instructional Approach for Structuring Numbers

Our approach to instruction for facile addition and subtraction, based on structuring numbers, was developed from RME instructional design documented particularly in the work of Treffers (1991; 2001) and Gravemeijer (1994; 2000). The aim is to develop an instructional approach to automatising addition and subtraction to 20, which could connect to students' informal methods involving doubles, fives and 10s and could "make these arithmetic methods accessible to weaker pupils who persist in counting" (Treffers, 1991, p. 38). In accord with RME principles (Treffers & Beishuizen, 1999), learning and instruction are envisioned as a process of *progressive mathematisation*, in which students structure and invent their own informal, context-bound strategies, and the attentive teacher guides the activity of the student toward more formal, efficient structuring. The learning-and-teaching trajectory moves from calculation by counting, through calculation by structuring, to formal calculation and automatisation (Treffers, 2001). Structuring numbers is the bridge from counting-based calculation to formal, facile calculation.

Treffers (2001) observed that whole numbers can be structured, broadly, according to two models: a line or ordinal model, such as establishing predecessor and successor numbers in the number sequence and recognising 10 as a reference point in the sequence; and a group or cardinal model, such as grouping in doubles or fives. It is worth noting the attention to 5 as an important grouping in students' structuring, which has also been observed by others (e.g. Sugarman, 1997). The instruction aims to encourage students' structuring as a combination of line and group models.

An *emergent modelling* (Gravemeijer et al., 2000; Verschaffel, Greer, & Torbeyns, 2006) heuristic for instructional design proposes that, from a students' informal activity, a model of the initial task emerges. This *model of*, at least if favourably constructed, can develop into a *model for* more formal mathematical reasoning, as students shift their attention from the initial task to their own structuring activity. Drawing on the emergent modelling heuristic, we use instructional settings in which students can first structure context-bound models of combinations—such as identifying 5 red and 3 green dots as 8 dots—and then, reflect on their activity and generalise toward more formal reasoning about numbers—such as partitioning 8 into 5 + 3 to solve the written task 8 − 3 without counting. We use the ten-frame setting for the range 1–10 (Bobis, 1996; Treffers, 1991; Young-Loveridge, 2002), and the arithmetic rack for the range 1-20 (Gravemeijer et al., 2000; Treffers, 1991) (see Instructional Settings below), settings which suggest a combination line-group structuring. Wright and colleagues (2006) have developed instructional procedures with these settings specifically for one-on-one interventions. The instructional procedures for structuring numbers 1 to 20 used in the present study are described in the Method section below.

Structured number environment

Our instructional design does not have the purpose of teaching isolated skills or knowledge of basic facts. Rather, the design aims to develop structuring of numbers up to 20 into a relational framework or structured number environment. Gravemeijer (1994) acknowledges the inspiration of Van Hiele's suggestion that learning number requires a shift in broad levels of thinking, from a ground level where numbers are tied to observable quantities and physical actions, to a first level where relations between numbers are established and a relational framework is being constructed. For example, "on the first level, [four] is a junction in a relational framework. It might be two plus two, or two times two, or possibly five minus one" (Van Hiele, 1973, cited in Gravemeijer, 1994). The aim of the structuring numbers instructional sequence of Gravemeijer and colleagues was for students to "come to act in a quantitative environment structured by relationships between numbers up to 20" (2000), drawing on Greeno's (1991) notion of number sense as knowing one's way around an environment. Gravemeijer et al. (2000) explain that the aim is not to dispense with developing skills, but "instead of talking about skills as consisting of automated sub-skills, we would prefer to speak of skilled activity" (p. 244). Thus, the structuring numbers approach could be described as developing both the structured knowledge of number that supports facile calculation and, at the same time, the structuring activity that constitutes facile calculation.

Mathematical structure and using materials

It is important to clarify that mathematical structures are not considered inherent in the content of a task. Until a student has structured the content, the structure is not there for the student; a student will only act with the structures they have organised (Cobb, 1991; Gravemeijer, 1991). When the arithmetic rack is first put in front of a student, we ought not pretend that we have offered the student a doubles structure or a fives structure. As Gravemeijer asserts, "For the pupil who does not yet have this mathematical knowledge, there is nothing to see!" (Gravemeijer, 1991, p. 65). Those structures will need to arise in the child's activity with the rack. In the same way, when a student is doing arithmetic with bare numbers, we must not pretend that doubles and fives structures are simply there "in the numbers" somehow. The student needs to structure the numbers. Similarly, when we speak of modelling, the arithmetic rack itself is not the model; rather "a model as we have characterized it originates from students' ways of acting and reasoning in the starting-point situations" (Gravemeijer et al., 2000, p. 242). As Cobb (1991) has emphasised, maintaining this clarity requires vigilance in distinguishing the perspective of the student from the perspective of the adult observer.

Method

The Numeracy Intervention Research Project

The study reported here is part of a larger study, the Numeracy Intervention Research Project (NIRP). NIRP has the aim of developing assessment and instructional tools for intervention in the number learning of low-attaining 3rd- and 4th-graders (Wright et al., 2007). The NIRP adopted a methodology based on design research (Cobb, 2003).

There were three one-year design cycles for the NIRP. Each design cycle consisted of (a) initial development of the pedagogical tools, (b) use of the tools in an intervention program with teachers and students, (c) analysis of the learning and teaching in the program, and (d) refinement of the tools based on the analysis. Within each cycle, analysis and development were ongoing, in meetings of the researchers and project teachers, in analysis of assessments, and in teachers' daily lesson planning. The analysis of the learning and teaching in the intervention program was informed by a teaching experiment methodology (Steffe & Thompson, 2000).

The intervention program for each year involved eight or nine teachers, each from a different school, across the state of Victoria. In each school, 12 students were identified as low-attaining in arithmetic, based on screening tests administered to all 3rd and 4th graders. In each school (a) in Term 2, these 12 students were assessed in individual interviews; (b) in Term 3, eight of the low-attaining students participated in intervention teaching cycles;

and (c) in Term 4, the 12 students were again assessed in individual interviews. The teaching cycles lasted 10 weeks and involved approximately 30 teaching sessions of 25 minutes' duration. Two students were taught as singletons and six as trios. All interview assessments were videotaped, as were all of the instructional sessions with singletons. This process provided an extensive empirical base for analysis.

In total, the project involved professional development of 25 teachers, pre- and post-assessments of 300 low-attaining students, and intervention with 200 of those students. The assessment and instruction addressed several key aspects of number knowledge, including number word and numeral sequences, structuring numbers to 20, addition and subtraction in the range 1 to 100, conceptual place value, and multiplication and division (Wright et al., 2007). Each lesson typically addressed three or four of these key aspects.

Instructional Sequence: Structuring numbers 1 to 20

This report focuses on two case studies of students' progressive learning of structuring numbers 1 to 20. The purpose of this section is to provide an overview of the instructional program planned for the two students whose cases are described. The overview takes the form of an intended instructional sequence (Gravemeijer et al., 2000) consisting of 10 instructional topics. For each topic, the instruction as implemented was informed by a detailed and elaborated description, which included exemplars of instruction, explanations of purpose and descriptions of likely student responses to instructional tasks (Wright & Ellemor-Collins, 2008; Wright, Martland, Stafford, & Stanger, 2006; Wright, Stanger, Stafford, & Martland, 2006). As well, inherent in each topic and across the topics is a sequential approach to instruction involving progressive mathematisation (as described earlier). Below is a brief description of each topic.

1. Patterning and partitioning. This very preliminary topic is bypassed with many of the intervention students. The topic is included here because it is integral to the instructional sequence. The topic involves presenting students with standard spatial configurations, typically for numbers in the range 1 to 6 (such as the patterns on dice, dominos, or playing cards). Instruction focuses on students reasoning numerically about combining and partitioning numbers: for example, 5 is made up of two 2s and 1; 4 and 2 make 6.

2. Five-wise patterns for numbers 1 to 10. This topic focuses on ascribing number to five-wise patterns, typically presented on a ten-frame (see below) or using fingers. The pattern for 8, for example, has a row of five dots and a row of three, and the pattern for 4 has all dots in one row.

3. Pair-wise patterns for numbers 1 to 10. In similar vein, this topic focuses on ascribing number to pair-wise patterns, typically presented on a ten-frame (see below) or using fingers. The pattern for 8, for example, has two rows of four dots, and the pattern for 5 has a row of three and a row of two.

4. Complements to 10. This topic focuses on automatising the complements to 10, that is, the partitions of 10 (1 + 9, 2 + 8, etc.), initially using a setting of ten-frame complements cards (see below) or fingers, and progressing to bare numbers.

5. Adding two numbers with a sum in the range 1 to 10. This topic focuses on adding two numbers, with an extension task asking how many more to make 10. For example, 4 and 2 make 6, and 4 more make 10. The topic involves initially using a setting of ten-frame addition cards (see below) and progressing to bare numbers.

6. Doubles from 6 and 6 to 10 and 10. This topic focuses on automatising the doubles (6 + 6, 7 + 7, etc.) and involves initially using an appropriate setting such as an arithmetic rack.

7. Ten-wise patterns for numbers 1 to 20. This topic focuses on ascribing number to ten-wise patterns, typically presented on an arithmetic rack. The pattern for 8, for example, has five blue and three red on one row and the pattern for 17 has a row of ten beads and a row of seven beads.

8. Pair-wise patterns for numbers 1 to 20. In similar vein, this topic focuses on ascribing number to pair-wise patterns, typically presented on an arithmetic rack. The pattern for 12, for example, has two rows of six beads, and the pattern for 17 has one row of nine beads and one of eight beads.

9. Adding two numbers in the range 1 to 10. This topic focuses on adding two numbers, with an extension task asking how many more to make 20. For example, 8 and 5 make 13, and 7 more make 20. The topic involves initially using a setting such as an arithmetic rack and progressing to bare numbers.

10. Addition and subtraction in the range 1 to 20. Extends Topic 9 to sums where the sum and one addend are in the range 11 to 20. Also includes subtraction, including cases where the minuend and either the subtrahend or the difference is in the range 11 to 20.

Instructional Settings

Ten-frames 1–10. A 2 X 5 frame with a standard configuration of dots for a number in the range 1 to 10, either pair-wise (such as four dots on each row) or five-wise (five and three).

Ten-frame complements cards. Ten-frames with 10 dots of two colours (in the combinations: 9 and 1, 8 and 2, 7 and 3, 6 and 4, and 5 and 5), configured pair-wise and five-wise.

Ten-frame addition cards. The 25 frames having 1 to 5 red dots on one row and 1 to 5 green dots on the other.

Arithmetic rack. Two rods, each with five red and five blue beads. As on a counting frame, beads can be moved to one end of the rods to present certain configurations, such as 5-and-1 on upper and 5-and-1 on lower; or 10 on upper and 2 on lower.

Expression cards. Two addends in the range 0 to 10, in horizontal format (such as 2+7). The set of expression cards includes all 121 such expressions.

Data and Data Analysis

Robyn and Nate1 were selected as cases because they each made significant progress toward facile addition and subtraction in the range 1 to 20 (see also Ellemor-Collins & Wright, 2008a, 2008b). Robyn was nine years old and in the 4th grade when she participated in the study. Her intervention teacher was Ms Parkin. Nate was eight years old and in the 3rd grade when he participated in the study. His intervention teacher was Ms Moss. Their pre-assessment interviews were conducted in May, their post-assessments in October. Robyn's intervention comprised 29 individual lessons over 10 weeks from July to October. Nate's intervention comprised 24 individual lessons over 10 weeks from July to September.

For each student, we summarise their responses to addition and subtraction tasks in the range 1 to 20 in their pre- and post-assessment interviews to indicate their progress from counting strategies to non-counting strategies. We describe three episodes from Robyn's intervention lessons, and two episodes from Nate's lessons, and in each we formulate their learning in terms of structuring numbers.

The Case of Robyn

Robyn: Pre- and post- assessments

In her pre-assessment, Robyn did not have automated knowledge of complements to 10, or of double 7, 8 or 9; she attempted these tasks using counting by ones. Five one-digit tasks were presented in written horizontal format: 6 + 5, 7 + 6, 9 + 3, 9 + 6, and 8 + 7. She solved all by counting on by ones, the last task incorrectly. Three further written tasks in the range 1 to 20 were 13 + 3, 11 + 8, and 17 − 15. She solved the first two by counting on by ones, keeping track on her fingers. On the last task she made three attempts to count 15 counts back from 17 but did not establish an answer, explaining her difficulty that she "can't go past zero".

In her post-assessment, Robyn had automated knowledge of complements to 10 and of double 5 up to double 10. She solved the same one-digit written tasks (6 + 5, 7 + 6, 9 + 3, 9 + 6, and 8 + 7) using the non-counting strategies of near-doubles and compensation. She was not asked 13 + 3, 11 + 8, or 17 − 15. As a comparison, she successfully solved four two-digit tasks without counting by ones, including 43 + 21, 37 + 19, 86 − 24, and 50 − 27. In summary, in the range 1–20, Robyn showed increased fluency with combinations and partitions, a shift from counting to non-counting strategies, and increased facility and success with calculation.

[1] Anonyms have been used for the names of the students and teachers.

Robyn—Episode R1: Complements to 10 in a ten-frame setting

In her initial interview, Robyn solved complements to 10 tasks using counting by ones. In the first three weeks of intervention, Ms Parkin gave attention to this aspect of number knowledge. Drawing on instructional topic 4, she posed tasks involving partitioning 10 and finding complements to 10, using finger patterns and in ten-frame settings. Episode R1 is from Lesson 7, Week 2. Ms Parkin flashed cards from a set of blue-black ten-frame complements cards, with dots arranged 5-wise. Robyn's task was to say how many blue dots and how many black dots she saw.

Card 9-and-1 flashed.

Robyn: Nine blues and one black.

Ms Parkin: (Shows the card.) How did you work that out?

Robyn: Umm, I saw there was one black, so then obviously…

Ms Parkin: Excellent strategy.

Card 3-and-7 flashed.

Robyn: Three blues and…six blacks.

Ms Parkin: (Shows card.)

Robyn: (Looks at card) Oh, seven blacks.

Ms Parkin: Yep. 'Cause three and seven go together, don't they. What does six go with?

Robyn: Four.

Ms Parkin: Yes.

Card 8-and-2 flashed.

Robyn: Um. Eight blues and ni-, er, and two blacks.

Ms Parkin: (Showing card.) Is that right?

Robyn: Yep.

Ms Parkin: How did you check it?

Robyn: Umm…like that (covers end of card with palm to leave 2 X 3 rectangle of blue dots.) With six and then (uncovering end of card) two more. [This replicates a way of seeing a 5-wise eight which Ms Parkin had shown earlier in the lesson.]

Ms Parkin: Yep. How else?

Robyn: Umm…two, four…like that. Umm…ones.

Ms Parkin:	Ones. Mrs. D wouldn't want you to count by ones (shaking her head and smiling. Robyn laughs.) What else could you do?
Robyn:	Umm…(looks at card for 8 seconds) Go by fours.
Ms Parkin:	Mmm.
Robyn:	Like (indicating 2 X 2 square of blue dots) and those four (indicating remaining L-shape of four blue dots). Um, you could do that there's two there (placing two fingers beside the two black dots), two blacks, and there's obviously eight (briefly tracing around the rest of the ten-frame with a finger).
Ms Parkin:	Cool, I like that one. One more.
Robyn:	Umm…
Ms Parkin:	Use the shape of the tens-frame to help you (finger-tracing a rectangle around the card).
Robyn:	See that there's five up the top (pointing to upper row)…
Ms Parkin:	(Nods)
Robyn:	And then three down the bottom (pointing to lower row)…and then… (points to two blacks).
Ms Parkin:	Yay…That's what I wanted.

When the 9-and-1 card was flashed, Robyn identified nine dots as the complement of one dot. When the 3-and-7 card was flashed, Robyn needed to think for longer, she seemed first to identify the three blue dots, and then to need to work out the complement of black dots. That is, she could not identify both number patterns from one flashed viewing, and nor did she know that 7 was the complement of 3. In trying to work out the complement of 3, she answered incorrectly. When Ms Parkin showed the 3-and-7 card for Robyn to check her answer, Robyn could then readily identify the seven-dot pattern. She also knew, without a corresponding ten-frame, that 6 goes with 4. Thus, she indicated that she could read the dot pattern for a number and had developed some, but not all, tens-complement structures in the ten-frame setting. Robyn's uncertainty with the flashed 3-and-7 card persisted over the next five lessons.

In the discussion of the 8-and-2 card, Robyn gave five different ways of checking the 8-pattern without counting by ones. Each way required making a neat partition of the dots, and knowing 8 as composed of smaller numbers, such as 4 and 4, or 5 and 3. Thus, in the context of the ten-frames, Robyn was beginning to use a part-part-whole construction of numbers, and showing knowledge of some number combinations.

Robyn—Episode R2: Adding two numbers with a sum in the range 1 to 10 in a ten-frame setting

This episode is from Lesson 11, Week 3. Ms Parkin began Topic 5, using red and green ten-frame addition cards for the first time. Ms Parkin flashed each card, and Robyn's task was to say how many red dots, how many green dots, and how many altogether. She readily identified the numbers of red and green dots. The sums generally took a few seconds, and were correct. In explaining how she knew the answer for the 5-and-4 card was 9, Robyn said, "Because there's one more missing." The answer for the 3-and-4 card was 7, "because I know that six and six is three…er, three and three is six, and just add one more on." Later in the lesson, in reference to a 7-dot ten-frame, Robyn said "I know that's seven because that's three empty spaces."

Thus, when explaining her thinking to Ms Parkin in the context of the ten-frames, Robyn structured the numbers, using doubles and using 10. It is significant that, at this point, Robyn was finding number combinations in the range 1 to 10 using neither counting nor instant recall. We imagine she was using her increasingly facile knowledge of the dot patterns and number relationships, which she had structured meaningfully for herself. Further, explaining her number thinking to Ms Parkin in part-whole terms had become standard practice for Robyn, and made sense to her.

Robyn—Episode R3: Adding two numbers in the range 1 to 10 in a written setting

In Week 6, Ms Parkin introduced a set of expression cards with sums in the range 11 to 18, to address Topic 9. During the first lesson with the cards, Ms Parkin tried to introduce specific non-counting strategies: jump-though-10, and near-double. The following episode is from the second lesson with the cards, Lesson 17 in Week 6. They worked on 37 cards, and through this extended activity, Robyn seemed to move from incompetence to competence with non-counting strategies for these tasks.

For some tasks early in the episode, Robyn could not find a workable strategy. Ms Parkin's response was to make supportive comments. This is described for the tasks 6 + 5 and 9 + 6.

Card 6 + 5.

Robyn: Um. 6 + 5. Um, well I know that six plus five is…you know that there's five and add six more on, which is 12?

Ms Parkin: Hmmm. Some of the known facts that we know—do you know any doubles that are near that? (Points to the two numerals.)

Robyn: Ahh, 5 + 5.

Ms Parkin: 5 + 5—What's that?

Robyn: 10.

Ms Parkin: And then add on… (Pointing to the 6.)

Robyn: One more, makes 11.

Ms Parkin: Makes it 11? (Robyn nods.) Good. There's another double that you could use…

Card 9 + 6.

Robyn: 9 + 6. I know that…nine…plus nine is… (looking at the card for six seconds) oh wait, nine plus nine is

	18..and…take away... (looking for four seconds) one…no wait…
Ms Parkin:	Do you think that 9 + 9 double is the easiest way to do that one? (Pointing to the 9 and 6.)
Robyn:	No.
Ms Parkin:	What's another easier way? (After some discussion of the near-doubles strategy, Ms Parkin continues.) Could you use nine being near 10?
Robyn:	Yes.
Ms Parkin:	Is that the best fact?
Robyn:	Yep.
Ms Parkin:	So what could you do?
Robyn:	Umm, you know that nine is near 10, plus six more, then take away…which is 16, take away one is 15.
Ms Parkin:	Yep.

After Card 9 + 6, Robyn was able to recognise tasks amenable to the near-doubles or a close-to-10 strategy. She 'thought out loud', calculating the subsequent 10 tasks as follows: 9 + 2 as 10 + 2 − 1; 8 + 7 as 8 + 8 − 1; 8 + 9 as 8 + 8 + 1; 7 + 6 as 7 + 7 − 1; 6 + 7 as 6 + 6 + 1; 5 + 6 as 5 + 5 + 1; 4 + 9 as 10 + 4 − 1; 8 + 8 as known double; 5 + 9 as 10 + 5 − 1; 7 + 8 as 7 + 7 + 1. We describe the responses to the next task, 7 + *9*.

Card 7 + 9.

Robyn:	Seven plus nine…seven…seven plus seven is 14, add two is 16.
Ms Parkin:	Good. Is there another way you could do that one?
Robyn:	Umm, you know that nine is closest to 10, umm, and then plus seven is 17, then plus two…no, take away two…take away one.
Ms Parkin:	Leaving…
Robyn:	16.
Ms Parkin:	You're getting very good at expressing your ideas.

Ms Parkin posed 18 more tasks, all of which Robyn solved successfully and with increasing ease. For the final six tasks, Robyn did not vocalise the steps of her solutions. Rather, she quickly stated the answer, and then she explained her solution. This is described for tasks 4 + 8 and 5 + 8.

Card 4 + 8.

Robyn:	4 + 8 is…12.
Ms Parkin:	Quick. Four times tables?

Robyn: Yep.

Card 5 + 8.

Robyn: 5 + 8…12…13.

Ms Parkin: Why is it 13?

Robyn: I know that eight, eight to 10 is two, plus another three is 13.

In the early tasks, Robyn tried to use one of the procedures she had been shown—near-doubles—without a sense of the number relationships in the particular sum. After the 9 + 6 task, she paid attention to the number relationships, recognising a near-double or a near-10. Furthermore, in doing 9 + 2 and 4 + 9 using 10 as a reference, she did not use the adding-through-10 strategy that Ms Parkin had tried to teach. Rather, she spontaneously used compensation: 10 + 2, then take away 1; 10 + 4 then take away 1. We would argue that she was structuring the numbers to solve the task, rather than following procedures. Ms Parkin's suggested strategies involving doubles and 10 helped Robyn arrive at this structuring. For example, Ms Parkin's prompt for Card 9 + 6, "Could you use close-to-10?" suggests a structuring of the numbers, without specifying a strategy such as adding-through-10 or compensation. We think it was Ms Parkin's structuring of the numbers in her examples, rather than the rehearsing of a procedural set of steps, which was most helpful to Robyn's learning.

Initially, Robyn used near-doubles and compensation strategies, but later she also used jump-though-10 and times tables appropriately. We contend that Robyn's new fluency in structuring numbers into combinations and partitions underpinned her success. For example, she had automatised the doubles, so near-doubles became routine for her. In calculating 7 + 9 as 7 + 7+ 2, Robyn used 7 + 7, and recognised that 9 is 2 more than 7. For the alternative calculation of 7 + 9 as 10 + 7 − 1, she could re-organise the numbers and determine a different compensation. To connect 4 + 8 to four times tables, presumably she saw 8 as 4 and 4. For 5 + 8 as 8 + 2 + 3, she found the tens-complement of 8, and partitioned 5 into 2 and 3. In general, she structured tasks in terms of known combinations, and took her knowledge of combinations for granted as she made calculations and provided explanations. Overall, there was a sense of growing confidence with using her structured knowledge and non-counting strategies in a written setting. We see this as another critical step in Robyn's development of facile additive thinking.

The Case of Nate

Nate: Pre- and post-assessments

In his pre-assessment interview, Nate could say how many more to make 10 for 5, 9, and 7 fluently, and for 2 and 4 with four seconds of thinking for each. He found partitions of 7 using his fingers, but then found partitions of 6, 12, and 19 without fingers or counting. Asked the doubles 8 plus 8, 9 plus 9, and 7 plus 7, he thought for five seconds for each answer, but apparently did not use counting. For 9 plus 9 he answered "19". Five tasks were presented in written horizontal format. He solved 6 + 5 using a near-doubles strategy. 9 + 6, 8 + 7, and 11 + 8 were solved by counting-on by ones, using fingers to keep track of his counts. The subtraction 17 − 15 was solved counting back 15 counts from 17. A written test included 12 addition and subtraction tasks in the range 1 to 31, in horizontal format. Nate incorrectly answered four subtractions: 15 − 9 (9), 14 − 11 (2), 9 − 4 (3), and 23 − 17 (7). In summary, Nate knew some useful combinations in the range 1–20 such as complements to 10, doubles, and 10s-combinations, but still tended to use counting strategies for unknown calculations, and was error-prone to some extent.

In his post-assessment interview, Nate was more fluent on the complements to 10, partitions, and doubles tasks, and did not use counting on these. He made one error, initially stating 9 and 11 as two numbers to make 19 and then saying "18 and 1". Asked for another partition of 19 he answered "12 and 6". He answered the five tasks presented in written format quickly and successfully, using non-counting strategies. On the written test, he answered all 12 addition and subtraction tasks correctly. In summary, in the range 1–20, Nate showed increased fluency with combinations and partitions, a shift from counting to non-counting strategies, and increased facility and success with calculation.

Nate—Episode N1: Complements to 10 in a ten-frame setting

In the first two weeks of instruction, Ms Moss used the five-wise and pair-wise sets of ten-frame cards to present tasks from Topics 2 and 3, patterns for numbers 1 to 10. She would show each card from a set, and Nate's task was to name the number. Later, she changed to flashing each card. Nate was generally successful and facile on these tasks. In Lesson 5, Ms Moss introduced the ten-frame complements cards. Nate's task for each card was to say how many of each colour, for example, "six and four; two and eight." In Lesson 6, Week 2, Ms Moss flashed the pair-wise 1–10 set and the five-wise 1–10 set; then she extended the task, asking Nate to say for each card both the number of dots and how many more to make 10. Nate was successful on these tasks. He answered the first three tasks rapidly: "Five and five! Nine and one! Two and… two, eight," then commented with animation:

Nate:	Oh, isn't that like the…(tracing two lines and dots on the desk) two dots…
Ms Moss:	It's like my coloured ones, isn't it? (Indicating the blue and orange ten-frame complements cards.)
Nate:	Yeah.

Nate seemed to make an association between tasks involving cards partitioned in dots and blank squares, and tasks involving cards partitioned in two colours. This suggested a degree of fluency with the patterns in the ten-frame setting. After these tasks, Ms Moss showed and flashed the pair-wise ten-frame complements set, and flashed the five-wise ten-frame complements set twice. Nate answered without errors, and with increasing ease. We suggest that at this point he could see, for example, the pattern for 2 and the pattern for 8 as parts of a pattern that encompassed the partition of 10 into 2 and 8. Nate had developed significant knowledge of the complements to 10, that is, knowledge of an aspect of structuring the numbers.

Nate—Episode N2: Adding 9 and another number in the arithmetic rack setting

This episode is from Lesson 19, Week 6. Following an initial segment involving doubles and 10-plus tasks presented verbally (Topics 7 and 8), Ms Moss presented a set of 9-plus tasks, using the arithmetic rack (Topic 9). First, with 9 on the rack, Ms Moss asked "Nine and three", and moved 1 and 2 on the rack. Nate seemed to be confused. After four such tasks, they discussed why Ms Moss was making the addend in a 1-and-X form, and she presented three more tasks. Next, Ms Moss screened the rack. She called out the additions—"Nine and four more"— and after Nate answered, she lifted the screen for him to check. Nate was successful with these tasks and became more engaged, looking up from the screen and thinking hard. He was not using counting by ones, and it seems likely he was adding through 10 and perhaps visualising the rack in doing so. Finally, Ms Moss posed eight 9-plus tasks verbally. Nate was successful with these, generally answering within one second. Following these 9-plus activities, Ms Moss presented similar 8-plus tasks, first on the arithmetic rack and then verbally. Nate was successful on these tasks, apparently with less certitude than he showed with the 9-plus tasks. On the verbal task, "Eight and five more", Nate nodded his head three times before answering "13", then stated that he had counted by ones. This suggests that his use of non-counting strategies was not yet fully routine. Rather, using a non-counting strategy was, to some extent, engendered by the setting (that is, the arithmetic rack).

It seems that, in the beginning of the episode, when Ms Moss pre-arranged the beads (made the addends in a 1-and-X form), Nate had significant difficulty in regarding, for example, one bead on the upper row and four on the lower row as standing for the addend 5, and therefore he had difficulty organising his responses. However, after several tasks he overcame this difficulty and when Ms Moss began to screen the rack, Nate was successful on these tasks. A possible explanation is that he began to mentally structure the addends. The initial segment may have drawn his attention to the 9-and-1 structure, or the 1-and-X structure. It seems that he made sense of the tasks through becoming aware of these structures. When Ms Moss posed 9-plus tasks verbally, it appears that Nate had routinised his approach to these tasks. He seemed to be aware that, when a number in the range 2 to 9 is added to 9, the answer is a teen determined as one less than the number added. Further, this awareness no longer depended on an instructional context involving the rack. Nate was now reasoning formally about numbers rather than beads on a rack.

Discussion

We seek to better understand how structuring numbers might be realised in students' activity, and how their structuring of numbers contributes to their developing facility with addition and subtraction. Thus we have selected episodes where we have formulated the students' activity as structuring numbers. We review here these examples of structuring numbers. In doing so, we also note the level-raising character of structuring, as the students' structuring activity becomes less tied to the settings, and more generalised or formal.

Robyn's Structuring

Episode R1. Robyn could ascribe number to ten-frame dot patterns that had been flashed. Such an association of dot pattern with number is already a simple form of organising the numbers. Further, looking at the 8-and-2 card, Robyn was able to identify different partitions of the eight dots, suggesting a further structuring of numbers in terms of combinations: that 5 and 3 is eight, or that two out of the ten-frame "obviously" leaves eight. This partitioning activity was closely tied to the ten-frame setting: Robyn was describing how to organise the dots on the card, using hand gestures to indicate pairs, squares, the whole rectangle, and so on. Some of these combinations may have been available to Robyn when the ten-frame was no longer visible. For example, from a flashed 9-and-1 card, she named both numbers 9 and 1, and suggested that in seeing the one, "then obviously" the complement is nine.

Episode R2. Robyn's task shifted from identifying a number of dots, or a pair of numbers of dots, to an additive task, finding the total of two rows of dots. Her thinking as she describes it involved structuring the number patterns: five and four is nine because there's one missing from the ten-frame; three and four is found as one more than three and three. Thus, the structuring was at a more complex level than in episode R1, re-organising two numbers into a sum number. Also, the structuring occurred without the ten-frame visible: We suppose that Robyn was visualising patterns on the ten-frame card. We could describe her activity as modelling the earlier tasks of combining and partitioning the dots on a visible card. The task still in the context of the ten-frame setting, and her thinking likely still referred to that setting.

Episode R3. Robyn initially had difficulty solving these bare number addition tasks. Her increasing facility came with increasing organisation of the addends in terms of 10s and doubles. As we argued, though the teacher Ms Parkin began with suggestions of particular solution procedures, the suggestions supported Robyn's attention to structuring the numbers; Robyn's calculations were an activity of structuring the numbers, rather than following procedures. Robyn commuted addends (e.g., 4 + 9 calculated as 9 plus 4); used eight as "close to 10" and constructed additions with compensations; we formulate this activity as structuring numbers. Significantly, in making and explaining these calculations, Robyn took many combinations for granted, such as doubles, complements to 10, 10-plus combinations, 9 as 7 and 2. Her structuring of numbers into these combinations was no longer tied to the ten-frame or arithmetic rack setting. Rather, we regard her thinking here as an example of developing a relational framework or structured number environment, as described as the instructional aim with reference to Greeno (1991) and Gravemeijer (2000).

Nate's Structuring

Episode N1. Nate's task progressed from simply identifying a number of dots, to identifying the tens-complement of the dots. As suggested in the description of the episode, Nate recognised that he could attend to the blank squares as he did to coloured dots. This was a minor generalisation of his structuring of the numbers to a subtly different setting, which he seemed to experience as an exciting moment of insight. In turn, in the context of the ten-frames, he seemed to have structured each of the pairs of complements to 10 into an increasingly integrated complement relation, coming to know 8-and-2 as one pattern.

Episode N2. Nate seemed to make progress with the 9-plus tasks by structuring the addends, perhaps into a 1-and-X or X-teen structure. Significantly, Ms Moss's initial pre-arranging of the beads was not sufficient for him to organise his responses. We argue that he needed to actively structure the addends himself; the sense of the additions arose for him from his activity of structuring the numbers. In answering the verbal 9-plus tasks, his structuring became independent of the arithmetic rack setting.

Characteristics of Structuring Numbers

Level-raising and common sense. In the theoretical framework we described a level-raising characteristic of structuring. In our view, an indication of level-raising of learning occurs when students take for granted knowledge which, at an earlier time, was problematic for them—for example, one missing from the ten-frame is "obviously" nine, or seven and seven is fourteen, used to solve seven and nine. We could argue that the new knowledge is becoming, in Freudenthal's (1991) terms, "common sense". Freudenthal regarded learning mathematics successfully as, from one perspective, bringing more sophisticated structure into the realm of common sense (p. 7). The structuring observed in these case studies seems in accord with this view—the students' constitution of number became significantly more structured, and these new structures became common sense for the students.

Mathematical structure and using material. The structuring activity observed in the context of the ten-frame and arithmetic settings is closely associated with the images of the settings and the students' visualisation of those images. For example, in Episode R1, we noted how Robyn's part-whole descriptions were closely tied to the rectangle, rows, pairs, and other spatial patterns she could see in the ten-frame. As Treffers and Beishuizen (1999) suggest, the images "invite specific strategies" (p. 35) and make more "transparent" (p. 33) the students' structuring into doubles and other combinations. Bobis (1996) and Moser Opitz (2001) observed the potential of similar image-based settings for developing students' knowledge of combinations and partitions in the range 1 to 20.

These findings are broadly in concert with several theories which propose that imagery is central in developing mathematical structure (e.g. Mulligan, Mitchelmore, & Prescott, 2006; Pirie & Kieren, 1994). Nevertheless, we note again that we cannot take for granted that students will see mathematical structures in images. In R1, it was Ms Parkin's questioning which supported Robyn to find alternative partitions of the eight dots. In N1, it was Nate's insight which established the equivalence of blank square patterns and dot patterns. In N2, Nate could not at first make sense of Ms Moss's arrangement of the 1+X beads. As Gravemeijer (1991) asserts, "It is not the material that transmits certain knowledge. In this approach, understanding and insight are supported by the context, which can serve as a situation model. The material is used to elicit (mental) arithmetic actions. Close study of the actual occurrence of such acts is necessary" (p. 75-6).

Structuring as distinct from learning about structure.

In the episodes, we can make the distinction, described in the theoretical framework, between structuring numbers and learning about structures. In Episode N1, Nate was not learning the list of complements to 10; rather he was constructing 8 as a complement of 2, 7 as a complement of 3 and so on. In Episode R3, Robyn was not learning that her two strategies are compensation and near-doubles; rather, she was re-organising addends as 10-plus or doubles combinations. In subsequent lessons, Nate may have come to list more or less systematically, the complements to 10, and Robyn may have developed a nomenclature for her strategies. At first, those structures became part of the constitution of the mental objects 1 to 20, and the mental operations of addition and subtraction, but the structures were not strongly constituted mental objects themselves. Thus, rather than say the students had been learning about structures, we prefer to say they had been learning about numbers, and part of what they have learned involved structuring the numbers.

Advanced structuring of numbers constitutes facile addition and subtraction

The aim of the instruction was to support students to become facile in addition and subtraction in the range 1 to 20. As described in the background section, we can characterise facile addition and subtraction in terms of four interrelated aspects of number knowledge/activity: (a) the part-whole construction of number; (b) the use of non-counting additive strategies; (c) automated knowledge of key number combinations and partitions; and (d) a relational number sense, relating unknown combinations to known combinations. We argue that students' structuring activity constituted development of each of these aspects.

(a) By structuring the numbers into combination and partition relationships, initially in the context of the ten-frames, and then increasingly in bare number contexts, the students established a part-whole construction of numbers. There was a striking shift in each of these cases from their pre-assessments, where their arithmetical activity generally involved counting by ones, to their lessons and post-assessments, where their arithmetical activity involved combining and partitioning numbers as parts of other numbers.

(b) Particular part-whole combinations became automatised number knowledge for these students. For example, the complements to 10 became taken for granted, first in the ten-frame context, and later in bare number tasks. We have described this automatisation as an aspect of the structuring of the numbers.

(c) In Episodes R2 and R3, Robyn used non-counting strategies to solve addition tasks. Her solutions involved structuring the numbers in the tasks, rather than merely attempting to imitate a demonstrated procedure. Similarly for Nate in Episode N2, adding numbers to nine, the non-counting addition arose for him from his activity of structuring the numbers.

(d) In Episode R3, Robyn's increasing proficiency arose as she developed more sense of how to relate the addends to her number knowledge—to build to 10, or to doubles. This developing relational number sense was an aspect of her structuring activity: Robyn was increasingly organising the addends in line with her structured number knowledge, anticipating which number relations would accomplish a solution.

To summarise, each of the aspects of facile addition and subtraction is constituted through structuring numbers. Considered another way, facile addition and subtraction, as we understand it, is constituted as an advanced activity of structuring numbers.

Conclusion

We have formulated the activity of two students in the context of intensive intervention as structuring numbers. The analysis developed detailed illustrations of what structuring numbers 1 to 20 involves. Structuring numbers involves developing a rich network of number relations. Important structuring of numbers includes making doubles combinations, and combinations using 5 and 10 as reference points.

Structuring numbers in terms of such combinations may initially be closely tied to instructional settings, but students can progress to independence from the settings. As in this progression to independence, structuring numbers involves level-raising. Structuring numbers can be distinguished from learning about number structure.

We contend that extensive structuring of numbers constitutes a progression from counting strategies to facile addition and subtraction. Indeed, facile addition and subtraction, consisting of activities like connecting, re-organising, combining, and compensating numbers, can be understood as an advanced activity of structuring numbers. Hence, if students are not facile with addition and subtraction, then intervention instruction that encourages students to structure the numbers can lead them to the arithmetic facility they need. The case study students' final success with calculation affirms Treffers' claim that "if one can structure numbers in several ways, their operational calculation follow naturally" (2001, p. 46). We think it is important to affirm that, with attentive teaching, low-attaining students can learn to structure numbers: They need not be left at the dead end of persistent counting.

The rich account of low-attaining students' progression with structuring numbers 1 to 20, developed here, can in turn inform our refinement of the instructional design. Further research could develop accounts of students' structuring of numbers in other domains, such as addition and subtraction in the range 1 to 100 and multiplicative reasoning.

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References

Australian Government (2008). *National Numeracy Review Report*. Canberra: Commonwealth of Australia.

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In J. T. Mulligan & M. C. Mitchelmore (Eds.), *Children's number learning* (pp. 17-33). Adelaide: Australian Association of Mathematics Teachers/Mathematics Education Research Group of Australasia.

Bryant, D. P., Bryant, B. R., & Hammill, D. D. (2000). Characteristic behaviors of students with LD who have teacher-identified math weaknesses. *Journal of Learning Disabilities, 33*, 168-177, 199.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 19-22.

Cobb, P. (1991). Reconstructing elementary school mathematics. *Focus on Learning Problems in Mathematics, 13*(3), 3-33.

Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: Analyzing students' learning in social context,* Journal for Research in Mathematics Education, Monograph No. 12, 1-16. Reston, VA: NCTM.

Denvir, B., & Brown, M. (1986). Understanding of number concepts in low attaining 7-9 year olds: Part 1. Development of descriptive framework and diagnostic instrument. *Educational Studies in Mathematics, 17*, 15-36.

Ellemor-Collins, D., & Wright, R. J. (2008a). From counting by ones to facile higher decade addition: The case of Robyn. In O. Figueras, J. L. Cortina, S. Alatorre, T.

Rojano & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of PME32 and PME-NA XXX* (Vol. 2, pp. 439-446). México: Cinvestav-UMSNH.

Ellemor-Collins, D., & Wright, R. J. (2008b). Intervention instruction in Structuring Numbers 1 to 20: The case of Nate. In M. Goos, R. Brown & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Brisbane, Vol. 1, pp. 179-186). Adelaide: MERGA.

Ellemor-Collins, D., Wright, R. J., & Lewis, G. (2007). Documenting the knowledge of low-attaining 3rd- and 4th- graders: Robyn's and Bel's sequential structure and multidigit addition and subtraction. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart,Vol. 1, pp. 265-274). Adelaide: MERGA.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht, The Netherlands: Kluwer.

Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Gervasoni, A., Hadden, T., & Turkenburg, K. (2007). Exploring the number knowledge of children to inform the development of a professional learning plan for teachers in the Ballarat Diocese as a means of building community capacity. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart, Vol. 1, pp. 317-326). Adeaide: MERGA.

Gravemeijer, K. P. E. (1991). An instruction-theoretical reflection on the use of manipulatives. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 57-76). Utrecht, The Netherlands: Freudenthal Institute.

Gravemeijer, K. P. E. (1994). Instructional design as a learning process. In K. P. E. Gravemeijer (Ed.), *Developing Realistic Mathematics Education* (pp. 17-54). Utrecht, The Netherlands: Freudenthal Institute.

Gravemeijer, K. P. E., Cobb, P., Bowers, J. S., & Whitenack, J. W. (2000). Symbolizing, modeling and instructional design. In P. Cobb, E. Yackel, & K. J. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 225-273). Hillsdale, NJ: Lawrence Erlbaum

Gray, E. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics, 22*, 551-574.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170-218.

Heirdsfield, A. (2001). Integration, compensation and memory in mental addition and subtraction. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Utrecht, The Netherlands: Program Committee.

Hunting, R. P. (2003). Part-whole number knowledge in preschool children. *Journal of Mathematical Behavior, 22*, 217-235.

Mapping the territory: Primary students with learning difficulties. (2000). Canberra: Department of Education Training and Youth Affairs.

McIntosh, A. J., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*, 2-8.

Moon, B. (1986). *The 'new maths' curriculum controversy: An international story*. London: Falmer.

Moser Opitz, E. (2001). Mathematical knowledge and progress in the mathematical learning of children with special needs in their first year of school. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 207-210). Utrecht, The Netherlands: Program Committee.

Mulligan, J., Mitchelmore, M., & Prescott, A. (2006). Integrating concepts and processes in early mathematics: The Australian pattern and structure mathematics awareness project (PASMAP). In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 209-216). Prague: Program Committee.

Olive, J. (2001). Children's number sequences: An explanation of Steffe's constructs and an extrapolation to rational numbers of arithmetic. *The Mathematics Educator, 11*(1), 4-9.

Pearn, C. (1998). Is there a need for a mathematics intervention program in Grades 3 and 4? In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia, Gold Coast, Vol. 2, pp. 444-451). Adelaide: MERGA.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.

Principles and standards for school mathematics. (2000). Reston, VA: National Council of Teachers of Mathematics.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109-151). New York: Academic Press.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153-196). New York: Academic Press.

Rivera, D. P. (1998). Mathematics education and students with learning disabilities. In D. P. Rivera (Ed.), *Mathematics education for students with learning disabilities* (pp. 1-31). Austin, TX: Pro-Ed.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetic meanings and strategies*. New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum.

Sugarman, I. (1997). Teaching *for* strategies. In I. Thompson (Ed.), *Teaching and learning early number* (pp. 142-154). Milton Keynes, UK: Open University Press.

The national numeracy project: An HMI evaluation. (1998). London, UK: Office for Standards in Education.

Thompson, I. (1995). The role of counting in the idiosyncratic mental calculation algorithms of young children. *European Early Childhood Education Research Journal, 3*(1), 5-16.

Thornton, C. A. (1978). Emphasising thinking strategies in basic fact instruction. *Journal for Research in Mathematics Education, 9*, 214-227.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50*, 29-47.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project*. Dordrecht: Reidel.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21-56). Utrecht, The Netherlands: Freudenthal Institute.

Treffers, A. (2001). Grade 1 (and 2): Calculation up to 20. In M. van den Heuvel-Panhuizen (Ed.), *Children learn mathematics* (pp. 43-60). Utrecht, The Netherlands: Freudenthal Institute.

Treffers, A., & Beishuizen, M. (1999). Realistic mathematics education in the Netherlands. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 27-38). Milton Keynes, UK: Open University Press.

Verschaffel, L., Greer, B., & Torbeyns, J. (2006). Numerical thinking. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education*. Rotterdam: Sense Publishers.

van de Walle, J. A. (2004). *Elementary and middle school mathematics: Teaching developmentally* (5th ed.). Boston: Pearson.

van Hiele, P.M. (1973). *Begrip en inzicht* [Understanding and insight]. Purmerend, The Netherlands: Muusses.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics*, *26*, 25-44.

Wright, R. J., & Ellemor-Collins, D. (2008). *Structuring numbers to 20: An important topic in early number learning*. Manuscript submitted for publication.

Wright, R. J., Ellemor-Collins, D., & Lewis, G. (2007). Developing pedagogical tools for intervention: Approach, methodology, and an experimental framework. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart, Vol. 2, pp. 843-852). Adelaide: MERGA.

Wright, R. J., Martland, J., Stafford, A. K., & Stanger, G. (2006). *Teaching number: Advancing children's skills and strategies* (2nd ed.). London: Chapman.

Wright, R. J., Stanger, G., Stafford, A. K., & Martland, J. (2006). *Teaching number in the classroom with 4-8 years-olds*. London: Chapman.

Young-Loveridge, J. (2002). Early childhood numeracy: Building an understanding of part-whole relationships. *Australian Journal of Early Childhood, 27*(4), 36-42.

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