Equivalence and Equations in Early Years Classrooms

ELIZABETH WARREN, ANNETTE MOLLINSON and **KYM OESTRICH** explain how young children can develop powerful understandings of equivalence and equations.

Early algebraic thinking in a primary context is not about introducing formal algebraic concepts into the classroom but involves reconsidering how we think about arithmetic. Early algebraic thinking assists young students to engage effectively with arithmetic in ways that support engagement with arithmetic structure rather than arithmetic as a tool for computation.

The distinction between arithmetic thinking and algebraic thinking in the early years' context is best defined as: arithmetic thinking focuses on product (a focus on arithmetic as a computational tool) and algebraic thinking focuses on process (a focus on the structure of arithmetic) (Malara & Navarra, 2003). This distinction assisted us to distinguish between the two in classroom discussions, and to move from one to the other as the need arose.

In our work with young children (5 year-olds), at times we needed arithmetic to support algebraic thinking (e.g., computing two expressions to see if they were the same) while at other times we needed algebraic thinking to support arithmetic (e.g., adding 3 to 2 is the same process as adding 3 to 82, 3 to 1012, 30 to 20).

The power of mathematics lies in the intertwining of algebraic thinking and arithmetic thinking. Each enhances the other as students become numerate (Warren,2008). Using the distinction between arithmetic and algebraic thinking, a series of hands-on activities were collaboratively planned and implemented. The activities focused on the areas of equivalence and equations. The aim was to assist 5 year-olds to come to an understanding of the structure of equations, and in particular the use of the equal sign. The activities not only encouraged active learning (Crawford & Witte, 1999) but also reflected the principles of socio-constructivist learning (Vygotsky, 1962). In the case of equivalence and equations, many students in their primary years hold misconceptions with regard to the equal sign (Warren & Cooper, 2005). For many an equation only makes sense if the action occurs before the equal sign. For example, when asked to find the unknown for 7 + 8 = ? + 9, many students express this as 7 + 8 = 15 + 9 = 24.

With regards to equivalence in the early years, there are four key areas that students should explore.

1. Developing the comparative language that assists in describing equivalent and non-equivalent situations,
2. Developing an understanding that equals means that the two expressions are equivalent,
3. Representing equations in a variety of different formats including equations with more than one number on the left hand side (e.g., 2 + 5 = 3 + 2 + 2 and 7 = 5 + 2, and
4. Using the ‘balance principle’ to find unknowns.

Language of equivalence and non-equivalence

Initially these ideas were explored in a numberless world with a focus on developing the language used for describing equivalent situations, namely: “equal to,” “same as,” “not equal to,” and “different from.” This was achieved by using concrete objects and focusing on a variety of different attributes such as shape, size and colour. Students were also introduced to balance scales that were balanced when the objects on each side had the same mass. The beginning activities explored comparing two different sets, for example, two stacks of blocks, liquid in two containers, or mass in two sides of the balance scale.

Figure 1. Comparing different attributes and developing the language of equivalence.

During the classroom conversations, the language utilised to compare two sets of objects was emphasised. Each group of students was encouraged to explain why the sets were the same or different. The reasons they gave for the use of “same as” with the two stacks of blocks were: “The height of this stack [pointing to the first stack of blocks] is the same as the height of this stack [pointing to the second stack of blocks]. They are the same height. They are equal.”

By contrast the group who poured water into two different containers gave the following reason for their choice of the card “different from:” “The amount of water in jar A is different from the amount of water in Jar B.”

Understanding equals as equivalence

Equations can be modelled using balancing devices or strips of paper. Each side of the equations is called an expression. Balancing devices such as bucket balances or balance scales, and length models allowed students to manipulate objects to demonstrate equivalence. This also helped them understand the conservation of number — that the number of objects remains the same when they are rearranged spatially.

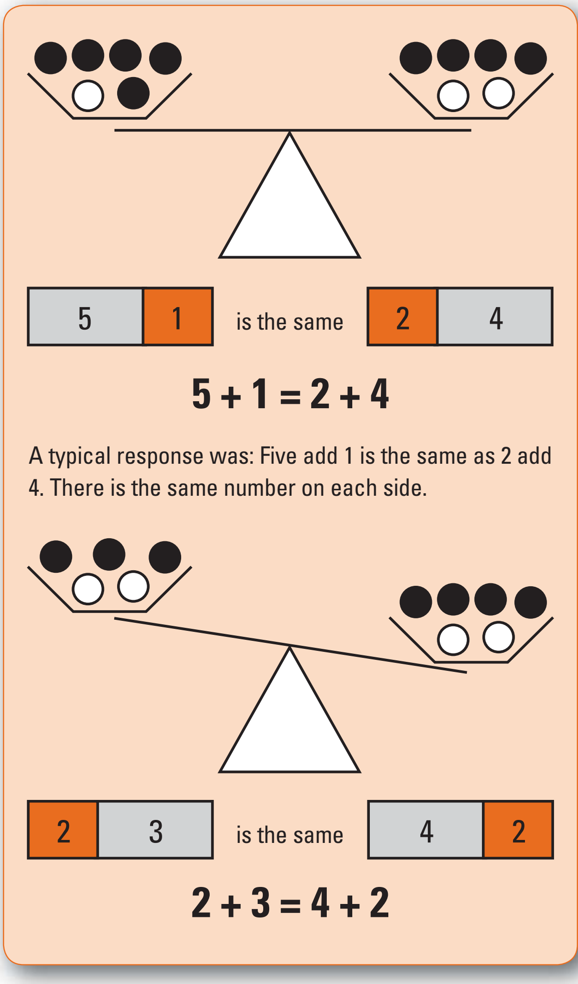

Figure 2. Using different representations to illustrate equations and in-equations.

"Three add 2 is different from 4 add 2. There is a different number on each side. Three add 2 is not equal to 4 add 2. It is different from 4 add 2". Arithmetic thinking is required when computing the value of the two expressions. Algebraic thinking is required when placing the appropriate language between the expressions. Equals is only applicable if the two expressions are equivalent, that is, have the same value. This generality applies to all situations no matter how large the numbers or how complex the computation for each expression.

Creation of equivalence stories using real world contexts

The students were asked to use play dough, paint, and pictures of animals and birds to create their own equivalent stories. Figure 3 presents some of the stories that they created.

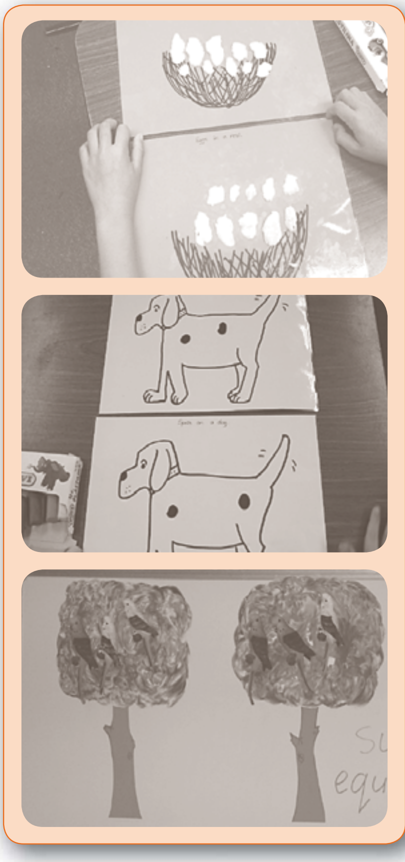

Figure 3. Stories involving equivalent situations.

Each student not only created equivalence stories but also explained why his/her story showed equivalence. Some typical responses were: "There are the same number of eggs in each nest, spots on each dog or birds in each tree." One student suggested that the story for the last picture was: "There are two trees in my backyard. The first tree has two budgerigars and one parrot. The second tree has two parrots and one budgerigar. There is the same number of birds in each tree."

Students were encouraged to verbalise their stories and represent them using symbols. Figure 4 presents some of their stories.

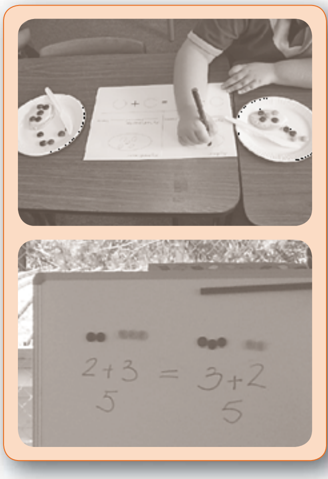

Figure 4. Representing equivalent stories using symbols.

These students often found it difficult to make up mathematical stories. However, it appeared that the more experience they had in expressing ideas, the more competent they became in using the language of mathematics to describe different story contexts.

Writing different equations

Four students were interviewed at the conclusion of the teaching. The students were chosen according to the teachers' beliefs that they represented a range of different abilities. One of the tasks required them to examine two Christmas trees with pears and bananas hanging in them, compare the two trees, and represent this comparison in an equation. They were given the frame ___+___=___+___ to write their answers. Table 1 summarises their responses to this task.

Abby ignored the frame and wrote the total obtained by adding all of the objects together (10). Her story exhibits a propensity to think arithmetically. Throughout the interview she continually computed problems, finding answers instead of engaging in comparing the two expressions and seeing if they were equal or not.

It is interesting that both Brianna and Ethan spontaneously introduced symbols as short hand for the object: p for pears and b for bananas or an iconic picture of each. In the secondary context this is commonly referred to 'fruit salad' algebra where the letter stands for an object instead of variable, and is thinking that we want to discourage. For example, a common misconception in the secondary context is that $3a + 3b$ stands for 3 apples and 3 bananas instead of a and b standing for any number. In the early years it is important to make the distinction between how we verbally describe number problems and how we represent these problems with symbols. While we say, "Three cars and five trucks," the convention is to represent this as 3 and 5. Number sentences are made up of numbers.

Olivia correctly shared that there are 2 bananas and 3 pears on the first tree and 1 banana and 4 pears on the second tree: "Two and three is the same as one and four."

The key issue exemplified by the students' responses is that appropriate activities and classroom dialogue, with a particular emphasis on expressing ideas using mathematical language, begins to assist young students to engage in the algebraic meaning of equals (a sign indicating a place to put the answer). Abby's responses also show that this way of thinking is difficult for some.

Table 1. Student's equations for the Christmas tree problem.

Student	Christmas three problem	Equations for Christmas three problem
Abby		5 + 5 = 10
Brianna		3P + 2b = 4P + 1b
Ethan		2[illegible] + 3[illegible] = 1[illegible] + 4[illegible]
Olivia		2 + 3 = 1 + 4

Understanding the balance strategy

The students' thinking was then extended to examine the 'balance strategy.' The balance strategy relies on students understanding that if we add or subtract the same amount (number) from each side of an equation it remains balanced. They initially explored this idea using balance scales with coloured bean bags of the same weight.

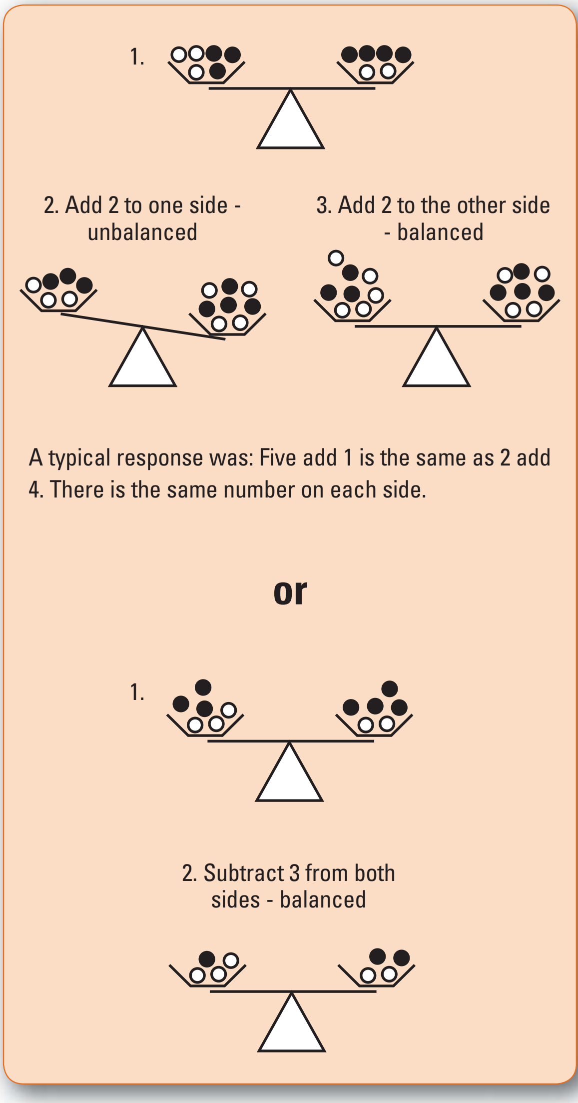

Figure 5. Adding or subtracting the same amount from both sides of an equation leaves it balanced.

The same ideas can also be modeled using the length model for numbers. The algebraic thinking is that no matter what the equation, if I add the same amount to both sides or if I subtract the same amount from both sides, the two sides of the equation remain equivalent.

Where to from here?

Using the balance strategy to solve for unknowns

The balance strategy is one of the most powerful strategies for solving for unknowns. It consists of two different thinking processes. These are (a) how do you isolate the unknown and (b) how do you balance the equation. The following problem illustrates this thinking.

Story – *I have some money in my piggy bank. If I had another $3 I would have $5. How much do I have in my piggy bank?*

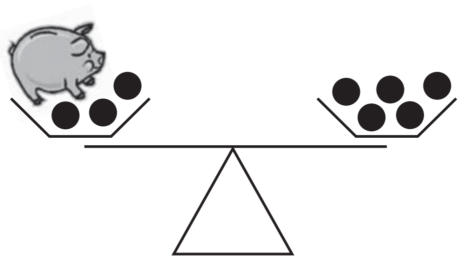

Figure 6. Using the balance strategy to model an equation with an unknown.

The problem is modelled in Figure 6.

When discussing the unknown amount of money in the piggy bank, ask students for suggestions about how they would like to represent this. Encourage them to create their own symbol. If they suggest 'p' discuss how other students might interpret this. Would they think that 'p' was for 'pig' rather than the amount of money in the piggy bank? One solution is to use a symbol such as "?" for the unknown amount.

Thinking – How do we isolate the unknown?
(Subtract 3 from the "left-hand side")
Will it still be balanced?
How do we make it balanced?
(Take 3 from the "right-hand side").
How much is in the piggy bank?

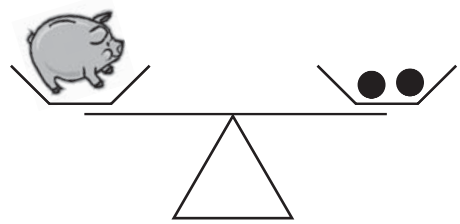

Figure 7. Using the balance strategy to find an unknown.

This problem can be modelled on a physical balance scale with an Unknown bag. Secretly place 2 bean bags in the Unknown bag and ask how you would find out what is in the bag. Some students automatically know it is 2 but the aim is not to know solutions by guessing and checking or even using number facts but to set up thinking that carries students through all levels of primary and secondary school. Ask them to work out ways of finding solutions using the scales and explain their solutions as they find them. We have evidenced two different ways of working out the unknown. One child suggested that we take the unknown bag from the scales and then keep taking bean bags from the other side until the scales are again balanced. The number of bags that are taken off the left hand side is the number of the unknown. Another suggested to continually take one bag from each side of the scales until you only have the unknown left on one side and bean bags on the other. The unknown would be the number of bean bags left on the other side.

Isolating the unknown involves recognising the inverse relationship between addition and subtraction (in the early years) and multiplication and division (in the middle years). Keeping the equation balanced involves understanding that if you do the same operations or combination of operations to each side of an equation it remains 'balanced'. The examples in Figure 8 demonstrate the power of this thinking in the middle years and secondary context, and hence the importance of developing it from the early years.

Example 1: $3x + 7 = 19$	
– 7 from both sides	$3x = 12$
Divide both sides by 3	$x = 4$
Example 2: $5x – 8 = 3x + 12$	
– 3x from both sides	$2x – 8 = 12$
+ 8 to both sides	$2x = 20$
Divide both sides by 2	$x = 10$

Figure 8. Examples of the balance strategy in the secondary middle grades.

Concluding comments

In the early years' classroom we are suggesting that algebraic thinking involves understanding what is meant by equivalence, that is, being able to describe equivalent situations using appropriate language, concrete models and symbols, and beginning to use the balance strategy to find unknowns for simple addition equations. Too often computational thinking dominates early years classroom conversations. While this does serve finding answers to problems, it does not assist us to engage in conversations about generalised arithmetic — conversations that eventually lead to formal algebra. In fact, misconceptions such as "= indicates a place to put the answer", and "There is always only one number of the right hand side of the equal sign, the answer", become so entrenched it is almost impossible to renegotiate equals as a symbol for equivalence.

References

Crawford, M. & Witte, M. (1999). Strategies for mathematics: Teaching in context. *Educational Leadership*, *57*(3), 34–38.

Malara, N. & Navarra, G. (2003). *ArAl Project: Arithmetic pathways towards favouring pre-algebraic thinking*. Bologna, Italy: Pitagora Editrice.

Vygotsky, L. (1962). *Thought and language* (E. Hanfmann & G. Vakar, Trans.). Cambridge, MA: Massachusetts Institute of Technology. (Original work published in 1934).

Warren, E. & Cooper T. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal*, *17*(1), 58–72.

Warren, E. (May, 2008). Early algebraic thinking: The case of equivalence in an early algebraic context. Keynote address *Future Curricular Trends in School Algebra and Geometry* The University of Chicago and The Field Museum Chicago, Illinois, USA.

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