NUMBER GUESSING

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It is instructive and interesting to find hidden numbers by using different positional numeration systems. Most of the present guessing techniques use the binary system expressed as less-than, greater-than or present–absent type information. This article describes how, by employing four cards having integers 1–64 written in different colours, one can guess the secret number held by the player. This game can be used as a teaching aid for demonstrating base-4 representation of numbers or the concept of isomorphism.

The framework

The game is simple: ask a student to select an integer between 1 and 64, and bet that you will discover it by asking three questions. Then show that person three cards and ask, "What is the colour of your secret number here?"

Assume that you get answers "Blue" for the first card, "Yellow" for the second card, and "Green" for the third card.

Then you say, "Your secret number is 28." When I tried this game with my students and friends, their reaction was always a great surprise at how the number could be guessed so easily.

It looks very natural for us to use a positional number system in base 10 because human beings use their fingers for counting; but mathematically it is completely valid to use any other base equal to or greater than two. For example, binary, octal and hexadecimal arithmetics are used in computing and programming. Time and angle measurements rely on sexagesimal (base-60) system. Although quaternary (base-4) numbers are not very common, there are some areas appropriate for their usage: it is possible to represent directions by a base-4 system.

Base-4 also can be used in genetic coding of DNA. The four nucleotides are abbreviated as *A*, *C*, *G* and *T*. Therefore nucleotide sequences can be represented numerically by coding these letters as $A = 0$, $C = 1$, $G = 2$, and $T = 3$. Proposed first by Adleman (1994), DNA or molecular computation emerged as a new research area. During the last decade, hundreds of research papers appeared describing biomolecular computation techniques for various challenging problems which require extremely large parallelism and memory in conventional electronic-based computers (Garzon et al., 2004; Jonosca, 2004).

It might be surprising to convey and get information by using different bases. If there are b different digits, 0, 1, …, $b - 1$ in our system, it is possible to write any integer N by using the positional numeration system $a_n a_{n-1} \ldots a_1 a_0$ which corresponds to

$$N = a_n b^n + a_{n-1} b^{n-1} + \ldots + a_1 b + a_0$$

in expanded notation.

For example $N = 61$ can be written in quinary (base-5) representation as $N = 221_5$ because

$$61 = 2 \times 5^2 + 2 \times 5^1 + 1 \times 5^0$$

The same number can be represented in quaternary base (base-4) as 331_4. This representation can be extended for expressing the fractional parts of real numbers by the use of negative exponents.

Application to number guessing

By using the information on the absence or presence of a number on five window cards Gardner (1956) developed a mathematical game to guess a person's age. An Internet search will lead to hundreds of sources and computer programs dealing with number guessing but almost all of them rely on using a binary base expressed as less-than–greater-than information. Among guessing sites, there are some exceptions that use bases other than binary: Penny (1997) uses base-3, by arranging numbers on three columns and asking the column in which the number is found. By three steps, the program discovers the secret number picked from a list of 21 numbers. The program does not use the full information because in base-3 with three digits it is possible to guess $3^3 = 27$ different numbers. Bogolmony (2003) uses 25 numbers arranged in a 5×5 square. In two responses the computer determines the location of the selected number by its row and column. To confound the spectators, the computer randomly reshuffles the numbers before each displaying. Bogolmony also uses 27 numbers arranged in three rows and discovers the selected number in three steps.

The game presented by Marmon (2004) works on three-digit integers in base-10 and is more complicated. The player presents a three-digit number at each trial; the program supplies information on the number of digits correctly guessed and the number of those in correct places. This problem is a special case of well-known deductive games such as "Mastermind" and "Bulls and Cows" in which a codebreaker tries to discover the patterns chosen by a codesetter. Some optimal strategies for these games can be found in Yue and Chang (2002), Chen and Lin (2004), and references therein. The authors calculate the probability of guessing an n-digit number given certain hints on the number of correct digits and propose optimal strategies.

We used the information expressed in base-4 to develop a pleasant game in which one player discovers a secret number selected by another player. By using three positions, one can write and discover 64 different numbers ranging from 0 to 63. Base-4 can be applied in discovering the numbers held in a very practical way because it may correspond to spinning a square card in four different directions or giving messages by showing four different colours. This base has digits 0, 1, 2 and 3. They can be replaced by other signals, for example:

- letters: *A*, *B*, *C* and *D*;
- colours: red, blue, yellow and green (R, B, Y and G);
- directions: up, right, down, and left (U, R, D, and L); or north, east, south and west (N, E, S and W).
- quadrants of a square: upper left, upper right , lower right, lower left (I, II, III, IV).

Therefore one can communicate to a friend by showing cards of different colours arranged in a definite order to convey information. For example, if the colours are arranged as GGR, this corresponds to number 330_4 or 60_{10}. The same information is given by turning a pointer in different directions: WWN. This idea is used to develop a game played by four cards, which enables us to discover 64 different numbers.

In the basic plate shown in Figure 1, integers 1–64 are arranged in a definite pattern. As it can be seen clearly, the 8 × 8 table is divided into four quadrants and each quadrant is divided into consecutive sub-sections. The upper left quadrant contains integers 1–16, each set of four numbers written in 2 × 2 cells in clockwise order. The same pattern applies to other quadrants of the table.

In order to represent four different digits, three cards are prepared as depicted in Figure 2. In the first card, the upper-left 16 cells are opened by removing the shaded area. Similar cuttings are applied to the second and third cards by taking off subsequent upper-left quarters. As seen in Figure 3, the numbers belonging to the first quadrant are written in red colour. The following quadrant letters are blue, yellow, and green respectively in the clockwise order. In the second card, each sub-quadrant is given suitable colours according to its clockwise order. The same pattern is applied in the third card. Although integers 0–63 are obtained in three digit quaternary numbers, for the sake of convenience we added 1 to each number, hence obtained integers 1–64

1	2	5	6	17	18	21	22
4	3	8	7	20	19	24	23
13	14	9	10	29	30	25	26
16	15	12	11	32	31	28	27
49	50	53	54	33	34	37	38
52	51	56	55	36	35	40	39
61	62	57	58	45	46	41	42
64	63	60	59	48	47	44	43

Figure 1. The basic plate.

In order to discover the number held by the player, show him/her three cards in sequence and ask the colour of the number behind the card. If the colour of the number is red, put the card on the basic plate so that the black dot will be situated upward. If the number is blue, the card will be turned to place the dot at right. For yellow and green numbers, the dot will be situated at bottom and left respectively. The correspondence between digits, colours, directions and quadrants is summarised in Figure 4.

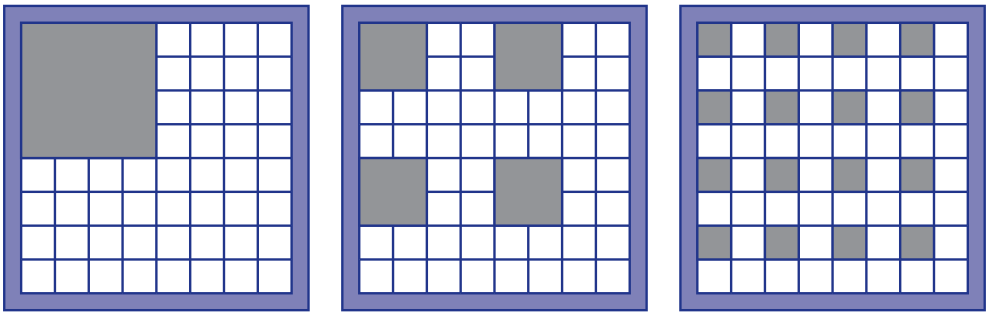

Figure 2. Template for the three window cards.

CARD 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RED
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	BLUE
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	YELLOW
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	GREEN

CARD 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
RED				BLUE				YELLOW				GREEN			

CARD 3

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
RED	BLUE	YELLOW	GREEN	RED	BLUE	YELLOW	GREEN	RED	BLUE	YELLOW	GREEN	RED	BLUE	YELLOW	GREEN

Figure 3. Tables showing the numbers written behind the first, second and third windows.

Digit	Colour	Direction	Quadrant
0	Red	Up	I
1	Blue	Right	II
2	Yellow	Down	III
3	Green	Left	IV

Figure 4. The correspondence between digits, colours, directions and quadrants.

Formally speaking, there is an isomorphism between sets defined by digits, colours, directions and quadrants therefore we are able to discover the number held by the player. In mathematics isomorphism is defined as a one-to-one correspondence between the elements of two sets such that the result of an operation on elements of one set corresponds to the result of the analogous operation on their images in the other set. It is possible to define a one-to-one onto function between isomorphic sets. In this sense, the number guessing game is a pleasant example with which teachers can introduce the concept of isomorphism.

Example

Assume that the player holds the number 34. Then:

1. Turn the dot in the first card down, because 34 is yellow in the first card.
2. Keep the dot in the second card up, because 34 is red in the second card
3. Turn the dot in the third card right, because 34 is blue in the third card.

Now all numbers are covered except 34.

The range of integers can be enhanced by using more than three cards. For example by using 5 cards, it is possible to discover $4^5 = 1024$ different numbers.

It may be more attractive to place pictures of objects, animals or fruits instead of numbers in the basic plate. In this case each picture will correspond to an integer and the procedure will remain the same, by replacing the numbers on the cards by objects with the same colours. By imposing suitable range restrictions, it is also possible to add or subtract base-4 numbers by spinning of cards. After some experience, one can guess the numbers without stacking and spinning cards. By multiplying the digits of Figure 4 by 16, 4 and 1 respectively and adding mentally, one can guess the secret number in a more impressive way. By some exercise it is even possible to guess the selected numbers mentally or by the help of a simple calculator in k steps if N colours are used to code N^k numbers.

Educational value and presentation

According to the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006), an excellent teacher must recognise a range of effective strategies and techniques for: teaching and learning mathematics; promoting enjoyment of learning and positive attitudes to mathematics, utilising information and communication technologies. In the domain of professional practice, “excellent teachers of mathematics arouse curiosity, challenge students’ thinking, and engage them actively in learning.” Excellent teachers motivate students to improve their understanding of mathematics and develop enthusiasm for, enjoyment of, and interest in mathematics.

Problem solving is a basic component of mathematics learning. Accumulation of formal knowledge about concepts and procedures of mathematics has no practical value unless students are able to use these assets for problem solving. A well chosen puzzle can be considered as a stimulating and challenging problem. By dealing with puzzles students will appreciate the value of problem solving by experiencing the fun and excitement of this process. The National Council of Teachers of Mathematics (NCTM) is the counterpart of the AAMT in the United States of America and it also recognises the significance of mathematical games and puzzle solving in the process of mathematical teaching. Perhaps this significance needs to be elaborated?

According to my experience, my colleagues and students find this number-guessing puzzle very interesting and intriguing because they consider the input information insufficient and irrelevant. How can you discover a quantitative value by using apparently irrelevant qualitative attributes such as colours? After some reflections and explanations, they can appreciate the equivalent patterns of numbers, colours, directions and quadrants. This reasoning experience helps them to understand the mathematical concepts involved. While this puzzle can be used as an illustration of different base representations of numbers, for more advanced curricula, it may also help when teaching the concept of isomorphism in sets and ordered fields.

In application, guided-discovery methods can be used by passing the activity of the lesson to students. In order to increase the motivation of students, the teacher may allow them to play with cards and discover certain patterns. In the first step, they will discover the position of different colours on the basic plate; for example, a student inspecting red numbers in the first window card can discover that they are all situated in the upper-left quadrant of the base card. In a similar manner, the student might perceive that the second card divides each quadrant to four sub-regions and that these sections have colours red, blue, yellow and green in a clockwise manner. Inspection of the third window card will reveal a similar pattern. According to my observations, the second step of discovery is the relationship between colours and spinning positions of the window cards. Students will realise that there is a correspondence between dot positions and colours in the following manner:

$$\begin{aligned} \text{up} &\equiv \text{red} \\ \text{right} &\equiv \text{blue} \\ \text{down} &\equiv \text{yellow} \\ \text{left} &\equiv \text{green.} \end{aligned}$$

After these steps, students will be able to identify the isomorphism between colours and directions. After a class discussion to reach final conclusions, some mathematically-gifted students might carry the reasoning further and can allude to base-4 representation of numbers, but discovering this representation will generally require guidance by the teacher. This is the third step of discovery and, with the help of the teacher, students will realise that by using the three digits 0, 1, 2 and 3, one can produce 64 different numbers in positional representation. The only difference from the base-4 representation is that here 1 is added to each number obtained.

Negative attitudes of children to mathematics can be overcome by presenting them with more enjoyable teaching material. The student must get reward for his/her achievements. By learning how this game can be used, students can demonstrate their “magic ability” to their parents or friends. They will be the focus of attention in some social gatherings. Intellectually stimulating activities can also provide pleasure, entertainment and create a positive perception of mathematics.

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