

Appreciating Mathematical Structure for All

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We take *mathematical structure* to mean the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set. Because we take the view that appreciating structure is powerfully productive, attention to structure should be an essential part of mathematical teaching and learning. This is not to be confused with teaching children mathematical structure. We observe that children from quite early ages are able to appreciate structure to a greater extent than some authors have imagined. Initiating students to appreciate structure implies, of course, that their appreciation of it needs to be cultivated in order to deepen and to become more mature. We first consider some recent research that supports this view and then go on to argue that unless students are encouraged to attend to structure and to engage in structural thinking they will be blocked from thinking productively and deeply about mathematics. We provide several illustrative cases in which structural thinking helps to bridge the mythical chasm between conceptual and procedural approaches to teaching and learning mathematics. Finally we place our proposals in the context of how several writers in the past have attempted to explore the importance of structure in mathematics teaching and learning.

Theoretical Frame

We take *mathematical structure* to mean the identification of general properties which are instantiated in particular situations as relationships between elements. These elements can be mathematical objects like numbers and triangles, sets with functions between them, relations on sets, even relations between relations in an ongoing hierarchy. Usually it is helpful to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced. Mathematically, the definition of a *relation* derives from set theory as a subset of a Cartesian product of sets. Psychologically, a *relationship* is some connection or association between elements or subsets which have been themselves been discerned. When the relationship is seen as instantiation of a property, the relation becomes (part of) a structure. For example:

The relation between a whole number and its double can be denoted by $\{[n, 2n]: n = 1, 2, 3, \dots\}$ or by $n \rightarrow 2n$ among other ways; being a pair consisting of a number and its double is a property instantiated here among the whole numbers, but other numbers are possible.

The set $\{[1, 2], [2, 5], [3, 2], [1, 4], [4, 4]\}$ is also a relation, though not one that deserves a special name, and unlikely to be an instantiation of a specific property other than that it specifies this relation!

Recognising a relationship amongst two or more objects is not in itself structural or relational thinking, which, for us, involves making use of relationships as instantiations of properties. Awareness of the use of properties lies at the core of structural thinking. We define structural

thinking as a disposition to use, explicate and connect these properties in one's mathematical thinking.

We adopt a phenomenological approach to studying opportunities for stimulating learners to appreciate mathematical structure, at every age and stage of their exposure to mathematics and to mathematical thinking. We start from the position that mastering procedures is an important component of taking advantage of opportunities to make mathematical sense, but that it is of little value to learners if it is simply a procedure, because as the number of procedures increases, the load on memory and retrieval becomes more and more burdensome. When procedures are accompanied by even a minimal appreciation of the mathematical structures which make them effective and which provide criteria for appropriateness, learning shifts to focusing on re-construction based on re-remembering (literally) rather than relying totally on photographic or rote memory.

The notion of mathematical structure pervades modern mathematics, reaching its height in the work of Birkhoff and MacLane (1958) and subsequent developments, including the Bourbaki enterprise of codifying and inter-relating the structures of known mathematics (Beaulieu, 1990; Mashaal, 2006). Roots of a search for structure can be found as far back as Euclid, through Gauss and then in the profusion of developments in the 19th century, including Peano's axioms for arithmetic, various axioms for non-Euclidean geometries, groups, rings, fields and so on (Cohn, 1965). We take, as the essence of this modern movement, the identification and isolation of properties used as the sole basis for reasoning, so that any deductions apply to any instantiation of those properties. For example, using objects called Points and Lines where Lines are sets of Points, there are two properties which axiomatise projective geometries:

- For each pair of distinct Points, there is a unique Line containing them;
- For each pair of distinct Lines there is a unique Point common to them.

These two properties have the following duality: Objects called Points and objects called Lines can be interchanged to give new theorems, and may not even have the appearance either of points or lines.

The underlying theoretical frame being used here is a distinction between different forms, states, or structures of attention (Mason, 2003, Mason & Johnston-Wilder, 2004):

- Holding wholes (gazing),
- Discerning details (making distinctions),
- Recognising relationships (among specific discerned elements),
- Perceiving properties (as generalities which may be instantiated in specific situations),
- Reasoning on the basis of identified properties.

The suggestion is that there is a subtle but vital difference between recognising relationships in particular situations, and perceiving relationships as instantiations of general properties which can apply in many different situations. This implies, we argue, that structural thinking must be thought of as a part of a continuum. Put another way, because language is necessarily general, it is very difficult to tell from a learner's words whether they are dwelling totally in the specific and the particular,

are vaguely aware of the particular as a special or particular case of something more general, or are aware of the particular as an instantiation of a general property. A useful language for discussing this issue is provided by variation theory (Marton & Booth, 1997; Marton & Trigwell, 2000; Marton & Tsui, 2004) whose roots go back to Aristotle. Marton claims that learning is associated with discerning variation among proximal events. Human beings naturally detect similarity through becoming aware of variation. The critical features for variation to be detected seem to be that there is juxtaposition of variation in close proximity of time if not place, or some other reason for attending to the multiplicity, that the variation be in some but not too many dimensions at once, and that the range of variation displayed be comprehensible. Thus varying four different aspects at the same time, and using elements which are unfamiliar, is unlikely to promote awareness of possible variation.

This idea is particularly powerful in mathematics, because variables and their variation are our stock in trade. Furthermore, in mathematics we like to vary whatever *can* vary, so rather than considering given dimensions of variation we also think about what dimensions can possibly be varied, and in what ways they can vary. Marton's use of dimensions of variation is too general, so we use *dimensions-of-possible-variation* and *range-of-permissible-change* to capture the qualities of variation arising in mathematics (Watson & Mason, 2003; Mason & Johnston-Wilder, 2004). Thus, for example, when considering counting, a dimension of possible variation is the magnitude of the number, and the range of permissible change is limited to whole positive integers – although magnitude of number in other contexts has other possible ranges of change. The cardinal number arrived at when counting only has meaning if the learner experiences situations with various cardinalities. These distinctions in attention and the role of variation will become clearer in the examples which follow.

In the rest of this paper we provide a wide range of examples from school mathematics where awareness of structure makes a significant difference to understanding. We then look in more detail at how structural awareness supports significant shifts between arithmetic and algebraic thinking. Finally we discuss pedagogic implications, indicating some earlier contributions to this issue.

Examples of Structures Supporting Understanding and Appreciation

We believe that attention to structure runs through the whole of mathematics, and that shifts of attention make a difference to how mathematics is seen. But we are not giving instructions about how to make this happen. Instead, we offer a range of examples which support our contention that appreciation of mathematical structure is vital for understanding, and well within the grasp of learners at all ages, even if it is not explicit or articulated. These examples are drawn from various aspects of mathematics: patterns, children's arithmetic, angle sums, quadratics, multiplicative structure and geometric structure.

Replicating and Constructing Repeating Patterns

As evidence of the capabilities of young children, we note particularly the work of Papic (2007). She was approached by some teachers of pre-school children (<5-year-old) who wanted to exploit children's natural play so as to draw out lessons that would contribute to their mathematical thinking. By alerting themselves to pedagogic possibilities, particularly in terms of getting children to reflect on their pattern-recognising and pattern-making activities, teachers developed ways to challenge learners to be more systematic and more structured than they might otherwise have been. The children they worked with showed significantly more sophisticated behaviour concerning the recognition and construction of patterns than did a parallel control group. In particular, they got children to reflect on their pattern-making activities and they developed ways to challenge the learners to be more systematic and more structured than they would otherwise have been. Along the same lines, Cooper and Warren (2008) and Warren and Cooper (2008) show how it is possible to construct tasks which alert young children to the difference between repetitions in patterns, and growth patterns (such as multiplicative and exponential growth). For example, in the pattern RB, RBB, RBBB, RBBBB, the R is repeated each time, but the B grows in a steady manner.

Some children benefit from having their attention directed to these two aspects of patterns, what grows and what stays the same, enabling them to respond appropriately to more sophisticated patterns than would otherwise be the case. A pedagogically effective approach is repeatedly to invite learners to say what they see as being the same, and what different about, different parts of a pattern, or different patterns. In variation theory, it is suggested that we respond to near simultaneous variation because then the contrast between change and invariance is easily visible. Attention can be directed (without being explicit) to features which are changing and features which are invariant, thus supporting awareness of dimensions-of-possible-variation and relevant ranges-of-permissible-change. Becoming accustomed to considering invariance in the midst of change, a ubiquitous mathematical theme, enculturates learners into a productive disposition (Kilpatrick, Swafford & Findell, 2001).

The teachers involved in Papic's (2007) study prompted students to become aware of repetition and growth as structures which can then be used to extend sequences. Such students were being exposed to multiple situations in which attention is usefully drawn to relationships between components in the form of repetitions and gradual changes. The mathematical way of describing how to draw attention to repetition and change is through controlling variables and, as Warren and Cooper (2008) demonstrate, it is through controlling variables that teachers can guide the attention of students.

Structural generalisation is quite different from empirical counting. As Rivera (2008) points out, any pattern-recognition is founded on an abductive assumption that "things that seem to be the case will continue to be the case", and that "patterns are present". Many mathematical-looking tasks involve inviting people to extend 'patterns' and to predict the n th term. In order to be mathematical tasks, there has to be prior agreement or

articulation of the actual underlying structure that generates the given sequence.

Thus the sequence 1, 2, 4, 8, ... is under-specified until it is related to some structural situation which generates the sequence (Mason, Burton & Stacey, 1982). In a more complex setting, asking for the number of regions formed by drawing all chords between n points distributed in general position around a circle provides a structurally generative action which can then be formulated and expressed in general, passing through the sequence above. Steiner's problem, which Pólya (1965) used in his film *Let Us Teach Guessing*, asks for the number of regions of space formed by n planes in general position. Pólya invites students to specialise to simple cases and to try to proceed inductively, not just to obtain a sequence, but to become aware of an underlying structure that can be used for the n th case. The students abductively conjecture the same geometric sequence which, however, soon collapses because the sequence is actually generated by polynomials. In this example, if the dimension-of-variation is taken to be the sequential positional number, rather than the number of points on the circle, learners are directed towards pattern-generation instead of towards the underlying structure.

To ask for predictions about the sequence $A B B A A B B A A B B A \dots$ is to force or enculturate learners into abductive assumptions about the way patterns usually work, but in the absence of any structural means for continuing the sequence, questions about the position of the n th A or the letter in the n th position are meaningless. However, if a structural rule is given (after the first A , alternate two A s with two B s; alternatively, repeat the pattern $ABBA$) or if structural information is given (the repeating pattern has already appeared at least twice), then the sequence is uniquely specified and it makes sense to count and predict (Mason et al., 1982). Enculturation into a mathematical expectation or anticipation of structure involves getting learners to articulate the structural basis for possible patterns as a matter of routine. For example, the fact that $(-1) \times (-1) = 1$ arises from mathematicians' explicit desire to extend structural properties of arithmetic such as associativity and distributivity from whole numbers to integers.

Children's Methods

Inviting children to find 'quick ways' to do arithmetic calculations such as adding the same to both numbers to reach an easier calculation ($47 - 38 = 49 - 40$) and the many variants, can be an entry into appreciating structure. The issue at any point is whether other learners appreciate that what they are discovering is a *method*, based on the key ideas of equivalence, compensation and attention to operations, rather than a single particular clever move. There is no need even to use particular numbers as the following example shows:

Two numbers have been chosen but we do not know what they are. We are about to subtract the second from the first, but before we do, someone adds 1 to them both (or perhaps adds 3 to the first and subtracts 2 from the second). What will be the difference in the differences?

There is a basic awareness based on physical manipulation of objects which tells people the answer without having to do particular cases, even with the extension to separate adjustments (Lakoff & Nunez, 2000). Curiously, extending a corresponding task with division is much harder, presumably because multiplicative awareness receives less attention and is encountered only after addition has been well established (Davis, 1984):

Two numbers have been chosen but we don't know what they are. We are about to divide the first by the second, but someone first multiplies them both by 3 (or perhaps multiplies the first by 3 and the second by 2). What is the ratio of the quotients?

The important part of the task is not the structural appreciation that adding one to both numbers makes no difference to the difference, but the push to exploring dimensions-of-possible-variation. What can be changed in the question and still the answer is the same? What if an answer itself is taken to be a dimension of variation? We have to decide what has to be changed in the question, and how to change it to get the answer. What if the operation itself is taken to be a dimension of variation so addition can be changed to subtraction, multiplication or division? Structural appreciation lies in the sense of generality, which in turn is based on basic properties of arithmetic such as commutativity, associativity, distributivity and the properties of the additive and multiplicative identities 0 and 1, together with the understanding that addition and subtraction are inverses of each other, as are multiplication and division. By working on tasks which focus on the nature of the relation rather than on calculation, students' attention is drawn to structural aspects as properties which apply in many instances. A more detailed analysis of connections between relational thinking in arithmetic and mathematical structure can be found in the next main section.

Angle Sums of Planar Triangles

The sum of the interior angles of a planar triangle is 180° . A standard approach to 'teaching' this theorem is to get learners to measure angles of triangles using a protractor, resulting in a range of sums of the order of $180^\circ \pm 10^\circ$. Eventually the teacher is forced to declare that the answer should be 180° , but this is likely to convince only those children who are trying to please the teacher and accept everything they are told. The experiment itself fails to convince anyone because the variation is due to measurement error, not to dimensions-of-possible-variation in the structural sense. It may however provide kinaesthetic support for what it asserts.

The epistemological basis for knowing the sum of the angles of a triangle lies neither in the material world, nor in the social world, but in the structural world of mathematics and the imagination (Hilbert & Cohn-Vossen, 1952). It is a consequence of an assumption about the material world, namely that, in the plane, keeping track of the rotation of a direction indicator as you transport it around the boundary of a triangle (indeed any non-self-intersecting polygon) always yields one full revolution (also known as the 'turtle-turning theorem': see Abelson & diSessa, 1980). The sum of the

interior angles can be deduced by reasoning on the basis of this explicitly acknowledged property.

The turtle-turning theorem is structurally perfectly adequate as the sole basis for reasoning about the exterior angle sum and, being based on personal bodily experience, links the kinaesthetic with the cognitive through reflective abstraction (Piaget, 1970), that is, through becoming aware of coordinated actions. It is an underlying structural reason for the result. The actual angle sum is not something that needs to be memorised, since familiarity will develop with use and with the confidence of being able to reconstruct it using the body-based notion of turning. Because it is used on imagined triangles (though it could be initiated by having someone traverse a big triangle while someone else records the turn, not by measuring but by replicating while standing at a fixed point), the result is a property that applies to all triangles, not just the few illustrated in the textbook.

This type of approach is an instance of *productive thinking*, to use a term coined by Wertheimer (1945, 1961) to which we shall return later, because the notion of traversing a closed planar figure is not restricted to triangles, nor even to polygons; and it can be extended to figures in which the turtle turns through other whole numbers of full revolutions (winding numbers other than 1) such as with self-crossing polygons. There are some hidden assumptions, however. Because you are effectively transporting an angle from one place to another (the angle turned through so far) you have to assume that this movement does not change the measure of the angle (as indeed it does in some less familiar mathematical spaces).

The turtle-turning observation about the total directional change is an example of what Gattegno (1987) referred to as an *awareness*, a basis for action, what Papert (1980) called syntonic awareness, and what Simon (2006) calls key developmental awarenesses. Lakoff and Nunez (2000) go further and propose that all mathematical concepts can be traced back to bodily awarenesses. A concomitant observation is that angles are invariant under scaling, or in other words, the size of an angle does not depend on the lengths of the arms (Balacheff, 1987, 1988). Bringing such structural awarenesses to the surface through carefully constructed tasks is quite different from lessons based on “today we shall have naming of parts”¹ of polygons, or types of polygons, since the only structural aspect is the naming of polygons by the number of their vertices. Technical names emerge quite naturally when they facilitate communication; but when learning names becomes an end in itself it interferes with learners developing a sense of, and appreciation for, mathematical structure.

When applied to simple planar polygons with n sides, the interior angle sum is seen to be $(n - 2)$ straight angles. It is all too easy to imagine worksheets which simply tell students the formula and ask them to fill out a table using substitutions, or which provide a table and expect students to induce the formula from some examples. Substituting various values for n is an exercise in arithmetic, perhaps, but has little to do with appreciating structure. However, seeing (experiencing) what happens when triangles are glued together along edges to create polygons, and seeing (experiencing)

¹ Opening line of a poem called *Naming of Parts* by Henry Reed.

how the sum of the interior angles of the triangles relates to the sum of the interior angles of the polygon, reinforces appreciation of a related structure, in which complex shapes are built up from triangular ones, just as complexes are built from simplexes in homology theories. The appreciation of structure has to do with experience of generality, not reinforcement of particularities. It is not a subject for empirical-inductive accumulation.

Completing the Square

Completing a quadratic expression so that it is displayed as the scaling and translating of the square of a quantity is often seen as a bit of theory to offer to the quicker thinkers while everyone else is engaged in the principal task of practising factoring or using 'the formula', presumably because this is what is expected in assessment. But it is precisely the process of completing a square in general which constitutes the underlying structure, which in turns affords the possibility of the important awareness that 'all quadratic graphs look like $\pm x^2$ ', by translating and scaling. This appreciation can be expressed alternatively by noticing that with a single template of a parabola it is possible, in principle, to draw any parabola by appropriate use of scaling, translation and/or reflection. In other words, by re-labelling the axes (allowing the positive y -axis to point downwards), a single parabola becomes the graph of any parabola. Without at least a sense-of the process, it is very difficult to re-construct when needed, so students resort to memorisation or, worse, to mnemonics to aid rote memory.

Rehearsing the technique on multiple examples, even with carefully judged variation in critical dimensions (the particular coefficients) and in their corresponding ranges-of-permissible-change, is most likely to attract students' attention to the technique rather than the structural generality. Thus students may adopt the practice of 'completing the square' as a sequence of actions. They may develop fluency through multiple rehearsals. But this sort of trained behaviour proves to be un-robust when conditions change (for example recognising $x^4 + 3x^2 + 14$ or $2 \sin^2 x + 3 \cos x - 5$ as quadratics, or on encountering Tartaglia's solutions of the cubic and quartic equations). More powerful is to be *aware* of having a general technique even if it consists of carrying out an automated practice. This awareness can arise through having attention drawn out of the mere doing, so as to reflect on what is being achieved mathematically beyond getting the answers in the back of the book. Much more powerful again is having a sense of what completing the square does and how it works through an appreciation of its geometrical and algebraic manifestation, so that the details can be re-constructed or varied to suit the situation.

The symbolic form $y = a(x + b/2a)^2 - b^2/4a + c$ can be much more easily perceived as a translation in the x direction, a scaling, and then a translation in the y direction, than can $y = ax^2 + bx + c$. On the other hand, the format $y = a(x - r)(x - s)$ makes the roots explicit (and extends to polynomials of higher degree). There are connections between these that are often overlooked, such as that $-b/2a$ is the mean of the roots, and that $(b^2 - 4ac)/a^2$ is the square of the distance between the roots ($r - s$) (Watson & Mason, 2005). There is semantic as well as syntactic content: stressing the syntactic leads to emphasis on techniques, while stressing the semantic leads to emphasis on

structural relationships. The important feature about structural relationships is *not* to convert them into content to be learned, but rather to treat them as awarenesses to be brought to the surface, possibly through the use of carefully-varied examples, language, layout and gesture, and integrated into functioning so that learners can reconstruct details when they need them. When the process has become familiar, there is an intermingling of recall of syntax or form, and semantic re-construction based on understanding.

Multiplicative Structure

It is a well known phenomenon that many secondary school students, when asked for the factors of a number presented in the form 3^25^3 will first multiply the numbers out to get a numeral, and then turn around and try to factor it. They simply do not see 3^25^3 as a number, but only as an instruction to calculate. Zazkis (2001) found with pre-service elementary school teachers that she could provoke them to perceive the prime-power form as a structural presentation of a number by using very large numbers whose very size negates any desire to calculate. Familiar routines are more likely to come to the fore than fresh or unfamiliar ways of perceiving, so using examples which block familiar routines may in other situations prove to be useful for provoking students to perceive a different structure. An instance of this is the following task about addition:

Under dictation, write down a collection of three and four digit numbers, in a column, but with the wrinkle that the high-order digits are to the right, so everything is written down 'backwards'. Once fluency has been achieved, add up the column so as to get the correct answer when read from right to left. Similarly, write down two three-digit numbers backwards and multiply them together by long multiplication.

The effect of breaking an automated habit draws the process to the surface. People often report that it causes them to re-think the technique.

There is considerable difference between the notion of 'structure of multiplication' seen as a multiplication table with changing differences as you move from row to row or column to column, and the multiplicative structure of numbers, arising when numbers are presented in terms of their prime factors, or as scalar multiples of each other. In multiplication tables the most obvious dimension of variation is often the additive relation between products; so to focus learners' attention on structure requires a deliberate challenge to the 'normal' kind of presentation. Furthermore, multiplicative relationships lie at the heart of the appreciation of ratio (see Vergnaud, 1983; Harel & Confrey, 1995), based on bodily awarenesses such as assigning portions of a number of objects equally amongst several people rather than repeated addition.

Multiplicative Closure

Euler (1810) demonstrated that numbers of the form $a^2 + b^2$ where a and b are integers are closed under multiplication, and went on to show that for a fixed k , numbers of the form $a^2 + kb^2$ are also closed, due to the fact that

$$(a^2 + kb^2)(x^2 + ky^2) = (ax + kby)^2 + k(ay - bx)^2 = (ax - kby)^2 + k(ay + bx)^2.$$

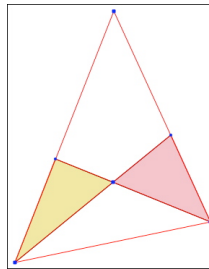
This result can be seen as an algebraic curiosity. But if a , b , x and y are seen as dimensions-of-possible-variation, with integers (or positive integers, or rationals) as the range-of-permissible-change, then the relationship becomes a property. A sense of structure begins to emerge.

Appreciating the ways in which sets of numbers satisfy relationships (e.g., closure of a binary operation) leads to the study of the consequences of such relationships taken as properties. Here, for each k , the sets of numbers expressible as the ratio of two numbers of the form $a^2 + kb^2$ (the denominator being non-zero) has the properties we now associate with groups. Thus a surprising algebraic relationship can be turned into a property to yield structure amongst certain integers and beyond (taking k to be irrational or real), and a structure amongst certain rationals. The 'structural move' which characterise so much of 20th century mathematics came about by selecting certain properties, taken as axioms, as the basis for further reasoning.

Geometric Structure

We have concentrated so far on algebraic structure, but we consider our remarks to be every bit as valid when applied to geometrical structure (including topology). As a single but far-reaching example, consider the following task based on a sample exam question from Latimer and Smith (1937, p. 193) drawn to our attention by Küchemann².

Two medians are drawn in a triangle, intersecting in the centroid. What can be said about the areas of the two shaded triangles, whose vertices consist of one vertex of the triangle, the midpoint of a side, and the centroid?



To make progress, it is essential to bring in outside knowledge. There is no point in measuring, as the diagram is unlikely to be accurate; what is needed is something that relates area to medians. A Pólya-esque approach might be to start with a simple case (Pólya, 1945, 1962) such as an equilateral triangle, where a relationship between the areas is clear by invoking the property of symmetry as a relation between the two regions. Then one might look for a way of varying what is known so as to approach what is not known. Another approach might be to cast around for what is known in the situation and connect that with what is wanted (Mason, Burton & Stacey, 1982). This process might lead to introducing a segment known to be

² Personal communication with Dietmar Küchemann, July, 2008.

parallel to the base; extending the shaded triangles by the same triangle (in either of two different ways) to produce triangles between parallels on a common base; after which the desired equality can be deduced. Each of these connections, each of these instantiations of a general property not only in the particular triangle depicted but in all triangles, is structural thinking in a geometric context. It involves shifts between the levels of geometric understanding delineated by the van Hiele (see van Hiele, 1986). They sought to identify levels of progress in students' geometrical understanding varying from simple visualisation, through analysis and informal reasoning to deductive reasoning and rigour. We prefer to think of how attention shifts between these levels in terms of the movement back and forth between different forms or states. These movements or shifts in how that attention is structured, rather than merely what is attended to, account for the emergence of the geometrical thinking of students and of experts. This is particularly helpful in thinking about the difference between inductive approach and deductive reasoning, which involve different ways of looking at objects. As with arithmetic and algebra, the student is expected to see or experience the general through the particular (cf. Whitehead, 1911 pp. 4-5), in other words, to appreciate the dimensions-of-possible-variation intended by the diagram and by the reasoning.

Relational or Structural Thinking in Arithmetic

In this section we probe more deeply into connections between structural thinking in arithmetic, on the one hand, and mathematical structure, on the other, to learn more about shifts from particular to structural understandings. The importance of structural understanding in these contexts is that they offer students a source of control which allows them to move beyond the particular situation. It will be clear in what follows that the extent of this control is open to growth – that is, it is open to increasing levels of generality. Several authors, including Carpenter and Franke (2001) and Stephens (2006), refer to the thinking underpinning this kind of thinking as *relational thinking*, but from our standpoint it might just as easily be called *structural thinking*. Structural thinking is in this sense productive – a term we referred to earlier and to which we shall return to later. The products of structural thinking can extend from being able to give several other instances of the same property to giving fully developed generalisations.

Number Sentences

Carpenter and Franke (2001), Stephens (2006), Jacobs, Franke, Carpenter, Levi and Battey (2007), Molina (2007), and Fujii and Stephens (2001, 2008) have studied in detail ways in which children as young as 6-years-old respond when asked to justify their decision about the validity of statements such as $173 - 35 + 35 = 173$.

Some children calculate their way to the answer and then decide; some start to calculate and then notice the familiar number to be subtracted and declare their decision; others look at the expression and declare immediately without apparently doing any calculation at all. To decide without any calculation is a form of relational thinking, of appreciating arithmetic

structure concerning, if not zero, then the effect of first subtracting and then adding the same quantity. It could be the manifestation of a fundamental awareness that taking and then replacing makes no change (Lakoff & Nunez, 2000). As such it would be an example of a theorem-in-action (Vergnaud, 1983). However students' appreciation of this relation may go beyond its use in this context; they may be aware that they are using a generality – indeed, they may be able to state that they are using an abstract relation.

When learners are justifying their decisions to each other, it is often very difficult to decide whether they are aware of a general property (the 173 and the 35 are instances of generality) or whether the 173 is mentally fixed but the 35 is an instantiation of taking and replacing, or an instantiation of $173 - a + a = 173$, or even of $b - a + a = b$, that is, whether the 173 and the 35 are seen as mere place holders or as quasi-variables (Fujii & Stephens, 2001; Lins & Kaput, 2004; Fujii & Stephens, 2008) or as mentally fixed *pro tem*. Some children can enact one or other of these relationships without being able to bring it to explicit articulation, and may not even use it robustly in all instances. Young children can sometimes articulate the general structural principles underlying the relationship, for example as “if you start with a number and you take away something and give it back you haven't changed the starting number”. Children may be fuzzily aware that this relationship holds for all a and b with which they are familiar, or even able to express it as a generality, yet they may not have encountered or considered situations where $a > b$ or where a and b are negative or rational. Thus general statements may adequately express limited structural understanding, based on restricted ranges of change.

One way to think about the different possibilities, and even to seek evidence for different awarenesses, is through the focus and structure of their attention. The way they describe what they are doing sometimes suggests not only what they are attending to, but different ways in which they are attending, whether to the particular, or through the particular to the general, or at the particular though the general. Another way of expressing the complexity of learner awareness is that, without further probes, it is difficult to know the range-of-permissible-change of which the learner is aware, and even which dimensions-of-possible-variation the learner is contemplating. By asking learners to construct similar examples, some light is shed on at least some of the features they appreciate as changeable as well as the range over which the learner accepts that the change can be made (Watson & Mason, 2005).

Sometimes too much concern is expressed about the abstruseness of letters used to denote as-yet-unspecified numbers. For example, Hewitt (1991) uses Greek letters to denote the constants in a complex calculation to the evident satisfaction and understanding of the whole of a class of 30 or so students in Year 7 (age 11-12). Having drawn their attention to the way they could find the number he was thinking of by undoing a sequence of calculations (“add 2 multiply by 3 divide by 5 ... my answer is 6 so what did I start with?”), the single lesson ended with a complex example using Greek letters in place of numbers which they all happily solved. The important part of the lesson was the development of structural awareness, through repetition and emphasis, of how addition and subtraction undo each other,

as do multiplication and division (being careful about zero of course). The development occurred in the context of finding an as-yet-unknown number, and was generalised using Greek letters in the same rhythm so as to indicate that what matters was the structural relationship between the operations, not the specific numbers. Structural awareness, or relational thinking in this context, involves explicit awareness of some range-of-permissible-change of some dimensions-of-possible-variation. These ranges-of-permissible-change can be extended when other kinds of numbers and number-like objects are encountered.

Missing Number Sentences

Recent research carried out by Stephens (2008), Stephens and Wang (2008), and Stephens, Wang and Al-Murani (2008) used three types of mathematical sentences to explore students' capacity to think about important aspects of mathematical structure. Type I number sentences used one missing number, Type II number sentences used two missing numbers and Type III sentences were modelled on Type II but used algebraic symbols. In the studies reported here, these three types of questions were used with 275 students in Australia, China and England ranging from Year 6 (10-11 years old) to Year 9 (13-14 years old).

Type I: one missing number. The first kind of number sentence (Type I) presents students with a number sentence with one number missing and asks them to find the value of the missing number and to explain briefly the reasoning they used to reach a solution. The authors used all four operations in Type I, and invited students to find the value of a missing number and to explain their thinking. For example:

$$\begin{aligned} \square + 17 &= 15 + 24 \\ 99 - \square &= 90 - 59 \\ 48 \times 2.5 &= \square \times 10 \\ 3 \div 4 &= 15 \div \square \end{aligned}$$

For each operation, four different problems similar to those above were used but with the unknown number being set in a different place for each of the four problems.

Some students relied on computation to solve these problems. In each case, they first computed the result of the operation involving the two known numbers, and then used this result to calculate the value of the missing number on the other side of the equal sign. Other students used compensation and equivalence. Irwin and Britt (2005) claim that the methods of compensating and equivalence that some students use in solving these number sentences may provide evidence of "what could usefully be described as structural thinking" (p. 169). They give, as an example, the expression $47 + 25$ which can be transformed into an equivalent expression $50 + 22$ by adding 3 to 47 and subtracting 3 from 25, thus making calculation easier. They claim that "when students apply this strategy to sensibly solve different numerical problems they disclose an understanding of the relationships of the numbers involved. They show, without recourse to literal symbols, that the strategy is generalisable" (Irwin & Britt, 2005, p. 171). It is however not always so easy to deduce from observed behaviour

whether learners are aware of the 47 and 25 as dimensions-of-possible-variation, of 3 as a dimension-of-possible-variation, or of the adding and subtracting as a special instance of more general compensation (another dimension-of-possible-variation). Structural thinking is much more than seeing a pattern, such as 'when one number increases by three the other goes down by three'. Where this merely recounts the pattern used in this particular problem with no sense of generalisation to other instances, it indicates recognition of a relationship in particular but not perception of property in general. A capacity to generate other instances that illustrate the same property is a feature of structural thinking. Often it seems that students act as if they have some such awareness, but it may be neither robust nor universal. Furthermore, their perceived range-of-permissible-change may be confined to positive whole numbers rather than to numbers more generally, whether involving negatives, rationals or decimals. A great deal depends on whether they are attending to and dwelling in the particular or in some sense aware of a property being instantiated, whether that in-dwelling comes from an awareness as a basis for their action in the form of a theorem-in-action, or from an emerging behavioural practice.

Where learners respond to direct suggestions to 'use compensation' or to 'add and subtract', or to indirect prompts to use a strategy before trying to do it directly, they are on the way to being influenced by careful scaffolding and fading (Seeley Brown, Collins & Duguid, 1989; Love & Mason, 1992) so as to be able to initiate these actions for themselves (van der Veer & Valsiner, 1991). Somewhere along the line, they display structural awareness.

A deep understanding of equivalence and compensation is at the heart of structural thinking in arithmetic. Students need to know the direction in which compensation has to be carried out in order to maintain equivalence (Kieran, 1981; Irwin & Britt, 2005). Indeed, we suggest that structural thinking is present only when students' explanations show that they understand the fundamental importance of the operations involved, make use of equivalence, the direction of compensation required to maintain equivalence, and how particular results are part of a more general pattern. According to Stephens (2008), students used a range of equally successful explanatory methods in their written responses to Type I sentences. Some students used arrows or brackets or other notation in ways which indicate a comprehensive understanding of equivalence and compensation. The use of arrows or directed lines to connecting related numbers, such as from 2.5 to 10, showing $\times 4$ above the line or arrow, and an arrow connecting 48 to the unknown number, with $\div 4$ joined to this line or arrow, was a simple and effective method of demonstrating the direction of compensation. Other students wrote their thinking in the form of mini-arguments (see Vergnaud, 1983) using expressions such as "Since 17 is two more than 15, the missing number has to be two less than 24 in order to keep the balance". Other students chose to make a similar argument starting with a relationship between 17 and 24. Relational thinking can be expressed using a wide range of methods and forms, but in all cases these forms and methods draw attention to the fundamental ideas of equivalence, and compensation as required by the particular operations. These features are equally important to elucidating the structures of the three types of mathematical sentences we refer to.

Type II: two missing numbers. One of the difficulties encountered in using Type I number sentences in a written questionnaire is that some students who may be quite capable of using structural thinking nevertheless choose to solve Type I sentences by computation. While they may find computation attractive and easy, these students are not to be confused with those who are *restricted* to solving such sentences computationally. This important distinction can, of course, be explored by means of an interview; by asking, for example, "Could you have solved this number sentence in another way?". But there are other ways of pushing students beyond computation using written responses. In studies referred to above, this was achieved through the use of Type II number sentences using two unknowns, denoted by Box A and Box B, and employing one arithmetical operation at a time. Type II questions are exemplified in parts (a) to (d) in Figure 1 below. Using a similar template, other questions were devised involving subtraction, multiplication and division.

<p>Think about the following mathematical sentence:</p> $18 + \square = 20 + \square$ <p>Box A Box B</p> <p>(a) Can you put numbers in Box A and Box B to make three correct sentences like the one above?</p> <p>(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?</p> <p>(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?</p> <p>(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.</p>

Figure 1. Type II number sentence involving addition.

Almost all students were able to make up three replicas of each mathematical sentence using specific numbers. In dealing with addition, some students used large numbers such as 1,000,000 in Box A and 999,998 in Box B; and others used decimal numbers and fractions. There were students who chose quite simple numbers such as 3, 4, and 5 in Box A which they associated with 1, 2, and 3 respectively in Box B. Those who used more complex numbers in Box A and Box B usually had no difficulty in describing in part (b) the relationship between the numbers in Box A and Box B and in successfully answering the subsequent questions. But the same was true for many who had used relatively simple numbers in their exemplifications of the mathematical sentence in part (a). What actually discriminated between students' accomplished and not-so-successful responses to parts (c) and (d) was how they answered part (b). Almost all students were able to identify some *pattern* between the numbers in Box A and the numbers they had used in Box B. But simply seeing a *pattern* may not be productive in perceiving *structure* as a property to be instantiated elsewhere.

Some students identified what might be called a non-directed relation

between the numbers used in Box A and Box B, saying, for example, "There is 2 difference", or "They are 2 apart", or "There is a distance of 2". Some qualified this non-directed relation by saying, "There is always 2 difference". Others noticed a directed relation between the numbers used but attached no magnitude to the relation, saying, for example, "Box A is bigger than Box B". Others expressed a direction but without referring to Box A or Box B, saying, for example, "One number is always higher than the other number by 2", or "One is two more than the other". These responses illustrate clearly the difference between seeing only particular features of a relationship and what we would call structural or fully referenced relational thinking.

In each of these cases, students had noticed some relationship between the numbers in Box A and Box B, but their descriptions suggest that they were attending to a specific feature of the relationship that could be expressed comprehensively as "The number in Box A is two more than the number in Box B". On the other hand, it may be that when they came to articulate what they were aware of, their attention was diverted to a part rather than some more comprehensive whole. Many might not have been familiar with the kind of relationships which prove to be productive in mathematics. To be productive, relationships have to be fully referenced – in this instance, there has to be unambiguous reference to the numbers represented by Box A and Box B; and the magnitude and direction of the relationship has to be specified – just saying that one is bigger than the other, or that there is a difference of two is not enough. Students had their own ways of elaborating comprehensive descriptions; with some using logical qualifiers such as "must be" or "has to be" instead of "is", whereas others added a phrase like "in order for the sentence to be correct" or "for both sides to be equivalent". There were others who chose to write the relationship in symbolic form, writing an equation involving Box A and Box B, or in some cases just A and B. The presence of logical qualifiers and the use of symbolic forms is evidence that students had grasped a source of control that comes from fully referenced relational thinking. These students appeared to be attending more carefully to what we recognised as the structure of the mathematical sentence than those above whose statements pointed to some but not all of the features essential for equivalence.

What we find very illuminating in all the questionnaire responses in the studies reported above is that no student who referred to only partial features of the relationship between the numbers in Box A and the numbers in Box B answered part (d) successfully. Of course, many attempted to answer this question but their answers were always incomplete. Some students answered "No", but then added that it would be necessary to have numbers in Box B that "will allow both sides to balance". Others thought that it would be impossible without using negative numbers. Still others continued to rely on the partial features that they had used in answering part (b) in order to answer part (d).

In summary, these kinds of responses may not be so much incorrect and erroneous as incomplete. They fall short in various ways of being productive. They add weight to the distinction we want to underline that seeing some relationship or pattern is not the same as recognising a mathematical structure. We do need to point out that a mathematically complete description of the relationship between the numbers in Box A and

the numbers in Box B, as required for part (b), did not guarantee a successful answer to part (d). Some who had correctly answered part (b) appeared to be worried about the range of variation that might be required for part (d) to be correct. Nevertheless, there was a strong association between a correct response to part (b) and part (d). Furthermore, when students used similar partial or incomplete descriptions to describe the relationship between the numbers in Box A and the numbers in Box B in related questions involving subtraction, multiplication and division, they were also unable to successfully answer the corresponding part (d) question, "If you put any number in Box A, can you still make a correct sentence?"

Type III: symbolic sentences. Following part (d) students were given a sentence involving literal symbols c and d in place of the boxes and where the numbers were slightly different. In the case of addition (Figure 1) a symbolic relationship of the form $c + 2 = d + 10$ was used in a part (e). Students were asked, "What can you say about c and d in this mathematical sentence?" Once again, none of the students who had given one of the 'partial descriptions' of the full relationship between the numbers in Box A and the numbers in Box B successfully answered this question. Many chose to give a particular set of values, such as $c = 10$ and $d = 2$ in the case of addition. Those who were able to answer part (e) successfully had all given a complete and correct response to part (b) and part (c), and most had given a correct and complete response to part (d). Among successful responses, there was, moreover, a high level of consistency between the language and terminology used to explain students' answers to part, (d) and (e). For example, where students had answered (d) using a symbolic relationship they almost always used a symbolic expression to describe the relationship between c and d ; and where they had used written descriptions in answering part (d) and part (e) they used similar words and phrases in both expressions. One student commented that the c and d were "just like Box A and Box B". This suggests an aspect of structural understanding that could be explored more deeply in interviewing students who gave correct answers to parts (d) and (e). Referring to the Type II number sentence and its corresponding Type III symbolic expression involving c and d , students could be invited to comment on the statement: "These two sentences look different. Are they so different? Can you comment from a mathematical point of view on any similarities you notice about them?"

Pedagogic Consequences: Bridging the Mythical Chasm

Distinctions between procedural and conceptual thinking (Hiebert, 1986) are so numerous in the literature that they have become an accepted, if mythical, commonplace in mathematics education. We suggest that any sensible approach to teaching combines work on understanding concepts with work on mastering procedures, and combines tasks designed to stimulate learners to express their own thinking using technical terms with tasks designed to highlight the use of important routines. Keeping the notion of mathematical structure in mind, together with seeking the structural underpinnings of any proposed tasks, provides ways for students to experience these important elements. It is important to realize that structural thinking cannot be described as either being present or absent. Rather it develops over time

with different levels of complexity in different mathematical contexts. In this next section we show how some well known descriptions of learning support the detail we have presented so far in this paper.

Biggs and Collis (1982; see also Biggs, 1999) distinguished five levels of student responses to probes based around the idea of relationships:

- Pre-structural: students accumulate isolated facts and elemental procedures
- Uni-structural: they make associations but without significance or meaning
- Multi-structural: multiple links are found but without overall meaning or significance
- Relational: students appreciate relations between parts and an overall narrative
- Extended abstract: connections are made within a topic and beyond.

Their taxonomy enabled them to distinguish students' depth of understanding of different topics by classifying the nature of responses to probes. There are close similarities with the different structures of attention described earlier. One significant difference is that these levels are seen as progressive, whereas the attention states we have described are non-sequential: sometimes attention shifts rapidly between states, though at other times its structure remains relatively constant for a few minutes.

Halford (1999) has similarly tackled the issue of progressive structural complexity from a developmental point of view. He proposes that the development of intelligence is characterised by dealing with relations not just between two things but several, and eventually relations between simpler relations. For teaching purposes, an arithmetic of structural complexity such as is proposed by Biggs, Collis and Halford serves as a reminder that as students encompass more concepts, and develop a maturity in the use of concepts, their attention expands to encompass more than simple associative links and connections. They become more what we are calling structurally aware and hence, with experience, more likely to respond structurally in the future. What is of interest here is that these approaches do not distinguish between procedures and concepts. They focus on complexity of structure from which, we argue, both conceptual understanding and procedural competence can emerge.

In a recent fresh attempt to delineate key components of mathematical proficiency, Kilpatrick and colleagues (Kilpatrick et al., 2001, p. 117) proposed five intertwined strands: adaptive reasoning, strategic competence, conceptual understanding, productive disposition and procedural fluency. They include comprehension of mathematical concepts, operations and relations under the heading of conceptual understanding, which can be interpreted to include both relationships between mathematical concepts, relationships among concepts, theorems and procedures, and structural awareness. Procedural fluency includes both facility in using procedures and a repertoire of procedures to use, or in the terms used here, of actions to be initiated based on core awarenesses. A productive disposition is encouraged where students are stimulated to use their own powers, including using structural awareness and memory to reconstruct (and re-member) rather than relying solely on rote memory for formulae and procedures. Furthermore, their reasoning is more likely to be

adaptive where there is greater flexibility, in contrast to the limitations of training in specific behaviour following taught procedures on ritualized exercises. Again, procedural fluency is more likely where the guidance as to what procedures to invoke is informed by structural awareness rather than simply by surface features of tasks.

Productive Thinking

A case for mathematics education based on structural or, as he called it, *productive thinking* was made more than sixty years ago by Wertheimer (1945/1961), one of the instigators of Gestalt psychology. For example, in justifying a formula for the area of a rectangle, Wertheimer contrasts several: the first is $a \times b$; another is the expression $\frac{a-b}{\frac{1}{b} - \frac{1}{a}}$ where a and b denote the

length and width (1961, p. 29). In every case the formula gives the same result as $a \times b$, but whereas it has no apparent connection with area, the familiar formula can be shown to be related to the process of dividing a rectangle into constituent units of area, and aggregating a total number of rectangles, each of unit area, which comprise the larger shape. Of course, a person who simply says that $a \times b$ gives the area of a rectangle without being able to explain why, may well be engaging in purely instrumental or procedural thinking. However, Wertheimer's point is that the second formula and its variations (replacing the subtractions by additions, or becoming even more convoluted), while always giving a correct value of the area of any rectangle, cannot, as presented, be related to any pertinent feature of the shape under consideration³. In other words, they have no structural relationship to the area of a rectangle. They are not expressions of the area, though they may be correct expressions for the area. To paraphrase Wertheimer, structural explanations must have the potential of referring to perceptual-structural features of what is being explained.

We take Wertheimer as supporting the view that to develop a productive disposition almost certainly requires more than success at the application of memorized methods to routine tasks. It depends on a developing identity as someone who engages with mathematical problems not simply as relaxing pastimes like crossword puzzles, but through appreciation of mathematical structures and relations between structures.

Instrumental and Relational Thinking

Using Skemp's (1976) distinction between relational and instrumental thinking, it is easy to point to instances where a student says, "This is how you do it, but don't ask me to explain why". For example, Fischbein and Muzicant (2002) in their study of equivalent algebraic expressions provide several very clear examples where students used procedural or instrumental thinking when asked to decide whether successive steps carried out on

³ Jo Tomalin and Sue Elliott (private communication) have recently found a geometrical justification of the formula as given.

simple algebraic expressions were correct and led to equivalent expressions. They reported on some Year 9 and 10 students, who, having completed several years of standard secondary school algebra, were given a statement

$$p(x) = \frac{2-x}{3} + \frac{2+x}{2} + 4$$

and presented with a simplification carried out by a hypothetical student Dan who had transformed this expression to become $p(x) = 2(2-x) + 3(2+x) + 24$. Students were then asked, "Was Dan correct? Yes/No." In subsequent interviews with students, the authors noted that many students agreed unhesitatingly with what Dan had done, accepting the elimination of the common denominator. Common responses to this and related questions were, "I eliminated the denominator. That is what we do in class".

Fischbein and Muzicant align their use of "structural" with Skemp's "relational" and their "procedural" with his "instrumental". Structural thinking, they suggest, needs to be defined with respect to "axioms, theorems, definitions, general concepts and properties (which) control the interpretation and use of more specific concepts and problem solving procedures" (p. 51). Structural thinking is the disposition to use, explicate and connect these properties in one's mathematical thinking.

Conclusions

Many authors in the past have drawn attention to learners' capacities to think relationally or structurally. However, it is not enough for a teacher to be aware of structure, whether arithmetic, geometric or some combination. It is certainly not wise to perform a didactic transposition on structural relations in order to try to convert them into instructions to learners in how to behave, that is, how to answer assessment questions on structure. Appreciating mathematical structure, and making use of it, is not a technique or a procedure to be taught alongside addition and subtraction. Rather, mathematical structure is an awareness which, if it develops in and for students, will transform their mathematical thinking and their disposition to engage. This can only happen if teachers are themselves not only aware of structural relationships, but have at hand strategies and tactics (such as those described in this paper) for bringing structural relationships to the fore. We maintain that this applies at every age.

Teachers who are themselves explicitly aware of structural relationships, who are aware of perceiving situations as instances of properties (rather than as surprising and unique events), are in a position to promote similar awareness in their learners. They can urge their learners to justify their anticipations and actions on the basis of properties which have been discussed and articulated, rather than on the basis of inductive-empirical experience. Working in this way could promote and foster the development of a structural or mathematically sound epistemological stance in preference to empiricist (it always seems to work), socio-cultural (adults act as if it works), or agnostic (it just is) stances.

While over-concentration on procedures usually diverts attention away from underlying structure, we have argued that the so called procedural-conceptual divide is a consequence of pedagogical decisions rather than a

necessary psychological experience, and that making pedagogical decisions from a structural point of view can overcome many of the difficulties arising from over-reliance on procedures or conceptual thinking exclusively. Furthermore, treating some learners as incapable of understanding structure simply reinforces their natural focus on procedures rather than directing their attention to conceptual structure.

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