A mathematical solution to the motorway problem

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The motorway problem (e.g., Isenberg, 1975) is an interesting one in optimisation. Suppose that there are four cities, A, B, C, and D, located at the four corners of a square with sides of length 1 (see Figure 1). A network of motorways is to connect all four cities in the shortest distance possible in order to optimise construction costs (Dutta, Khastgir & Roy, 2008). What configuration of motorways will yield the minimum total distance?

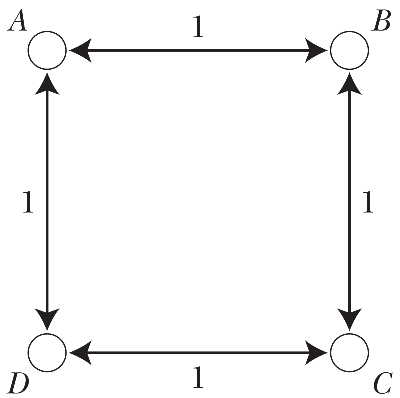

Figure 1. Four cities A, B, C and D located at vertices of a square of side length l.

Clearly, the square configuration produces a total network length of 4. If one side is removed, then the total length would be reduced to 3. Still, the length can be further reduced to $2\sqrt{2} \approx 2.828$ if the configuration is in the shape of an X. However, the configuration that will produce an even shorter total network length is shown in Figure 2. What is the size of the angle θ that will minimise the total length of the network of motorways? The author hereby suggests a mathematical solution to the motorway problem.

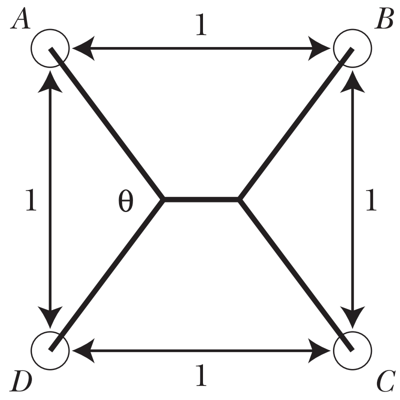

Figure 2. Four cities with connecting network of motorways.

Let L be the total path length of the network. Then, $L = m + 4n$, as shown in Figure 3. Each side n is equal in length due to the symmetry of the shape
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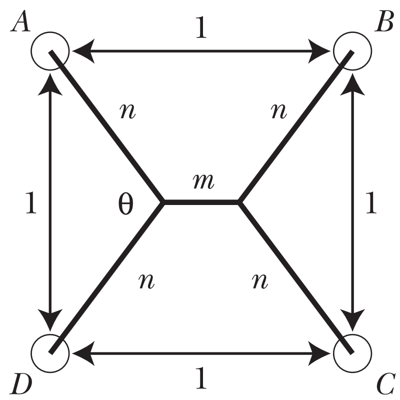

Figure 3. Motorway network with lengths shown.

In order to find the length of n, we must construct the right triangle as in Figure 4. Due to the symmetry of the shape, it is clear that q and $\overline{AD}$ in Figure 3 are each bisected by the broken line segment of length p, shown in Figure 4.

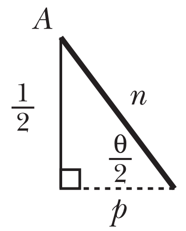

Figure 4. Right triangle to find the length n.

By trigonometry, therefore, it follows that

$$\sin\frac{\theta}{2} = \frac{\frac{1}{2}}{n}$$

which implies that

$$n = \frac{\frac{1}{2}}{\sin\frac{\theta}{2}} = \frac{\operatorname{cosec}\frac{\theta}{2}}{2}$$

Meanwhile, by the Pythagorean theorem, we know that

$$p^2 + \left(\tfrac{1}{2}\right)^2 = n^2$$

which implies that

$$p = \sqrt{n^2 - \tfrac{1}{4}}$$

Substituting n into this result yields

$$\begin{aligned}
p &= \sqrt{\left(\frac{\operatorname{cosec}\frac{\theta}{2}}{2}\right)^2 - \frac{1}{4}} \\
&= \sqrt{\frac{\operatorname{cosec}^2\frac{\theta}{2}}{4} - \frac{1}{4}} \\
&= \frac{\sqrt{\operatorname{cosec}^2\frac{\theta}{2} - 1}}{2} \\
&= \frac{\sqrt{\cot^2\frac{\theta}{2}}}{2} \\
&= \frac{\cot\frac{\theta}{2}}{2}, \text{ because } \frac{\theta}{2} < \frac{\pi}{2}.
\end{aligned}$$

Since $m = 1 - 2p$, as shown in Figure 5, it follows that

$$m = 1 - 2 \times \frac{\cot\frac{\theta}{2}}{2} = 1 - \cot\frac{\theta}{2}$$

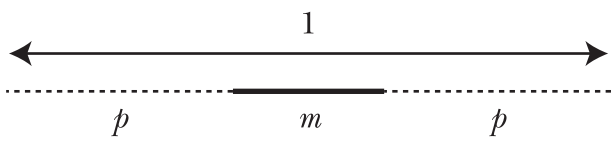

Figure 5. Relationship between m and p.

Now that m and n have been defined, we can find the formula for L:

$$\begin{aligned}
L &= m + 4n \\
&= \left(1 - \cot\tfrac{\theta}{2}\right) + 4\left(\frac{\operatorname{cosec}\frac{\theta}{2}}{2}\right) \\
&= 1 - \cot\tfrac{\theta}{2} + 2\operatorname{cosec}\tfrac{\theta}{2}
\end{aligned}$$

To find the angle θ that minimises L, we must first find $\frac{dL}{d\theta}$:

$$\begin{aligned}\frac{dL}{d\theta} &= -\frac{-1}{2\sin^2\frac{\theta}{2}} + \frac{-\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} \\ &= \frac{1}{2\sin^2\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} \\ &= \frac{1-2\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}\end{aligned}$$

Next, we must set $\frac{dL}{d\theta} = 0$ and solve for θ:

$$\frac{1-2\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = 0$$

Since $\sin\frac{\theta}{2} \neq 0$ for $\theta > 0$, we now have:

$$\begin{aligned}1-2\cos\tfrac{\theta}{2} &= 0 \\ 2\cos\tfrac{\theta}{2} &= 1 \\ \cos\tfrac{\theta}{2} &= \tfrac{1}{2} \\ \tfrac{\theta}{2} &= \cos^{-1}\tfrac{1}{2} \\ \tfrac{\theta}{2} &= \tfrac{\pi}{3},\ \tfrac{5\pi}{3},\ \ldots \\ \theta &= \tfrac{2\pi}{3},\ \tfrac{10\pi}{3},\ \ldots\end{aligned}$$

Clearly, $\theta = \frac{2\pi}{3} = 120°$.

Hence, 120° is the angle that minimises the total length of the motorway network, as shown in Figure 6. In a very interesting experiment, Isenberg (1992) used a demonstration of soap films and soap bubbles to find this angle.

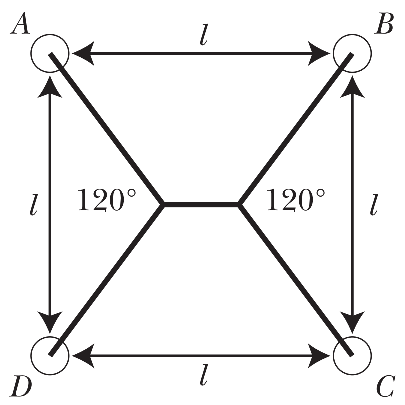

Figure 6. Motorway network with minimising angle.

Then, out of interest, what is the total length of the network? Substituting $\theta = \frac{2\pi}{3}$ into L yields:

$$\begin{aligned} L &= 1 - \cot\tfrac{\theta}{2} + 2\operatorname{cosec}\tfrac{\theta}{2} \\ &= 1 - \cot\tfrac{\pi}{3} + 2\operatorname{cosec}\tfrac{\pi}{3} \\ &= 1 - \frac{1}{\tan\frac{\pi}{3}} + \frac{2}{\sin\frac{\pi}{3}} \\ &= 1 - \frac{1}{\sqrt{3}} + \frac{2}{\frac{\sqrt{3}}{2}} \\ &= 1 - \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \\ &= 1 + \frac{3}{\sqrt{3}} \\ &= 1 + \sqrt{3} \end{aligned}$$

That is $L \approx 2732$ as confirmed by Jákil and Saupe (2006) and Kocevar (2005). This result is approximately 3.4% better than the next shortest distance in the X configuration mentioned previously.

In conclusion, the motorway problem is an excellent application in optimisation. As it integrates the concepts of trigonometric functions and differentiation, the motorway problem can be used quite effectively as the basis for an assessment tool in senior secondary mathematics subjects. For example, the motorway problem might be used as a problem-solving or modelling task in the Mathematical Methods VCE course (VCAA, 2005), as a practical investigation in the New South Wales HSC Mathematics Advanced course (Board of Studies NSW, 2008), or as an extended modelling and problem solving task in the new Queensland Senior Mathematics B syllabus (2008). Students would be able to demonstrate their abilities to use the most appropriate sequence and combination of procedures and strategies across topics to enable a complete solution to the motorway problem.

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