Introducing complex numbers

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One of the difficulties in any teaching of mathematics is to bridge the divide between the abstract and the intuitive. Throughout school we encounter increasingly abstract notions, which are more and more difficult to relate to everyday experiences. The subtraction of two apples from six is simple; but the subtraction of six from two causes some ripples of uncertainty. The division of twelve by three can be related to the sharing of sweets amongst siblings; but the division of twelve by five takes us a little out of our depth. Fortunately the introduction of negative numbers takes care of our apples and rational numbers mollify our sweets.

By the time senior schooling is reached, students of mathematics will have no difficulty in executing the four operations of arithmetic. However, if we consider the fifth operation of arithmetic—the extraction of roots—then the students may no longer be sure of themselves. For example, taking the square root of four gives answers 2 and –2, but neither of these numbers when squared gives the answer –4. How then does one handle taking the square root of a negative number? Let us first examine a familiar approach to thinking about negative numbers.

Understanding negative numbers: The number line

An intuitive approach to understanding negative numbers is with the aid of the number line. Here numbers are considered to be vectors: they have a magnitude (e.g., 2 or 17) and a direction (e.g., left or right; facing north or facing south). Let us take the concept "facing north" to be the positive numbers. We should then say that multiplication by negative one takes north facing numbers and turns them into south facing numbers; –1 acts as an *about turn.* Thus since multiplication is commutative we can write the following:

$$5 \times (-2) = 5 \times (-1) \times 2 = (-1) \times 5 \times 2 = (-1) \times 10 = -10$$

and likewise

$$(-3) \times (-3) = (-1) \times 3 \times (-1) \times 3 = (-1) \times (-1) \times 3 \times 3 = 9.$$

In the latter, two about turns mean we have gone from positive to negative and back to positive, whence the mantra "minus times minus makes a plus."

Geographically we have a starting point and can move as far as we would like in either the northerly or southerly direction. What though, of other directions—of east and west: how can these be related? If we were to introduce a concept of "east" we might as well start with a "unit east," in the same way that +1 and –1 were units for north and south respectively. If we denote by the unit east, what can be said of the properties of *i*? In most textbooks which introduce complex numbers, this analogy is used in more or less the same way. As Hardy (1908/1992, pp. 66–81) writes, rules for multiplication are then established, along with the notion of rotation, whence complex numbers follow. In the next section a less complicated approach is presented.

The number compass

If we are facing north, then multiplication by i gives a right-turn to east. Thence, another multiplication by i means we are now facing south, and the unit south is –1. So we have

$$i \times i = i^2 = -1$$

and we arrive at a "definition of *i*." If we take another right turn (multiply by i) we arrive at unit west

$$i^3 = -i$$

and one more right turn (multiplication by i) gets us back to north, or to +1, viz.

$$i^4 = 1.$$

So then, by allowing our numbers to represent more than the north–south dichotomy, we have "discovered" the other two cardinal points. Of course the discovery does not stop there! One can then define "unit north-east" as some number w: so that two multiplications of w (two acts of turning 45 degrees to the right gives unit east) can be written

$$w^2 = i = \sqrt{-1}$$

whence we deduce that

$$w = \sqrt{\sqrt{-1}} = \sqrt[4]{-1}$$

Continuing in this fashion we can identify any "direction on the compass" with the multiplication by some number, which has unit magnitude, but a varying angle. Here is certainly the time to introduce the Argand diagram, and the notation of writing numbers as complex exponentials, along with Euler's identity, complex conjugation and roots of unity; though now students can put to rest any shaky foundations they may have had in this area.

An aside: Irrational numbers

It was discovered in antiquity that there are no whole numbers *a* and *b* such that we can write

$$\frac{a}{b} = \sqrt{2}$$

According to legend, when this was first announced, Hippasus of Metapontum, the poor mathematician who announced the result, was thrown overboard since he had disturbed the sanctity of the numbers. Some of this myth and the surrounding mystery of the cult of Pythagoreans is explored from a mathematical perspective in brief in Edgett (1935) and in detail by Shanks (1993).

It is difficult to relate the notion of irrationality to intuition. For example, suppose you had a number:

$$\frac{1873}{199824}$$

You could in theory conceive of breaking the 1873 objects you were sharing into small pieces, 199824, and then apportioning them appropriately. However, a number like $\sqrt{2} = 1.4142\ldots$ with no repeating decimal expansion and no representation as the quotient of two whole numbers poses some problems. One could appeal to another Pythagorean perspective and imagine a right triangle with perpendicular sides of unit length—thus we can measure the hypotenuse and define that as $\sqrt{2}$. It is precisely this example which caused Hippasus to be cast overboard.

To *grasp* the concept of an irrational number, one really needs the notion of the limit from analysis, or results from higher algebra—the so called "Dedekind cut". So it is actually easier to conceptualise complex numbers than irrational numbers, although still we feel an inherent "realness" towards the latter, and a sense of alienation to the former.

Author's note

I have found this helpful when tutoring students both in Australia and in England. Indeed with younger students, I would draw a map of Australia with the A87 running from Adelaide to Darwin and through Alice Springs. This was the positive–negative axis and *i* would be the direction to turn off to head for Brisbane.

As an exercise, one can ask an advanced class to add complex numbers in the following way. Mark out on a map a route to travel, passing through various towns (assuming that most travel is done "as the crow flies," which is fortunately true for long distance travelling in Australia). This route can then be converted into a sequence of complex numbers to be added together, and this sequence can be given to the students, to see whether they can reconstruct the travel route.

In England, I would ask some of the first year undergraduates to give a mathematical and non-mathematical definition of a complex number, and found them very confident on the former but not on the latter. After this explanation, their affinity for the concept increased—as it should when forced to closely examine an often-used principle.

Conclusion

Certainly a complex number is not "tangible"—one cannot have $17i$ bananas; but by this same reasoning, negative numbers are not tangible either. If we are comfortable overcoming our intuition with the latter (normally by use of the number line), then we should be comfortable with the former (by extension to the number compass).

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