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# Icelandic 5th-Grade Girls' Developmental Trajectories in Proportional Reasoning

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Understanding of ratio and proportion is critical to the development of higher level mathematical skills. Following Carpenter, Gomez, et al.'s (1999) proposal of a four-level trajectory in the development of proportional reasoning, a 12-week investigation was undertaken of the developmental trajectory of proportional reasoning of girls in two 5th-grade classes in Iceland. Students in these classes were accustomed to instructional practices that encouraged them to devise and explain their own solutions to mathematical problems. Results of the study confirm the learning trajectory with the addition of a further distinct level of development between Level 2 and Level 3. Results showed that girls moved easily, with minimum scaffolding, from Level 1 to 2 and from Level 2 to 3. The transition to Level 4, which involves explicit awareness of 'within' and 'between' multiplicative relationships, took greater time and effort. Teacher awareness of the four-level learning strategy, with the new emerging Level 3, assists in the design of appropriate problems, class structure, and teaching strategies. Building on Lamon's notions of unitizing and norming as by Carpenter et al.'s (1999) developmental model, this study contributes to our comprehension of students' understanding of proportionality and how it develops.

An understanding of ratio and proportion is critical in the development of higher level mathematical reasoning (Hiebert & Behr, 1988; Lesh, Post, & Behr, 1988; Resnick & Singer, 1993). Three of the most useful types of elementary mathematical thinking relevant for the day to day world are (1) proportional thinking, (2) estimation, and (3) mathematical modeling activities which align with conceptual development in proportional thinking (Sriraman & Lesh , 2006). Although young children demonstrate practical understanding of foundational ideas in proportional reasoning, students are slow to attain mastery of these important concepts. This study investigates the developmental trajectory of proportional reasoning in girls in two 5th-grade classes in Iceland. It is a replication and an extension of a smaller study conducted by the first author and others in one combined 4th- and 5th-grade classroom in the United States over a two week period (Carpenter, Gomez et al., 1999). The four-level learning trajectory that is used as a framework emerged originally from this previous study. The present study collected data during a 12-week instructional unit in two 5th-grade classrooms in Iceland.

# Influences on Children's Proportional Reasoning

Studies of proportional reasoning development provide evidence for a range of influences on students' thinking about proportionality. These influences play varying roles in the development of students' understanding of rational numbers and proportion. Among these influential factors are an understanding of the contextual structure and number structure of such problems. Contextual structure refers to the situation described in the problem statement, while number structure refers to the multiplicative relationships within and between ratios. Multiplicative relationships can be integer or noninteger. A 'within' relationship is the multiplicative relationship between elements in the same ratio, whereas a 'between' relationship is the multiplicative relationship between the corresponding parts of different ratios (Figure 1). In this paper ratio is defined as the relationship between two quantities that have two different measure units. Understanding these relationships is a key marker in the development of proportional reasoning.

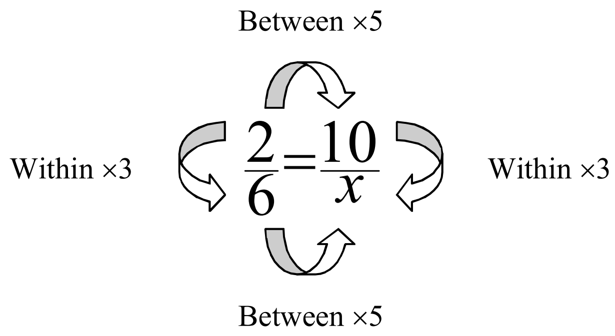



*Figure 1.* Within and between multiplicative relationships.

## *Students' Problem-Solving Strategies*

Researchers have suggested that growth in students' understanding of proportional reasoning can be described as a learning trajectory (Carpenter, Gomez, et al., 1999; Inhelder & Piaget, 1958). The term 'learning trajectory' refers to a predictable pattern of development in students' understanding of proportion. As students' strategies for solving problems become more mathematically sophisticated, their ability to solve difficult problems also develops. There is no consensus on whether the framework of Carpenter,

Gomez, et al. is simply a classification schema for students' solution strategies or whether the framework presents a developmental trajectory. We interpret it as the latter. We see every indication that describing each of the proposed levels of development in conceptual terms corresponds to distilling the strategies that students can or cannot employ to solve problems. Prior research such as that involving CGI has analyzed and categorized students' reasoning based on the strategies they employ in problem solving (Carpenter, Fennema, Franke, Levi, & Empson, 1999).

## *From Qualitative to Multiplicative Reasoning*

Studies of individual cognition and the development of proportional reasoning have identified three levels of strategies that students use as they grow in understanding of proportional relationships. These are qualitative; build-up reasoning or strategy that is based on repeated addition of the given ratio; and multiplicative reasoning (Behr, Harel, Post, & Lesh, 1992; Inhelder & Piaget, 1958; Kieren, 1993; Resnick & Singer, 1993). This section describes how each strategy might be applied to a missing-value task. The following problem is a missing-value problem.

*It is lunchtime at the Humane Society. The staff has found that 8 cats eat 5 large cans of cat food. How many large cans of cat food would the staff members need to feed 48 cats? (In an algebraic equation:* $\frac{8}{5} = \frac{48}{x}$ *.)*

Research with preadolescent students indicates that they have an informal and qualitative understanding of proportion long before they are capable of treating the topic quantitatively. Qualitative strategy is based on an intuitive knowledge of relationships without numerical quantification (Kieren, 1993). In other words, without being able to arrive at a specific number answer, students might conclude that for the above problem, they would need a lot more cans because 48 cats is a lot more than 8 cats.

Next is build-up reasoning, a strategy that requires quantification of the ratio relationships. To solve the problem above, a student might use a build-up strategy like the one in Figure 2 to arrive at the solution of 30 cans (Figure 2).

| | | | | | | |
|---|---|---|---|---|---|---|
| Cats | 8 | 16 | 24 | 32 | 40 | 48 |
| Cans | 5 | 10 | 15 | 20 | 25 | 30 |

*Figure 2.* Build-up strategy in the form of a ratio table for the problem $\frac{8}{5} = \frac{48}{x}$.

Build-up strategies are often observed during childhood and adolescence and are the dominant strategy for many students of these ages (Kaput & West, 1994; Tourniaire & Pulos, 1985). Resnick and Singer (1993) caution that build-up strategies enable students to solve ratio problems without recognizing the multiplicative relationship inherent in proportion. While the use of build-up strategies is an important milestone on the path toward proportional reasoning, developing more sophisticated reasoning is crucial for solving more complex problems and understanding multiplicative relationship. Multiplicative strategies allow students to recognize multiplicative relationships within and between ratios and can be applied to both integer and noninteger problems.

When solving simple proportion problems, two types of multiplicative strategies have been identified: 'within ratio' and 'between ratios' ( Noelting, 1980a; Vergnaud, 1983). The within-ratio strategy is based on applying the multiplicative relationship within one ratio to the second ratio to produce equal ratios. The between-ratio strategy is based on determining the multiplicative relationship between corresponding parts of the two ratios to create equal ratios (see Figure 1).

Earlier research focused on within-ratio and between-ratios strategies in analyzing students' reasoning (Abromowitz, 1975). Taking a different perspective, Lamon (1993, 1994) proposed two processes, unitizing and norming, as central to the development of proportional reasoning. Unitizing involves the construction of a reference unit from a given ratio relationship. Norming refers to the reinterpretation of another ratio in terms of that reference unit (Lamon, 1995):

> The unitizing and norming process may be an important mechanism by which more advanced reasoning evolves. Understanding ratio and proportion depends on one's ability to view a relationship as a single quantity and then to operate with it…. understanding the relative nature of quantities in a ratio may be just another level of sophistication built upon an already complex foundation of unitizing processes. (p. 113)

Consider the previous problem about the Humane Society. Using norming and unitizing, a student might interpret the target ratio as a

multiple of the given ratio $\frac{8}{5}$. Therefore, noting that $8\times 6=48$, she recognizes that she needs six groups of the 8-to-5 ratio unit in order to get an answer for 48 cats. For their calculations, students might use methods such as build-up strategies or direct multiplication.

A student using a between strategy, on the other hand, would consider a single quantity in the given ratio and operate on that quantity, recognizing that the same operation must apply to the corresponding quantity in the second ratio. Referring to the same equation, $\frac{8}{5}=\frac{48}{x}$, the student multiplies $8\times 6$ to get 48, then multiplies $5\times 6$ to get the answer, 30. The key difference is that when unitizing and norming, the student thinks of the ratio as a single, complex unit. The student can operate on the unit $\frac{8}{5}$ by adding, multiplying, or reducing, but each operation is understood as creating a new unit that preserves the relationship within the given ratio.

## *Developmental Trajectory*

Lamon's (1993, 1994) work provides a basis on which to create a fuller picture of the development of proportional reasoning. A model of developmental trajectory based on this work moves beyond the simple characterization of students' reasoning according to qualitative, build-up, and multiplicative strategies. Using Lamon's (1994, 1995) operation of unitizing and norming, Carpenter et al. (1999) identified four levels of development in proportional reasoning. The following sections outline the four levels of reasoning and give examples of possible strategies for each level.

At Level 1, students show limited ratio knowledge. They either perform random calculations or focus on the additive difference between the components of the ratios. For example, attempting to solve $\frac{8}{5}=\frac{48}{x}$, a student might see 8 as 3 more than 5 and so determine $x$ to be 45.

Level 2 is characterized by perception of the ratio as a single unit. Students at this level are able to combine the ratio units together by repeated addition of the same ratio to itself or by multiplying that ratio by a whole number, but they cannot solve proportion problems in which the given ratio has to be partitioned, such as problems in which the target ratio is a noninteger multiple of the given ratio parts of the ratio (e.g., $\frac{8}{12}=\frac{42}{x}$ or $\frac{8}{3}=\frac{2}{x}$).

Students reasoning at Level 2 primarily use a build-up strategy in which they build up the ratio with addition, multiplication, or some combination of operations. Classroom observations show how students move back and forth between additive and multiplicative strategies, depending on the

particular problem and their facility with whole-number multiplication.

At Level 3, as at Level 2, the ratio is thought of as a single unit. However, students at Level 3 can scale the ratio by nonintegers. This allows them to solve more complex problems involving both integer and noninteger multiplicative relationships. A typical Level 3 strategy combines a build-up strategy using either addition or multiplication with the reduction of the given ratio (see Figure 4). Consider the following problem:

*A hiking group is organizing a field trip, and they estimate that it will take 8 hours to walk 12 km. How many km did the group walk if they walked for 42 hours? (In an algebraic equation:* $\frac{8}{12} = \frac{42}{x}$ *.)*

Working towards the 42 hours required, a student could build up the unit from $\frac{8}{12}$ to $\frac{40}{60}$. As a separate operation, she then reduces the $\frac{8}{12}$ by four to get $\frac{2}{3}$, which she adds to $\frac{40}{60}$ in order to complete the problem (Figure 4).

| | | | | | | |
|---|---|---|---|---|---|---|
| Hours | 8 | 16 | 32 *(+ 8)* | 40 | 2 *(+40)* | 42 |
| Km | 12 | 24 | 48 *(+ 12)* | 60 | 3 *(+60)* | 63 |

*Figure 4.* Build-up strategy in the form of a ratio table for the problem $\frac{8}{12} = \frac{42}{x}$.

Other students might divide each component of the given ratio by some integer (in this case, 4) and create the unit $\frac{2}{3}$. They could then multiply by 21 to reach 42 hours, and find the solution of 63 kilometers. Or, students could multiply the given ratio $\frac{8}{12}$ by a fraction or a mixed number (in this case, by $5\frac{1}{4}$) to find the solution.

Students at Level 4 think of ratios as more than just unit quantities. In problem solving, they recognize the 'within' relationship in each ratio and the relationship 'between' the corresponding terms of both ratios. This understanding provides flexibility in student approaches to solving ratio and proportion problems. They are able to identify the relationships that will make the computation easiest. Thus, at Level 4, students are not limited to building up or operating on the unit as a whole. They are able to focus on the numbers and the numerical relationship in the problem rather than the contextual structure, in order to determine an appropriate and efficient strategy.

Classroom research reported in this study was designed to validate the learning trajectory of four levels of reasoning proposed by Carpenter et al. (1999) and to identify key factors in student transition from level to level. The following questions are considered:

How well does the four-level learning trajectory describe the pathway of a population of Icelandic girls before, during, and after they have engaged in a unit focused on proportional reasoning?

How does instruction focused on students' reasoning help students make the transition from level to level?

# Method

The data used in this study were collected during a 12-week instructional unit in 5th grade jointly designed by the first author and two 5th-grade teachers. Instruction took place from late January to mid-April. At this time, students were in fifth grade for a semester.

## *Participants and Settings*

The participants in the study were 26 5th-grade (age 10-11) girls in two classrooms at one of Reykjavik's public schools. The school's student population was predominately Caucasian with varying socioeconomic status and ability levels. The language spoken in the classroom was Icelandic. Both teachers, Karen and Margret, were experienced cognitively guided instruction (CGI) teachers and had participated in workshops on CGI. Classroom norms had already been established among students and teachers when the data were collected. One crucial expectation in the class was that students be able to explain their problem solving strategies. They were also expected to listen while other students explained their thinking and to ask clarifying questions of each other if needed. Students were often asked to look for alternative strategies to solving problems.

## *Data Collection*

Every math class throughout the course of the study was observed by the first author, taking on the role of 'participant observer' (Eisenhart, 1988). During data collection, students worked on 24 word problems that were created during instruction. To ensure variation in contextual structures, Lamon's categories of semantic types (1993b) were used. The tasks required proportional reasoning for both integer and noninteger multiplicative relationships. Sets of problems were developed which included two or three problems with the same contextual structure but different multiplicative

relationships. Problems were designed to further students' understanding of proportion and to aid recognition of the multiplicative relationships within and between the two ratios in the problem. Problem sets were designed to distinguish between Level 2 and Level 3 students and between Level 3 and Level 4 students.

Students were given a paper-and-pencil test prior to the study and again after the instructional period. The pretest and the posttest were designed using the same criteria as those for the instructional problems to check understanding of number structure and proportional reasoning. The pretest comprised 14 missing-value problems. The posttest comprised 12 missing-value problems. Students' problem solution strategies were collected.

During class students worked on problems individually and in groups and were encouraged to interact with their tablemates to compare strategies. Some groups were same-gender groups; others were mixed-gender groups. After each problem was completed, students presented their strategies to the whole class. Discussions took place in which the students were asked questions about the strategy, the solution, and the thinking process behind the strategy. Teachers also asked students to reflect on strategies that had been shared and compare them to their own. Written work and artifacts produced by students were collected. All whole-classroom discussions were videotaped and transcribed. Approximately 40% of students participating in group work at any given time were videotaped or audiotaped, and their discussions transcribed.

## *Data Analysis*

Data were analyzed to answer each proposed research question. The girls' written work—pretest, written work during classroom instruction, and posttest—was analyzed following the framework of the proposed four-level developmental trajectory. Differences in problem solving strategies used to distinguish students at different levels are described above. Group work was analyzed using the same criteria. The pretest offered insight into the girls' reasoning at the start of the unit. The posttest showed the level of reasoning after the unit of instruction.

The study looked for evidence that students were reasoning in ways not predicted by the proposed trajectory. An example of such evidence would be a girl who was able to solve complex problems in class (e.g. $\frac{3}{9} = \frac{x}{33}$, $\frac{3}{1950} = \frac{8}{x}$, and $\frac{7}{4550} = \frac{683}{x}$) by using the multiplicative relationship within the given ratio and applying it to find the unknown in the second ratio, without showing an understanding of the between relationship. Such a case would suggest that the four-level developmental trajectory was not robust. Another example of reasoning not predicted by the trajectory would be a strategy

that provided evidence of students otherwise at Level 2 or 3 treating the ratio as two different quantities. Such a strategy would imply that some girls were not working with the ratio as a unit (characteristic of Level 2 and 3 reasoning) but rather thinking about the unit as two different quantities on which to operate. Transcriptions from group work and whole classroom discussions provided data to show whether students' discussions corresponded with the levels of the developmental trajectory. Comparisons with transcripts were used to validate and explore student levels of thinking evidenced in written work.

Whole classroom discourse provided evidence of student strategies and was used to examine the robustness of the four-level developmental trajectory. An example of evidence confirming or contradicting the proposed trajectory could be found in a student's use of the multiplicative relationship within the given ratio in problems with an integer relationship within the ratio. If the student did not also show an understanding of the relationship between the ratios, then her use of the within relationship contradicted the learning trajectory. On the other hand, if students' discussions indicated that they used the within relationship because it was easier to use, Level 4 reasoning would be demonstrated, and the proposed trajectory would be supported.

Classroom instruction was analyzed to help understand changes in student thinking and strategies from pretest to posttest. Transcripts of classroom instruction and small group discussions provided evidence of the aspects of instruction and interaction that led to growth in student understanding.

## Results

The following section describes the relationship between the trajectory and actual events and progress in the classroom. Typical examples of students' strategies and explanations of reasoning are provided, as are examples of students' strategies and discussions while in transition from one level to the next.

### *Girls' Reasoning Prior to and After Instruction*

Table 1 provides a categorization of student levels of proportional reasoning on the pretest and the posttest. One unexpected result was that classification of students' solutions showed the need for a transitional level described as 'emerging Level 3'. Test results demonstrated that students on Level 2 were not able to solve any of the more complex problems that emerging Level 3 students were able to solve. On the other hand, emerging Level 3 students were not able to solve still more complex problems that

Level 3 students were able to solve without difficulty.

Table 1
*Number of Students at Each Level of Proportional Thinking at the Pretest and Posttest (N=26)*

| Posttest level | | | | |
|---|---|---|---|---|
| Pretest level | Level 1 | Level 2 | Level 3B | Level 4 |
| Level 1 | 1 | 1 | 7 | 0 |
| Level 2 | 0 | 0 | 9 | 1 |
| Level 3A | 0 | 0 | 5 | 1 |
| Level 3 | 0 | 0 | 0 | 1 |

Pretest and posttest were designed to discriminate between students at different levels of reasoning. Problems that could be solved by students reasoning on Level 2 had an integer relationship between the ratios and involved enlarging (e.g., $\frac{2}{8} = \frac{x}{24}$). Problems solvable by the transition Level 2 to Level 3 group (emerging Level 3) included all problems at Level 2, and problems with a scale-down number structure such as $\frac{8}{24} = \frac{2}{x}$. To solve these problems, one-step scaling down was needed, whereas other noninteger problems involved both building up and scaling down. The difference between Level 2 and Level 3 reasoning is the need to scale down or reduce the given ratio. During the emerging Level 3 stage, students are able to scale down by whole numbers but cannot use their knowledge of scaling to calculate noninteger problems. This is a distinct level of development between Levels 2 and 3. Problems that could be solved by students reasoning on Level 3 included all problems at lower levels and problems that have a noninteger relationship between the ratios, for example: $\frac{5}{6} = \frac{x}{21}, \frac{15}{10} = \frac{6}{x}$.

The data show that students reasoning on Level 2 were not able to solve the additional problem that emerging Level 3 students were able to solve, nor were the emerging Level 3 students able to solve the additional problems that Level 3 students were able to solve. This was true in all cases at both pretest and posttest.

No specific problems were designed to distinguish between Level 3 and Level 4. The difference was in the strategies that students used to solve the problem. Problems with an integer relationship within a ratio and a noninteger relationship between ratios (e.g., $\frac{4}{12} = \frac{11}{x}$) were particularly useful for making that distinction.

Table 1 also shows substantial growth from the pretest to the posttest in

reasoning skills, with progress a good match to the four-level trajectory, with the addition of the new 'emerging Level 3' (Level 3A). Students on each level could solve all of and only the problems for that level. Table 1 shows the number of students at each level of reasoning at the time of the pretest and at the time of the posttest. Almost all of the girls reached Level 3 reasoning. They could solve most of the problems but there was no evidence that they recognized the multiplicative relationships at an abstract level.

On the pretest, 35 % of the girls displayed Level 1 reasoning (Table 1). The most common strategy used was an additive differences strategy. Students found the additive difference within or between ratios and applied that difference within or between ratios to find the unknown quantity. For example, a problem represented in an algebraic equation as $\frac{2}{8} = \frac{x}{24}$ would result in an answer of 18 because $8 - 2 = 6$ and $24 - 18 = 6$, or $24 - 8 = 16$ and $18 - 2 = 16$. Around 40% of the girls exhibited Level 2 reasoning. The problems they could solve had an integer relationship between the ratios (e.g., $\frac{2}{8} = \frac{x}{24}$ see Table 1). On the other hand, they could not solve a problem that involved a noninteger relationship between ratios (e.g., $\frac{4}{12} = \frac{11}{x}, \frac{12}{24} = \frac{3}{x}$). Twenty-three percent of the girls were emerging Level 3. They could only solve problems such as $\frac{8}{24} = \frac{2}{x}$. One girl showed Level 3 reasoning on her pretest. She was able to solve problems such as $\frac{4}{12} = \frac{11}{x}$. All of her strategies represented build-up strategies, and she showed no sign that she recognized the nature of the multiplicative relationship that exists in a proportion.

Only 3 girls reached Level 4 thinking, whereas more than 80% reached Level 3 thinking. This is evidence that there is substantial jump to Level 4 thinking, which involves explicit recognition of within and between multiplicative relationships. The 2 girls who reached only Level 1 and 2 after instruction (see Table 1) had recognized learning disabilities prior to the study.

## *Student Transition Between Levels*

*Transition from Level 1 to Level 2.* The transition from Level 1 to Level 2 followed quickly on basic instruction. Of the nine girls who were categorized as being at Level 1 at the pretest, six solved the first problem of the teaching unit ($\frac{2}{6} = \frac{x}{36}$) successfully. Furthermore, 10 girls solved the second problem ($\frac{5}{8} = \frac{x}{48}$) successfully. The two remaining girls had learning disabilities identified prior to the study. Both had difficulty grasping the idea of proportion. Only one of them reached Level 2 on her posttest.

Nina represents close to 30 percent of the students. She was a typical Level 1 student at the time of her pretest. She solved problems by finding the additive difference between the ratios and then applying that difference to the second term in the known ratio to find the unknown term in the target ratio. She showed no signs of distinguishing integer and noninteger relationships and consequently her strategies did not vary depending on the multiplicative relationships.

In the beginning of the unit, Nina needed a little scaffolding to help her move away from the additive thinking she applied to the problem represented algebraically as $\frac{2}{6} = \frac{x}{36}$.

| | |
|---|---|
| Teacher: | What if you had 4 cans of food, how many cats could you feed? |
| Nina: | 8 cats. |
| Teacher: | Okay, we know that 2 cans of cat food can feed 6 cats. We get 2 more cans, and can they only feed 2 more cats? |
| Nina: | No, 2 cans can feed 6 cats, not 2. |
| Teacher: | Okay, what does that mean, then? |
| Nina: | Well, it is like if 2 cans can feed 6 cats, then another 2 cans can feed another 6 cats. |
| Teacher: | Think about that more and how you can solve your problem differently. I will come back to you. |

The teacher left Nina to grapple with her new ideas about the problem. At sharing time, Nina had not yet figured out how to go about solving the problem with her new knowledge. A couple of the strategies that were shared were build-up strategies in which students took the given unit $\frac{2}{6}$ and built it up unit-by-unit to reach the target number. Nina liked that strategy and utilized it with success. When the teacher returned to Nina, she had solved the problem using a build-up strategy. When the teacher asked her to explain what she had done, it became clear that she understood clearly what the numbers in the build-up strategy stood for.

Nina: First there were 2 cans and 6 cats, then next there would be 4 cans for 12 cats and—

Teacher: And why is that?

Nina: It's like first there were 2 cans and 6 cats, then there were 2 more cans and 6 more cats would eat that, and that is like having 4 cans and 12 cats.

When Nina started working on the second problem, she solved the problem with a build-up strategy without further teacher input.

Guidance from teachers and class discussion of different strategies conveyed new knowledge and insight, enabling students to move to higher levels of proportional reasoning. Less advanced students learned from listening to other students explain more efficient strategies.

*Transition from Level 2 to Level 3.* The transition from Level 2 to Level 3 also came without much difficulty for most students. Of the ten girls at Level 2 in the pretest, eight solved the first noninteger problem of the unit successfully ($\frac{6}{12} = \frac{15}{x}$). Three of the nine Level 1 girls also solved that problem correctly. The second noninteger problem of the unit ($\frac{3}{9} = \frac{x}{33}$) was more challenging. Half the Level 2 girls needed some assistance from the teacher or a peer to finish the problem. Three girls from the Level 1 group were also successful with some assistance. Almost 90% of the girls exhibited Level 3 thinking after the first week of instruction.

Gudrun was typical of 35% of students, starting at Level 2 and moving to Level 3 on the posttest. Figure 6 describes her reasoning in solving following problem:

*Jon, Gudrun, Alex, and Nina are planning to backpack in Iceland this summer. When planning they estimate that in 3 hours they can cover 9 km. If the walk at the same rate, how many hours will it take them if the trek is 33 km long* ($\frac{3}{9} = \frac{x}{33}$)?

| | |
|---|---|
| Hours / Km | I know 1 hour is 3 km. |
| 3 : 9 | |
| 6 : 18 | I took 1 hour and 3 km and then it was 11 hours and 33 km. |
| 9 : 24 | |
| 12 : 36 → 11 : 33 | |

*Figure 6.* Gudrun's strategy and explanations for solving $\frac{3}{9} = \frac{x}{33}$.

| | |
|---|---|
| Gudrun: | I found that like 3 are 9, 6 are 18, 9 are 27 and 12 are 36? |
| Teacher: | Yes. |
| Gudrun: | And 1 hour is 3 km. But I don't understand—it does not go up to 33. |
| Teacher: | Can you use this 1 hour and 3 km? |
| Gudrun: | Yes [pause], I see, it is like take this [pause] of course it is 11. Eleven then I just take 3 km here. |
| Teacher: | Yes, and then you will get? |
| Gudrun: | Thirty-three. The answer is 11. |
| Teacher | Can you write what you did with your solution? |
| Gudrun | Yes, I took 1 hour away. |

Gudrun used her familiar build-up strategy to start solving the problem. She soon realized that it did not work; her build-up passed the target number. She understood the ratio of 1 hour to 3 km but she did not know how to use that information to reach 33km. Gudrun had to find a way to take away 3 km. She needed little assistance to understand how she could use the ratio of 1 hour to 3 km, taking one unit of 1 hour and 3 km from 12 hours and 36 km to reach her answer of 11 hours.

In the amount of support required to finish this problem Gudrun was typical of the the 11 girls. All of them used a build-up strategy to reach the target number. The difficulty involved scaling down the 3: 9 ratio into equivalent ratios of 1:3, and how to use the 1:3 ratio to reach the answer.

*Transition from Level 3 to Level 4.* The greatest challenge for the girls was the transition from Level 3 to Level 4; only three girls reached Level 4 reasoning. Agnes, one of these 3 girls, was at emerging Level 3 on the

pretest. She was the only student in that group who solved $\frac{10}{4} = \frac{35}{x}$. The strategy involved 'halving' the given ratio $\frac{10}{4}$ to $\frac{5}{2}$ and then using build-up to reach the correct solution. Students can more easily recognize a relationship that involves a multiple of $\frac{1}{2}$ than relationships involving other fractions. Agnes could not solve problems that had a more complex multiple than $\frac{1}{2}$, such as $\frac{4}{12} = \frac{11}{x}$, even though the multiplicative relationship within the given ratio was a whole number.

During the course of the study, Agnes quickly adopted a flexible approach to problem solving. She moved comfortably between strategies, focusing on number structure and the multiplicative relationship between or within ratios. During instruction, Agnes developed Level 4 thinking and demonstrated that knowledge in her posttest.

The following example demonstrates Agnes's strategy solving a scaling problem represented algebraically as $\frac{8}{12} = \frac{x}{27}$. Agnes looked first at the relationship between 12 and 27 and tried various multiples. After several trial-and-error strategies to discover how 12 could get to 27, she gave up. Agnes then looked for another way to solve the problem by focusing on the within relationship between 12 and 8 (see Figure 7).

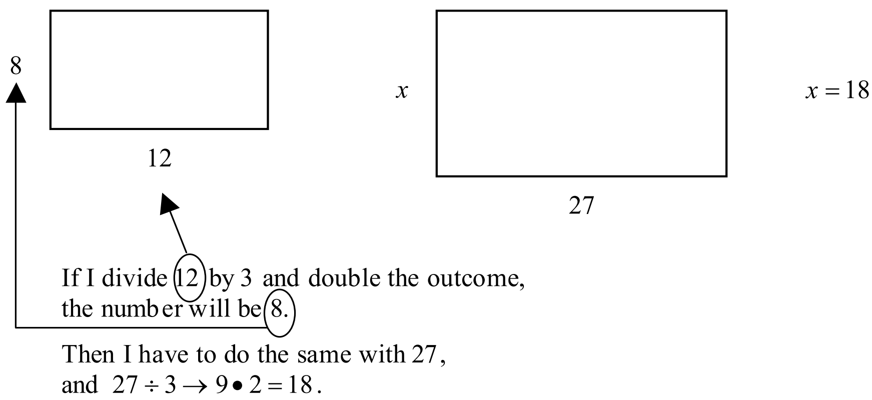



*Figure 7.* Agnes's strategy solving Problem 15a, $\frac{8}{12} = \frac{x}{27}$.

Agnes's use of this strategy shows she is working with familiar multiplication facts. She concluded that finding the relationship between 8 and 12 was easier for her than finding the relationship between 12 and 27. On the posttest, she further demonstrated flexible thinking in evaluating

number structure to determine which relationship would offer a more efficient solution. Figure 8 illustrates Agnes's solution strategy for solving the problem represented algebraically as $\frac{16}{4} = \frac{x}{25}$ on the posttest.

25 is not in the 4 times table
but 25 is in the 1 times table.

$\frac{16}{4} = 4$ and $\frac{4}{4} = 1$

$4 \bullet 4 = 16$

$25 \bullet 4 = 100$

Therefore, it is 25 to 100.

*Figure 8.* Agnes's strategy solving Problem 4 on the posttest, $\frac{16}{4} = \frac{x}{25}$.

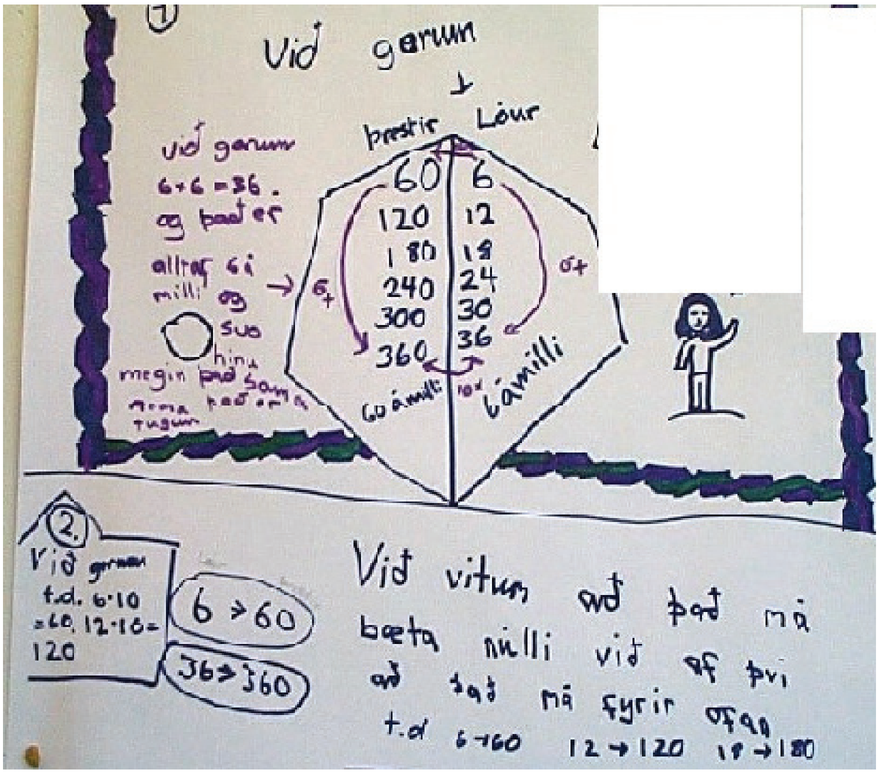
Agnes's thinking on the posttest reaches Level 4. According to the learning trajectory, students at Level 4 no longer think of ratios exclusively as unit quantities. They understand proportion in terms of multiplicative relations and recognize the relation both within the terms of each ratio in the proportion and between the corresponding terms of the ratios. This understanding allows flexibility in students' approaches to solving ratio and proportion problems.

The final two problems during instruction had an explicit whole number relationship both within the given ratio and between the ratios ($\frac{6}{60} = \frac{36}{x}, \frac{9}{99} = \frac{72}{x}$). These were designed to provide insight into students' understanding of the multiplicative relationships and, specifically, demonstrate whether students could recognize both the within and between ratio relationships. Students were asked to find two different strategies to solve these problems.

The first strategy used by the girls was a build-up strategy. With the visual image of the ratio table, they were able to look for patterns and relationships. Nina and Ester worked together on the problem $\frac{6}{60} = \frac{36}{x}$ (see Figure 9). After long discussions, they were able to explain their thinking to the class:

| | |
|---|---|
| Ester: | We did here—like 6, 12, 18, 24—and this is always 10 times more. |
| Teacher: | 6, 12, 18, 24 are 10 times bigger? |
| Nina: | No. Okay, 6 is $6 \times 10$ is 60, and $12 \times 10$ is 120. |
| Teacher: | And then you are looking at what in your table? |
| Nina: | Umm, that it is always 10 times. |
| Teacher: | What numbers are you looking at? |
| Nina: | This one 6, 60; then if we look at this, it is always 6 between and here is 60 between. |
| Ester: | We also did, like, $6 \times 6$ to get 36. |
| Teacher: | And what did you do next? |
| Ester: | Then we did $6 \times 60$, and then it was 360. |
| Teacher | So when you look at your table horizontally, then you say 10 times but when you look at it vertically, then it is 6 times? |
| Nina: | Yes. |
| Teacher: | Did you see one relationship before the other? |
| Nina: | No, it kind of all goes together. |

Neither Nina nor Ester was able to use their knowledge of within and between relationships on their posttest. All of their strategies on the posttest bear evidence of Level 3 thinking. Nevertheless, classroom discussion of the two different strategies and the two multiplicative relationships seemed to play a role in the collective construction of the multiplicative relationships involved in Level 4 thinking. This illustrates that less advanced students can learn from discussing more sophisticated or more efficient strategies than they commonly use and how a class can build on ideas that are discussed among its members.




Translation:
1) We did 6 x 6 = 36, and that is always 6 between, and then on the other side is the same except in decade numbers.
2) We did, for example, 6 x 10 = 60 and 12 x 10 = 120

  6 $\rightarrow$ 60

  36 $\rightarrow$ 360

*Figure 9.* Nina and Ester’s solution strategy for Problem 20, $\frac{6}{60} = \frac{36}{x}$ .

A more advanced strategy came from Agnes and her two group-mates, Hanna and Heba (Figure 10).

| | |
|---|---|
| Agnes: | We found that 36 divided by 6 is 6, and this 36 [a] is the second group of cardinals (birds in the problem), and this 6 [b] is the first group of cardinals. This 6 [c] is only that we had to find how many times we had to multiply the 60 so it will be in the same proportion. Then we also did 6 times 10 is 60, and therefore we also had to do that to 36, and 36 times 10 is 360. |
| Teacher: | Why is that a different strategy? |
| Agnes: | Here in Number 1, we are finding the relationship between 6 and 36 but in Strategy Number 2, we are finding the relationship between 6 and 60. |
| Teacher: | So what is the ratio, then? |
| Agnes: | Between the small and the big group is 6 times 6 to 36, and between cardinals and robins is 10 times 6 to 60. |

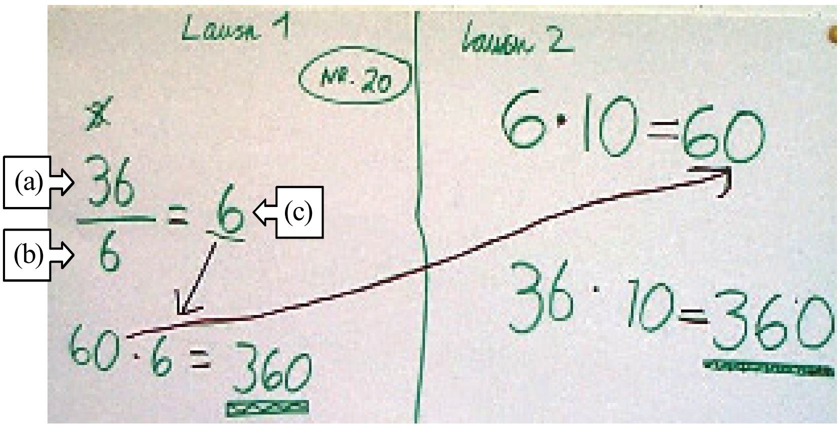



*Figure 10.* Agnes, Hanna, and Heba's strategy solving Problem 20, $\frac{6}{60} = \frac{36}{x}$ .

Their explanation does not refer to the ratio as a unit; rather, the students' focus is on the multiplicative relationships and how the numbers relate to each other.

The following day there were lively discussions about the ratio between the cardinals and the robins and the ratio between the small and the big group. As a class, students were able to support each other's reasoning, and articulated that for every one cardinal there were 6 robins; and for every bird in the small group, there were 10 birds in the large group. However, in the posttest, there was no evidence to suggest that individual students could use this knowledge of multiplicative relationships. Only 3 of the 26 girls showed

flexibility in their strategy use and were able to evaluate which multiplicative relationship was easier to work with.

## *Evidence Contradictory to the Four-Level Learning Trajectory*

The girls' strategies for problems 5, 7, and 8 ($\frac{3}{9} = \frac{x}{33}$, $\frac{3}{1950} = \frac{8}{x}$, $\frac{7}{4550} = \frac{683}{x}$ respectively) during instruction were examined closely to discern problem solving strategies used before and after instruction. The problems have an integer multiplicative relationship within the given ratio but not between the ratios. The strategies for problem 5 ($\frac{3}{9} = \frac{x}{33}$) were all consistent with the learning trajectory. For instructional problems 7 and 8 ($\frac{3}{1950} = \frac{8}{x}$, $\frac{7}{4550} = \frac{683}{x}$), all of the girls found the ratio unit (i.e., the price per ticket) but none of the strategies provided any evidence that the students were aware of the within relationship or applied that relationship to find the unknown in the target ratio. Rather, the students used the build-up strategy to solve the problem, using the ratio unit as their base unit. The contextual structure of both these problems (which asked for the price of movie tickets) may have had a significant influence on strategy.

There were a few incidents in which student explanations provided evidence that a between strategy was used; that is, the students found the multiplier between the ratios and applied that multiplier to the second term of the given ratio to find the missing term of the other ratio. Consider, for example, problem 4 ($\frac{5}{7} = \frac{x}{28}$): "Because I knew it was 28 km, I knew that 4 times 7 is 28, and then 4 times 5 is 20." Most students who had reached Level 3 explained their strategies and solutions for these problems in ways similar to the examples. The relationships between the ratios in these examples were easily within the girls' knowledge of number facts. When the multiple was more complex, students did not show any ability to use this strategy unless they had reached Level 4 reasoning. This might not contradict the learning trajectory because the simplicity of the relationship allowed the girls to recognize a familiar pattern.

## *Development of Proportional Reasoning and the Four-Level Learning Trajectory*

As mentioned before the learning trajectory originally emerged from a study conducted in one combined 4th- and 5th-grade classroom in the United States over a two week period. For the vast majority of the students in this study, reasoning and strategies aligned with the proposed trajectory, but with strong evidence for an additional level (emerging level 3) between Level 2 and Level 3.

Students reasoning on Level 1 used incorrect strategies on all problems. Students reasoning at Level 2 were able to solve problems with an integer relationship between ratios. Students on Emerging Level 3 were in addition able to solve problems involving one step partitioning. Students on Level 3 were able to solve more complex problems involving noninteger relationships between ratios. Students at Level 4 were distinguished by awareness of both within and between multiplicative relationships, and flexibility in strategy use. A similar pattern of learning development was found during instruction in Iceland as was evident in the United States' classroom in which the four levels were originally developed.

Students' explanations of their problem solving strategies also aligned with one of the fundamental claims of the four-level trajectory: that is, students initially think of the given ratio as a unit that is operated on as a whole. The ratio table, a tool reported as commonly used at early stages in proportional reasoning (Carpenter et al., 1999; Kaput & West, 1994), was used consistently by girls at early stages in this study. This confirms students' understanding of the ratio as a unit as described by Carpenter et al.

Students at emerging Level 3 successfully resolved, in addition to all Level 2 problems, problems with a scale-down number structure such as $\frac{8}{24} = \frac{2}{x}$ . The difference between Level 2 and Level 3 reasoning is the need to scale down or reduce the given ratio. During the emerging Level 3 stage, students are able to scale down by whole numbers but they cannot use their knowledge of scaling to calculate noninteger problems. Emerging level 3 is an identifiable level of development between Levels 2 and 3.

This study examined the robustness of the four levels of the developmental trajectory. For this population of students, the existence of the four levels of reasoning was validated, with the addition of a further level, that is, emerging level 3, between Levels 2 and 3.

## *Transitions*

The results support the claim that proportional reasoning is a difficult concept for students to master. Students moved quickly from Level 1 to Level 2, and without difficulty from Level 2 to Level 3A and from Level 3A to Level 3B. Moving beyond Level 3 was a much larger step than the previous transitions. One way to interpret these results is to look at them from the Vygotskyan perspective of the Zone of Proximal Development (ZPD; Vygotsky, 1978). That girls moved easily, with minimal scaffolding, from Level 1 to Level 2 and from Level 2 to Level 3, suggests that the knowledge needed to operate on those stages was within their reach (i.e., within their ZPD).

Classroom experience recorded in this study provides evidence that asking students to find different strategies to solve problems involving both between and within relationships was highly beneficial in assisting students to make the transition from Level 3 to 4. Students initially used a build-up strategy to look for patterns. These patterns later enabled the students to articulate the multiplicative relationships within and between ratios. Even so, at posttest, only three girls seemed to individually recognize the relationships both between and within ratios.

Carefully sequenced levels of complexity in problems were crucial to the development of proportional reasoning. Teachers implemented problems that were specifically designed to target students' reasoning at particular times. The four-level trajectory assisted teachers to predict what influence different number structures would have on students' reasoning and strategies and allowed them to plan effectively for student growth. Awareness of emerging Level 3 will enable educators to refine this process further.

Scaffolding that teachers provided during instruction further supported the development of proportional reasoning. The learning trajectory assisted teachers in understanding the difficulties associated with any given problem's number structure, and how students might react to new challenges and complex problems. Particularly in the case of girls who demonstrated problem solving strategies that are characteristic of Emerging Level 3, the teacher's role seemed crucial in providing scaffolding to support a transition to more sophisticated levels of reasoning. By recognizing the levels on which the students are operating, the teacher can strategically create problems and make instructional decisions to encourage a move to the next level.

Finally, the classroom norms of collaboration and discussion provided students with opportunities to raise questions and to challenge themselves and others. This study confirms that the sociomathematical norms (Yackel & Cobb, 1996) of including explanations, justifying rationales for strategies used, and mutual listening are significant in supporting students in developing proportional reasoning. Viewed through the lens of the sociocultural perspective of learning, this classroom structure allowed developing levels of reasoning, invisible in traditional approaches to problem solving, to come to the fore. As a result, teachers were able to provide appropriate support to help students make transitions from one level to the next.

One interesting result was the limited number of girls who reached level 4. Even after repeated discussions about multiplicative relationships, many girls preferred to use build-up strategies. This preference meant ratio tables were both a help and a hindrance in furthering the girls' development of

proportional reasoning. However, their reluctance to let go of well-understood strategies for one that was easier computationally may be viewed from a positive perspective. Boaler (1997) has argued that girls look for the reason behind each mathematical action. Rather than operating on Level 4 where they may have applied superficially simple operations, girls perhaps preferred to keep strategies for which they clearly understood the rationale.

## Conclusions

This study supported the four-level developmental trajectory, and at the same time provided clear evidence of a further distinct level between Levels 2 and 3. Pretesting and posttesting demonstrated the importance of teacher scaffolding and sociomathematical norms in supporting transitions along the learning trajectory. Study of more diverse student populations is needed to provide further verification of the four-level developmental trajectory, and to confirm the validity of emerging Level 3. Better understanding of the learning trajectory will assist teachers in making instructional decisions, and support improved learning outcomes for students.

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