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Attending to feeling: Productive benefit to novel mathematics problem-solving

Carol R. Aldous
Flinders University *carol.aldous@flinders.edu.au*

What does attention to feeling have to do with solving problems in mathematics? Can feeling be used to navigate a path to a solution? What is meant by a feeling anyway? To what kind of problem does this productive benefit refer?

A study of 405 middle school students solving two novel mathematics problems found that individuals utilising a feeling or free-flowing approach to reasoning were more likely to be successful in reaching a solution than those who did not. Indeed, feeling cognitions were found to have both a direct and indirect effect on the generation of a solution depending on whether mainly spatial or verbal processing was required. This finding is consistent with neuroscience research.

Problem solving, feeling cognitions, mathematics, causal model

INTRODUCTION

An introspective account of novel problem solving, by noted mathematician Poincaré, early last century, found aesthetic feeling being used to guide the development of new orders of mathematics. In reflecting on this process Poincaré (1924,p.385, original 1908) stated

> If I have the feeling, the intuition so to speak, of this order, so as to perceive at a glance the reasoning as a whole, I need no longer fear lest I forget one of the elements, for each of them will take its allotted place in the array, and that without any effort of memory on my part[Poincaré, 1924 #545,p.385, original 1908]

Feeling and intuition, by this introspective account, played an important role in mathematical reasoning. For Poincaré, attending to feeling (or intuition) figured just as highly if not more highly than attending to memory. Indeed, Poincaré is recorded as saying “It is by logic we prove, it is by intuition that we invent” adding that “Logic, therefore remains barren unless fertilised by intuition”(Miller, 1992, p394.).

In this context it is perhaps relevant to note the analysis of Poincare’s account by well known psychologist Ghiselin (1963)

> What he refers to is not feeling in the sense of emotional excitement, but an affective response to an intellectual order still eluding rational grasp (Ghiselin, 1963, p. 359)

ON DEFINING FEELING

Thus the definition of feeling, referred to in the above account, and the one adopted in this paper is specific, meaning a ‘feeling of a new intellectual order’ or a ‘feeling of cognition’. This feeling represents a form of information processing that signals the formation of a new structure or coherence still eluding conscious appraisal. Taken in this way the term feeling does not refer to the larger emotive states (such as anger, fear or despair) although they must inevitably be related, but rather to the more subtle affective states, such as feelings of knowing, associated with inspiration and intuition in the novel problem-solving process.

Although Poincaré is often cited as providing evidence of affective representation being used in mathematics problem solving, it should be stated that many other mathematicians also report aesthetic feeling being used to make choices, guide actions and make sense of objects and patterns(Sinclair, 2004).

The problem for mathematics educators, however is not merely to verify the presence or otherwise of such feeling cognitions among students studying mathematics but also to ascertain what relationship, if any, such feelings may have to success in solving novel problems. Only by so doing can intervention programs be devised and changes to pre-service and in-service teacher education programs be made.

PURPOSE OF THE STUDY

This paper describes the testing of a causal model of productive problem solving in data collected from students solving two novel mathematics problems in which measures of non-cognitive and cognitive approaches to reasoning are used. In this way it is hoped that any relationship between cognitive and non-cognitive systems of reasoning and success in novel problem solving may be ascertained.

The goals of this article are situated within a larger research project seeking to uncover cognitive and non-cognitive systems of reasoning used in solving novel problems. In particular, within a wider education context, the ways are being sought by which students can be constructively assisted through the novel problem solving process.

METHODS OF ANALYSIS

Developing a Structural Model of Novel Problem Solving

In designing a structural model of novel problem solving it was necessary to identify a set of variables and their associated constructs, hypothesised to have a bearing on successful mathematics problem solving. The constructs selected were adapted from Carroll's Model of School Learning (Carroll, 1963) and Keeves' Cycle of Performance (Keeves, 1986). The constructs include:

- antecedents;
- personal abilities;
- perseverance and motivation;
- opportunity to learn, and
- achievement outcome.

A sixth construct

- approaches to reasoning was added.

The latter construct was designed to tap both cognitive and non-cognitive systems of reasoning including those of feeling and intuition. It involved the use of a self-report instrument known as the Systems of Reasoning Questionnaire (SRQ) (Aldous, 2001) from which a set of five scales tapping cognitive and non-cognitive systems of reasoning was derived.

A list of variables (latent and manifest) and associated constructs, included in the model is given in Table 1. The final model is given in Figure 1.

The latent variables of **Gender**, **Age** and type of school attended be it single sex or co-educational (**Stype**), comprise a set of antecedents believed to have an influence on problem solving. The latent variable (**Space**), tapping two and three-dimensional spatial ability formed the ‘Personal abilities’ construct. The latent variables liking and playing mathematics games (**Play**) and intrinsic motivation (**Motiv**) comprise the ‘ Perseverance and motivation’ construct. The ‘Opportunity to learn’ construct is composed of the latent variable (**Experience**), which is formed by manifest variables measuring past participation and prior practice in solving novel mathematics problems. The construct ‘Achievement outcome’ is composed of scores on two novel mathematics problems. These problems are referred to as the Cute Numbers and Birthday Cake problems.

Table 1. Variables in the Causal Model of Novel Problem Solving

Construct	Latent Variables	Manifest Variables
Antecedents	**Gender**	1 item
	Age	2 items
	Stype (School type)	1 item
Personal Abilities	**Space** (2 and 3 dimensional spatial ability	2 items
Perseverance and Motivation	**Play** (Maths games)	3 items
	Motiv (Intrinsic motivation)	4 items
Opportunity to Learn	**Experience** (prior practice and past participation)	2 items
Approaches to Reasoning	**Strat** (Strategic approach)	10 items
	Free (Free-flowing approach)	6 items
	Spat/Vb (Spatial-verbal approach	8 items
	Feel (Feeling approach))	10 items
	Syst (Systematic approach)	10 items
Achievement Outcome	**Score** (Score on problem)	Cute Numbers Birthday Cake

Measuring Feelings of Cognition

Five latent variables within the Approaches to reasoning construct were used to measure cognitive and non-cognitive systems of reasoning. These are the Strategic approach (**Strat**), the Free-flowing approach (**Free**), the Spatial-verbal approach (**Spat/Vb**), the Feeling approach (**Feel**) and the Systematic approach (**Syst**) to reasoning. The scales for the Strategic and Systematic approaches to reasoning tap cognitive aspects of reasoning, while the scales for the Free-flowing and Feeling approaches to reasoning tap non-cognitive aspects of reasoning. The scale for the Spatial-verbal approach to reasoning taps both cognitive and non-cognitive systems of reasoning and involves simultaneous and successive synthesis depending on whether mainly spatial or verbal processing is being used (Aldous, 2005).

Data Collection and Instrumentation

Data were collected from 405 middle school students in Grades 7 to 10 who were participants in a national program known as the Mathematics Challenge for Young Australians. The Mathematics Challenge for Young Australians is organised by the Australian Mathematics Olympiad Committee with the purpose of encouraging as many students as possible to engage with challenging mathematics problems. Problems are purposefully designed by a committee within the organisation to be both novel and challenging. Entrants within the Mathematics Challenge had three weeks in which to answer six novel problems.

Students involved in the study reported in this paper were invited to answer the Systems of Reasoning Questionnaire(SRQ) upon completing two of the six novel problems. These were the Cute Numbers (**Cutenos**) problem and the Birthday Cake (**Bcake**) problem. Product moment correlations between each of the five approaches to reasoning factors and scores on tests of visual-spatial ability and verbal ability suggest that the Cute Numbers problem had a higher visuo-spatial processing component in the task than that for the Birthday Cake problem (Aldous, 2005).

Measures of two **(2ds)** and three-dimensional **(3ds)** spatial ability forming the latent variable (**Space**) were obtained using the Card Rotation Test (French, Ekstrom, and Price, 1963) and the Surface Development Test (French et al., 1963) respectively.

Measures of intrinsic motivation **Motiv** were obtained using a set of subscales designed by Deci and Ryan (2001) within the Intrinsic Motivation Inventory(IMI). In particular the scales of Interest and Enjoyment (**Intrst**), Perceived Competence (**Compt**), Effort and Importance (**Effor**) and Perceived Choice (**Choic**) were used.

Measures of **Play** were obtained by inviting respondents to indicate on a five-point scale their frequency of playing mathematics games and puzzles (**Magam**), their enjoyment in playing mathematics games and puzzles (**Lkgam**), and their enjoyment in tackling mathematics problems that were to them completely new and different (**Lknov**).

In forming the latent variable **Experience** respondents were invited to indicate the number of times they had previously participated in the Australian Mathematics Challenge (**Preprt**) as well as giving an indication of the degree of opportunity to explore past Mathematics Challenge problems (**Priprac**).

Indicators used in the formation of the latent variable Strategic approach to reasoning (**Strat**) involved the problem solving behaviours of preparation (**Pbr51**,**Pbr50**), strategic exploration (**Cgr41**, **Pbr53**, **Nta43**), incubation (**Pba52, Pba56**), insight and illumination (**Pbr58**, **Awa17**) and reflection(**Pbr54**).

The items used in the construction of the latent variable Free-flowing approach to reasoning (**Free**) centre around behaviours that reflect the use of semi-conscious and non-conscious forms of reasoning to solve a problem. These behaviours include defocused attention (**Cga39**, **Cga40**), associative thinking (**Cga27**), cue utilisation (**Ntr26**), breadth of attention (**Cga36**) and associative illumination (**Pba57**).

The items used to form the latent variable Spatial-verbal approaches to reasoning (**Spat/Vb**) mirror the nature of thinking employed by an individual be it spatial, verbal or both spatial and verbal in searching for a solution. Thus item (**Ntar49**) relates to alternating visual and verbal thinking, item (**Ntar48**) to simultaneous visual and verbal thinking, items (**Ntar44**, **Ntar45**) to sequential visual and verbal thinking, item (**Ntr49f**) to singular verbal thinking and item (**Nta25**) to singular spatial thinking. Items **(Ntr49g**, **Klga24**) relate to cue recognition connected verbal and spatial thinking.

Indicators formulating the latent variable Feeling approach to reasoning (**Feel**) reflect behaviours of an individual who recognises and follows a feeling, hunch or intuition about what to do in creatively solving a novel problem. Hence items (**Pba55**, **Cga20**, **Awa18**) pertain to intuition, item (**Cga19**) to imagination, item (**Cga37**) to originality and inventiveness, items (**Cga38**, **Awar15**) to intra-personal checking, item (**Cga22**) to intuitive insight and items (**Nta28**, **Nta47**) to cue activation and cue relevance.

Items used in the formation of the latent variable Systematic approach to reasoning (**Syst**) centre around behaviours that reflect the use of deliberate, methodical, analytical and conscious forms of reasoning. Items (**Cgr30**, **Klgr21**) relate to the retrieval of information stored in memory, item (**Cgr33**) to the utilisation of information, item (**Cgr34**) to the evaluation of information, items (**Ntr49e**, **Nt49h**) to familiarisation of information, items (**Cgr31**, **Awr16**) to logical, sequential thinking and items (**Cgr32**, **Cgr29**) to organisational, strategic thinking.

Sequencing the Variables

The variables included with the structural model of novel problem solving were placed in logical and temporal order based on a number of models. These were Carroll's model of school learning (Carroll, 1963), Keeves cycle of performance (Keeves, 1986), Shaw's model of the Eureka process (Shaw, 1989), and Hadamard's classic four-stage model of creative problem solving (Hadamard, 1945). The latter two models were used in sequencing the latent variables within the Approaches to reasoning construct. In particular the subsequence Strategic approach (**Strat**)→ Free-flowing approach(**Free**) → Spatial-verbal approach (**Spat/Vb**) → Feeling approach (**Feel**) →Systemic approach (Syst) formed a sub-model of cognitive and non-cognitive processing within the problem solving model.

Testing and Assessing Model Fit

The causal model of novel mathematics problem solving was tested in AMOS Graphics (version 4.01) (Arbuckle and Wothke, 1999) using multi-group analysis with maximum likelihood estimation procedures. All possible paths between latent variables were included within the initial model. Non-significant paths were then trimmed from the model one at a time in an iterative process until a stable well fitting model was reached. The specified and identified model of novel problem solving is given in Figure 1.

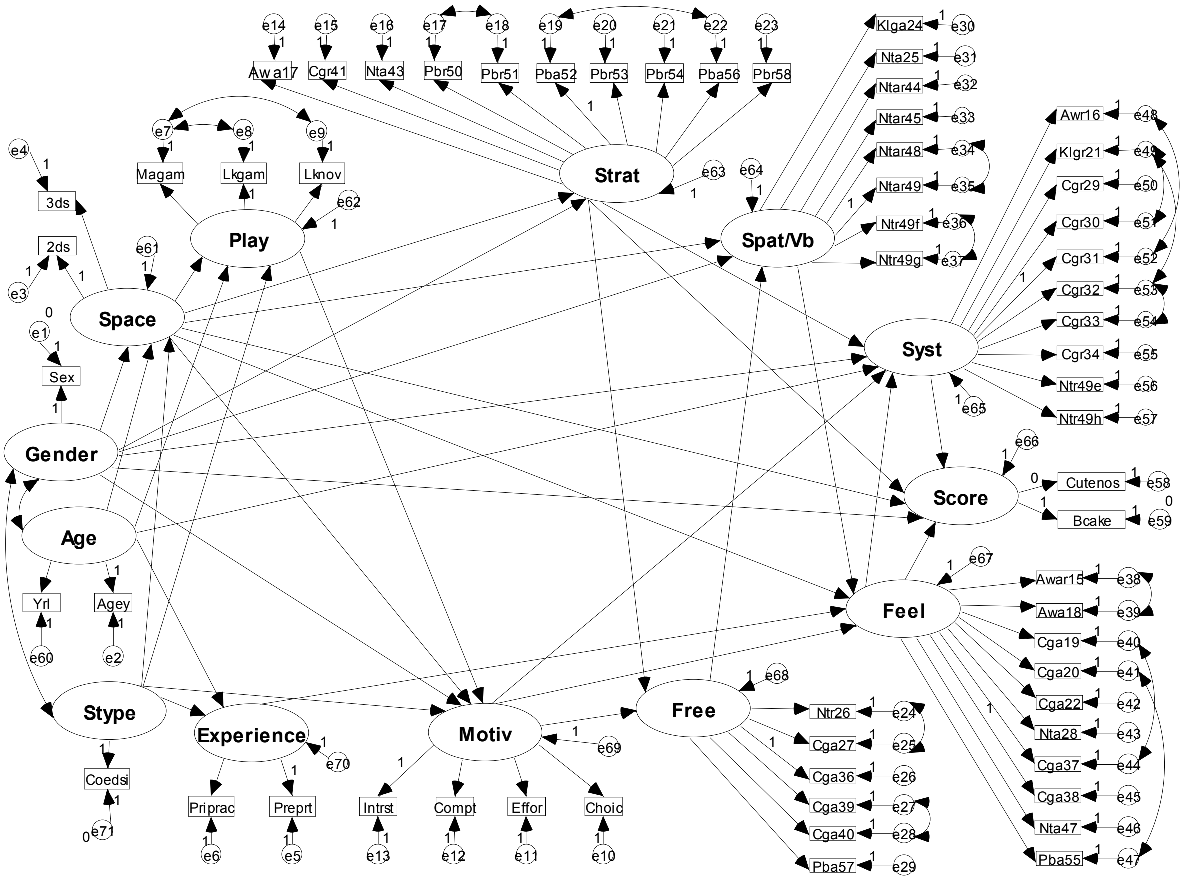

Figure 1. The final model of novel problem solving specified and identified by AMOS graphics (version 4.01)

A root mean square error of approximation RMSEA of 0.036 with a lower bound of 0.035 and an upper bound of 0.037 at the 90 per cent confidence interval indicated that the fit of the model to the available data was good (Byrne, 2001). Although a non-significant chi-square was not achieved, a normed chi-square of 1.28 indicated that the model was well fitting (Hair, Anderson, Tatham, and Black, 1995).

Thus, given that the final model of productive problem solving fitted the Cute Numbers and Birthday cake data in a highly adequate way, a comparison of the processes at work, particularly those pertaining to the Approaches to reasoning construct, between the two novel problems could then be carried out.

RESULTS

The AMOS Graphics path diagrams showing the standardised estimates for the Cute Numbers model and Birthday Cake model of novel problem solving are given in Figures 2 and 3 respectively. With three exceptions, there were no significant differences in the outer measurement paths between the two models and for this reason most of the discussion in this paper focuses on differences arising in the inner structural model between problems.

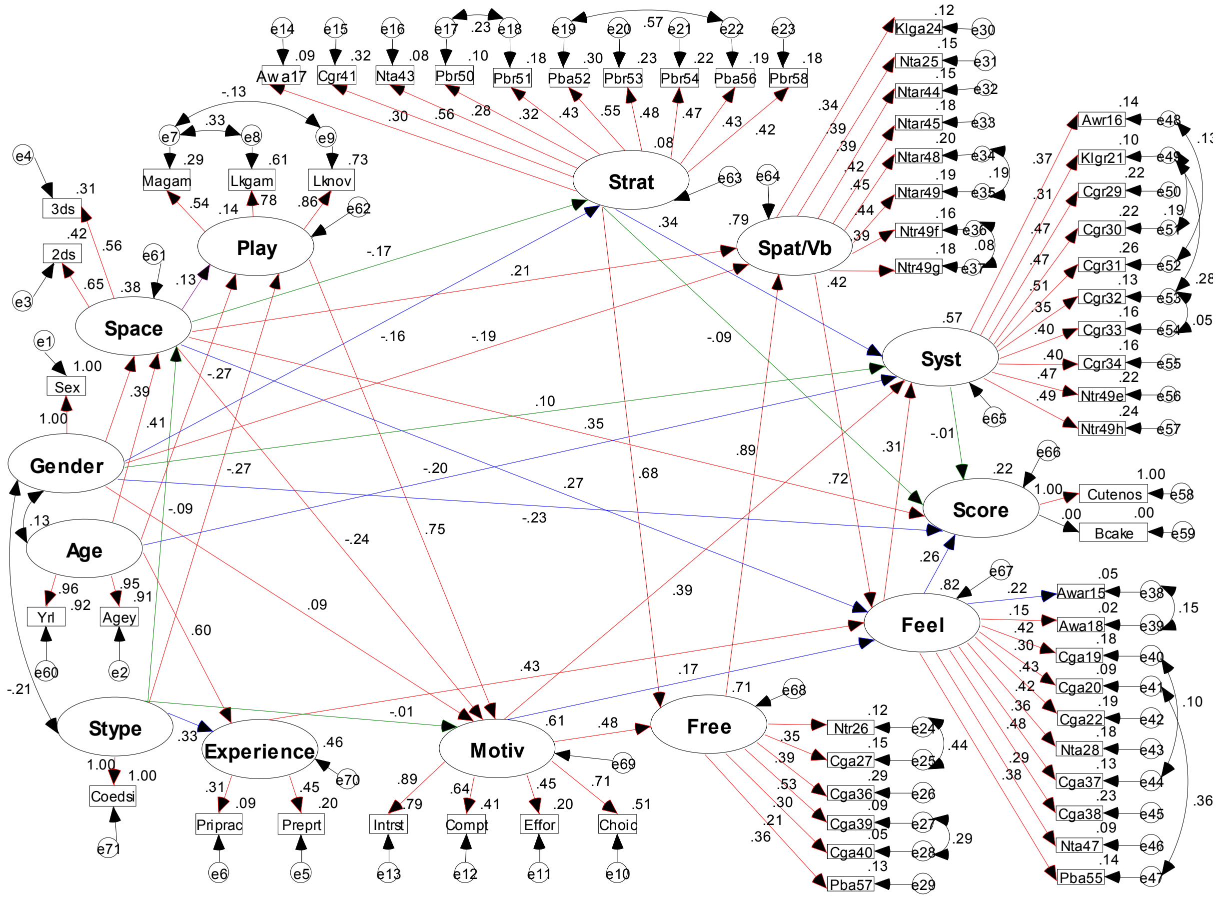

Figure 2. AMOS path diagram with standardised estimates for the Cute Numbers model

The three outer measurement exceptions involve item **(Cga37)** concerning originality and inventiveness, item **(Cga19)** on using imagination and item **(Awar15)** concerning associative reasoning and intra-personal checking within the Feeling approach to reasoning. In each instance the loading was greater in the Cute Numbers model of novel problem solving.

Table 2 presents the squared multiple correlations or the proportions of variance in the dependent variables explained by the collective set of predictors for the Cute Numbers and Birthday Cake models. The proportions of variance explained are also shown above the ovals of the dependent variables in Figures 2 and 3.

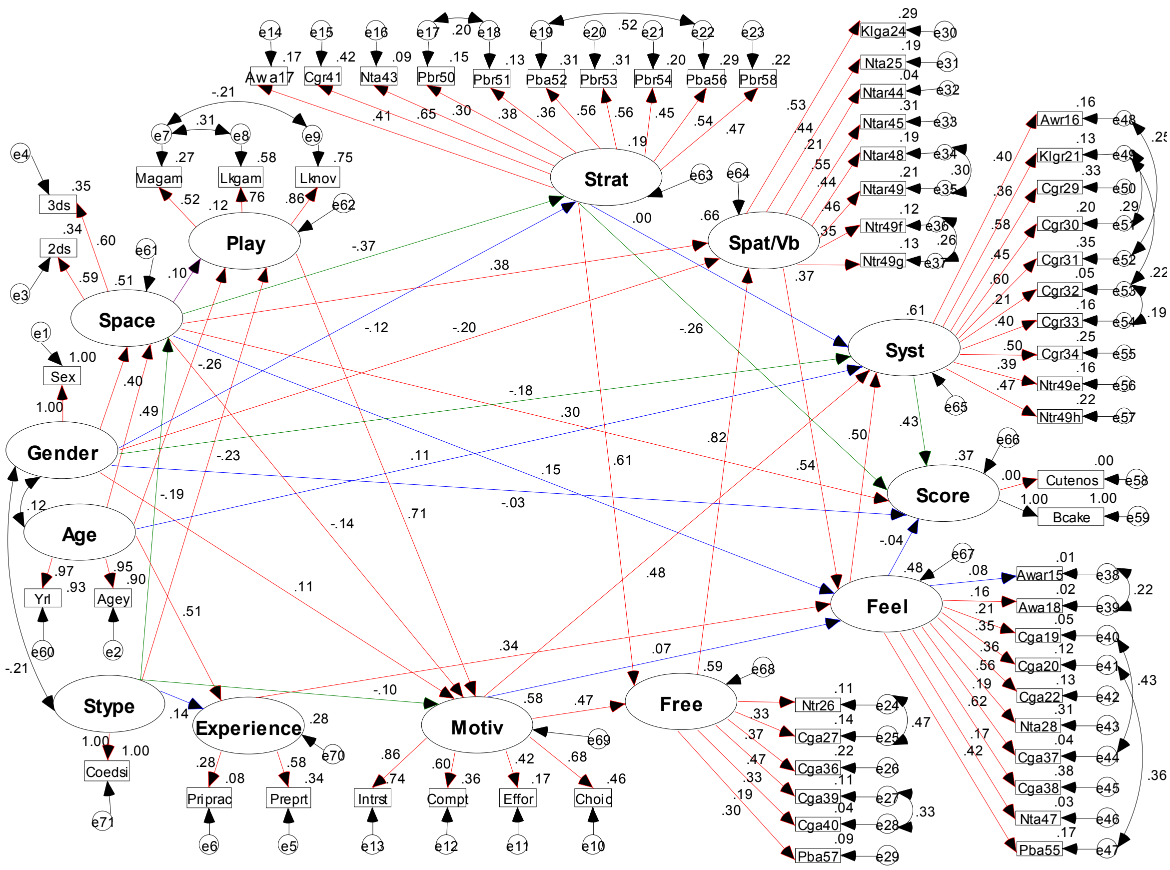

Figure 3. AMOS path diagram with standardised estimates for the Birthday Cake model

Table 2. Estimates of variance explained in the analyses of the Cute Numbers and Birthday Cake path models

	Squared Multiple Correlations	
	Cute Numbers	Birthday Cake
Latent Variable Produced	**Estimate**	**Estimate**
Space (Two & three dimensional spatial ability)	0.385	0.514
Play (Plays & enjoys novel maths games)	0.144	0.117
Experience (Previous participat^n & prior practice)	0.462	0.283
Motiv (Intrinsic motivation)	0.607	0.579
Strat (Strategic approach to reasoning)	0.079	0.192
Free (Free flowing approach to reasoning)	0.710	0.592
Spat/Vb (Spatial-verbal approach to reasoning)	0.793	0.658
Feel (Feeling approach to reasoning)	0.820	0.478
Syst (Systematic approach to reasoning)	0.571	0.607
Score (Score on the problem)	0.219	0.368
Mean	**0.479**	**0.439**

The emphases within the Cute Numbers model are seen in the significant role played by the Feeling factor **Feel**, the Free-flowing factor **Free** and the Spatial-verbal factor **Spat/Vb**. This is supported by the large variance explained in each of these factors in the Cute Numbers model. By contrast the emphases within the Birthday Cake model is seen in the significant role played by the Systematic factor **Syst**, the Spatial-verbal factor **Spat/Vb** and the Free-flowing factor **Free**, each supported by large variances. Further, the amount of variance explained by the Feeling factor **Feel**, within the Birthday Cake model (R^2 = 0.48) is considerably less than that explained by **Feel** in the Cute Numbers model (R^2 = 0.82). Overall, however, there is slightly more variance explained in the Cute Numbers model (mean R^2 = 0.48) than the Birthday Cake model (mean R^2 = 0.44).

While standardised solutions are useful for making 'within model' comparisons, unstandardised solutions are recommended for making comparisons between models. For this reason Table 3 displays the unstandardised results for each inner path in both the Cute Numbers and Birthday Cake models. A t-test conducted on each unstandardised path coefficient was used to locate any significant differences (Hair et al., 1995) between the Cute Numbers and Birthday Cake problems.

Table 3 shows that for each structural model the majority of corresponding paths are not significantly different from one other. However there are some notable exceptions. These exceptions involve the structural paths relating to the effects of **Space, Strat** (strategic approach to reasoning, **Spat/Vb** (spatial verbal approach to reasoning), **Feel** (feeling approach to reasoning) and **Syst** (systematic approach to reasoning) on **Score** (score on problem).

Comparing Structural (Inner) Models

The salient similarities and differences within each of the five endogenous constructs within the model are discussed below. In drawing comparisons between models, Cohen's (1992) system for classifying the size of an effect as being small, medium or large with standardised regression weights is used. In particular standardised estimates between 0.10 and 0.25 are considered small, estimates between 0.25 and 0.40 are classified medium and estimates greater than 0.40 are considered large.

Personal abilities

The Personal abilities construct is characterised by the latent variable **Space**. In both models there is a large effect of **Age** on **Space** with older students performing better on tests of two and three-dimensional spatial ability. However this **Age** (t =5.49) effect is significantly greater in the Cute Numbers problem. In a similar way there is a large effect of **Gender** on **Space** with boys tending to perform better on tests of two and three-dimensional spatial ability. As with **Age**, the **Gende**r (t = 5.37) effect is significantly more pronounced in the Cute Numbers problem.

Interestingly five of the ten significantly different paths between models involve the latent variable **Space.** This indicates there are differences in demand for spatial processing between problems. In particular the Cute Numbers problem has a high visual-spatial processing demand while the Birthday Cake problem is characterised by a high verbal processing demand.

Perseverance and motivation

The Perseverance and motivation construct is characterised by the latent variables **Play (**Maths games and puzzles**)** and **Motiv** (intrinsic motivation)**.** In each model there is a large effect of **Play** on **Motiv**, with students who like novel problems and who both play and enjoy mathematics games and puzzles being more intrinsically motivated.

Table 3. The results to the Structural model for the Cute Numbers and Birthday Cake data showing unstandardised regression coefficients

			The Structural Model										
			Cute Numbers N= 387					Birthday cake N= 360					
			Unstandardised				Standardised	Unstandardised				Standardised	t-value
Structural Path			Estimate	S.E.	C.R.	P	Estimate	Estimate	S.E.	C.R.	P	Estimate	Cutenos-Bcake
Space	<--	Age	6.454	1.041	6.202	0.00	0.407	0.706	0.103	6.849	0.00	0.487	5.49 *
Space	<--	Gender	12.513	2.102	5.953	0.00	0.392	1.167	0.204	5.72	0.00	0.401	5.37 *
Space	<--	Stype	-3.074	1.947	-1.579	0.11	-0.095	-0.559	0.179	-3.129	0.00	-0.190	-1.29
Play	<--	Space	0.006	0.004	1.597	0.11	0.130	0.049	0.046	1.052	0.29	0.099	-0.93
Play	<--	Stype	-0.401	0.084	-4.757	0.00	-0.268	-0.333	0.09	-3.709	0.00	-0.232	-0.55
Play	<--	Age	-0.200	0.048	-4.136	0.00	-0.274	-0.185	0.055	-3.391	0.00	-0.261	-0.21
Experience	<--	Age	0.128	0.022	5.711	0.00	0.596	0.140	0.024	5.875	0.00	0.513	-0.37
Experience	<--	Stype	0.144	0.039	3.677	0.00	0.327	0.079	0.044	1.791	0.07	0.142	1.11
Motiv	<--	Play	0.915	0.075	12.245	0.00	0.753	0.868	0.079	10.957	0.00	0.714	0.43
Motiv	<--	Gender	0.166	0.093	1.776	0.08	0.093	0.187	0.098	1.898	0.06	0.108	-0.16
Motiv	<--	Stype	-0.009	0.085	-0.109	0.91	-0.005	-0.183	0.089	-2.059	0.04	-0.104	1.41
Motiv	<--	Space	-0.014	0.004	-3.579	0.00	-0.242	-0.085	0.042	-2.007	0.05	-0.144	1.68
Strat	<--	Gender	-0.162	0.076	-2.124	0.03	-0.156	-0.126	0.081	-1.548	0.12	-0.117	-0.32
Strat	<--	Space	-0.006	0.003	-1.895	0.06	-0.173	-0.135	0.037	-3.653	0.00	-0.367	3.48 *
Free	<--	Strat	0.449	0.083	5.386	0.00	0.685	0.371	0.078	4.748	0.00	0.608	0.68
Free	<--	Motiv	0.182	0.033	5.545	0.00	0.477	0.178	0.037	4.778	0.00	0.470	0.08
Spat/Vb	<--	Free	1.002	0.196	5.103	0.00	0.889	1.018	0.228	4.469	0.00	0.825	-0.05
Spat/Vb	<--	Gender	-0.149	0.053	-2.817	0.01	-0.195	-0.165	0.065	-2.555	0.01	-0.204	0.19
Spat/Vb	<--	Space	0.005	0.002	2.242	0.03	0.205	0.105	0.032	3.318	0.00	0.379	-3.12 *
Feel	<--	Spat/Vb	0.643	0.137	4.694	0.00	0.716	0.518	0.121	4.285	0.00	0.543	0.68
Feel	<--	Experience	0.693	0.199	3.476	0.00	0.434	0.471	0.185	2.546	0.01	0.336	0.82
Feel	<--	Space	0.006	0.002	2.949	0.00	0.274	0.039	0.023	1.697	0.09	0.147	-1.43
Feel	<--	Motiv	0.067	0.030	2.231	0.03	0.173	0.032	0.035	0.929	0.35	0.072	0.76
Syst	<--	Feel	0.401	0.137	2.921	0.00	0.308	0.588	0.126	4.655	0.00	0.505	-1.00
Syst	<--	Motiv	0.194	0.040	4.814	0.00	0.387	0.250	0.039	6.414	0.00	0.481	-1.00
Syst	<--	Age	-0.090	0.032	-2.836	0.01	-0.201	0.049	0.028	1.749	0.08	0.110	-3.27 *
Syst	<--	Gender	0.088	0.049	1.794	0.07	0.098	-0.163	0.052	-3.159	0.00	-0.181	3.51 *
Syst	<--	Strat	0.296	0.076	3.870	0.00	0.343	0.002	0.056	0.041	0.97	0.003	3.11 *
Score	<--	Syst	-0.033	0.236	-0.141	0.89	-0.012	1.407	0.295	4.771	0.00	0.430	-3.81 *
Score	<--	Feel	0.933	0.363	2.570	0.01	0.263	-0.153	0.369	-0.414	0.68	-0.040	2.10 *
Score	<--	Gender	-0.559	0.166	-3.362	0.00	-0.228	-0.096	0.180	-0.534	0.59	-0.033	-1.89
Score	<--	Space	0.027	0.008	3.371	0.00	0.352	0.3	0.095	3.168	0.00	0.297	-2.86 *
Score	<--	Strat	-0.217	0.179	-1.209	0.23	-0.092	-0.713	0.199	-3.584	0.00	-0.260	1.85

* indicates t-values > 1.96 which are considered significant at the five percent level

Opportunity to learn

The Opportunity to learn construct is typified by the latent variable **Experience** (prior practice and past experience). There is a large effect of **Age** on **Experience** in each of the Cute Numbers and Birthday Cake examples with older students possessing greater experience.

Approaches to reasoning

The five latent variables, **Strat** (Strategic approach to reasoning), **Free** (Free-flowing approach to reasoning), **Spat/Vb** (Spatial-verbal approach to reasoning), **Feel** (Feeling approach to reasoning) and **Syst** (Systematic approach to reasoning) comprise the Approaches to reasoning construct.

There is a large effect of **Strat** on **Free**, of **Free** on **Spat/Vb** and of **Spat/Vb** on **Feel** in each model. While a large effect of **Feel** on **Syst** is observed in the Birthday Cake model, a medium effect of **Fee**l on **Syst** is present in the Cute Numbers model

In addition, although a large effect of **Syst** on **Score** is observed in the Birthday Cake model, a negligible, but negative effect of **Syst** on **Score** is found for the Cute Numbers model. However, this is counter balanced by a medium effect of **Feel** on **Score** in the Cute Numbers model but a negligible direct effect of **Feel** on **Score** in the Birthday Cake model. Further differences, in relation to the effects of the Approaches to reasoning variables on **Score,** are detailed under the construct Achievement outcome and in the Discussion section below.

In each model there is a medium to large effect of **Motiv** on each of **Free** and **Syst.** This indicates that intrinsically motivated students employ higher levels of the Free-flowing and Systematic approaches to reasoning. Further there is a medium to large effect of **Experience** on **Feel** indicating that the more experienced students draw upon greater amounts of the Feeling approach to reasoning.

The paths leading from **Space** to **Strat** (t = 3.48) and from **Space** to **Spat/Vb** (t = -3.12) are significantly different between models. Students with lower levels of two and three-dimensional spatial ability are more reliant on the Strategic approach to reasoning. By contrast students with higher levels of two and three-dimensional spatial ability, make greater use of the Spatial-verbal approaches to reasoning. These differences are more pronounced within the Birthday Cake example.

Three structural paths leading to **Syst** from **Age** (t= -3.27), **Gender** (t = 3.51) and **Strat** (t = 3.11) are also significantly different between models. Younger students are more likely to use the Systematic approach to reasoning in the Cute Numbers case but older students are more likely to use the Systematic approach to reasoning in the Birthday Cake example. In addition boys are more likely than girls to use the Systematic approach in the Cute Numbers case but girls are more likely than boys to use the Systematic approach in the Birthday Cake example.

Thus, the Cute Numbers example is characterised by younger boys using the Systematic approach, while the Birthday Cake example, is characterised by older girls using the Systematic approach to reasoning.

Moreover students in both the Cute Numbers and Birthday Cake examples who employ higher levels of the Strategic approach also employ higher levels of the Systematic approach to reasoning. However, this effect is more pronounced in the Cute Numbers example.

Achievement outcome

The Achievement outcome construct is characterised by the latent variable **Score.** This latent variable measures the score on either the Cute Numbers or Birthday Cake problem.

Three structural paths leading to **Score** are significantly different between models. These involve one from **Syst** (t = -3.81), one from **Feel** (t = 2.10) and one from **Space** (t = -2.86). In particular successful students within the Cute Numbers model use lower amounts of the Systematic approach to reasoning but higher amounts of the Feeling approach to reasoning. By contrast, successful students within the Birthday Cake problem use higher amounts or the Systematic approach to reasoning but lower amounts of the Feeling approach to reasoning. However the Feeling approach to reasoning does have a large effect on the Systematic approach in the Birthday Cake example. Further students with higher levels of two and three-dimensional spatial ability are more successful in both models although this effect is more pronounced in the Birthday Cake example.

There is a medium effect of **Strat** on **Score** in the Birthday Cake example but a negligible effect of **Strat** on **Score** in the Cute Numbers example. The effects on both tasks are negative. Thus students who proceed straight to **Score** from **Strat**, without passing through any intervening stage, particularly the illumination phase, as indicated by the latent variables of **Free** and more particularly **Feel**, are less likely to solve the novel problem successfully. This highlights the importance of intuition and intimation in solving novel mathematics problems successfully.

DISCUSSION

Evidence presented in the results above indicates that different processes are at work between the Cute Numbers and Birthday Cake problems. Non-cognitive processes dominate in the Cute Numbers problem while cognitive ones dominate in the Birthday Cake problem.

Of note is the chain of relationships arising between the process variables of novel problem solving. In particular, the path leading from **Strat** (Strategic approach to reasoning) to **Free** (Free-flowing approach to reasoning) to **Spat/Vb** (Spatial-verbal approach to reasoning), to **Feel** (Feeling approach to reasoning) to **Syst** (Systematic approach to reasoning) is significant in both models. The path is summarised below

Strat → Free → Spat/Vb → Feel→ Syst

However, while the next step to the outcome variable **Score** from **Syst** within the model, is significant in the Birthday Cake task, this is not the case in the Cute Numbers problem. A significant route from **Feel** direct to **Score** arises instead. However, the direct path from **Feel** to **Score** in the Birthday Cake problem is not significant. The successful paths for each problem are summarised below.

For the Birthday Cake problem:
Strat → Free → Spat/Vb → Feel → Syst → Score.

For the Cute Numbers problem:
Strat → Free → Spat/Vb → Feel → Score.

Thus the Cute Numbers task relies more heavily on associative intuitive forms of reasoning, while the Birthday Cake relies on systematic rule based forms of reasoning. However while the non-cognitive approach to problem solving adopted in the Cute Numbers task is independent of cognitive elements, the reverse is not the case. Successful problem solving in the Birthday Cake example depends on the interaction of both cognitive and non-cognitive systems of reasoning.

It should be pointed out that the cognitive processes involving **Strat** (Strategic approach to reasoning) are important to the successful solution of both problems. This requires that the operational path leading from **Strat** goes through the predominately non-cognitive sequence of **Free → Spat/Vb → Feel**. The path **Strat → Score** is not significant in the Cute Numbers problem, and although significant in the Birthday Cake task, the path is negative. This reveals cognitive processing to be dependent on the outcome of non-cognitive processing for its successful execution. In addition, while the path from **Strat** to **Syst** is significant in the Cute Numbers problem, the follow-on path from **Syst** to **Score** is not.

Thus, students who adopt a problem solving strategy without recourse to incubation such as that reflected in a state of defocused attention leading to some kind of intimation as represented by **Free**, or who do not realise a feeling of knowing that is indicative of some kind of illumination as represented by **Feel,** are not successful in solving the novel problem. However, students combining a Strategic approach to reasoning with a Free-flowing and Feeling approach to reasoning are successful.

The models presented here indicate that while highly similar processes are at work in solving novel mathematics problems there are different processing emphases expected within different problems. In particular the Cute Numbers problem demanding a high degree of visual-spatial processing was typified by feeling cognitions such as **Free** and **Feel** having a direct effect on the success of a solution. Systematic cognitions such as those of a **Syst** had little or no effect on the solution. By contrast the Birthday Cake problem with a high verbal processing demand was typified with an indirect effect of feeling cognitions and a direct effect of systematic cognitions in the problem-solving process.

CONCLUSION

The study documented in this paper set out to ascertain the relationship between cognitive and non-cognitive systems of reasoning and success in novel problem solving. It found, contrary to popular belief, that the non-cognitive feeling aspect of novel problem solving, rather than the cognitive rule based aspect, is what is crucial to success. This is shown by the fact that a solution can be found straight from **Feel** independent of **Syst.** Although a solution can be found from **Syst,** its involvement in the novel problem-solving context is not independent of **Feel**. Thus for a novel solution to manifest, due attention needs to be given to feeling.

This dependence of cognitive resources on non-cognitive ones, although surprising, is consistent with neuro-scientific research. Reason, Damasio (1994) has shown, is inseparably dependent on feeling. Individuals with lesions in a small frontal area of the brain impairing the connection between reasoning and feeling are unable to arrive at a conclusion regarding two rational alternatives. This is despite the fact that psychometric tests show their reasoning to be perfectly rational. Reduced emotion and feeling impairs the ability to make a decision leaving such individuals in the "predicament as *to know but not to feel*" (Damasio, 1994, p.45, (italics in original)). Thus if rationality is to be increased due attention must be given to feeling.

Therefore teachers of novel mathematics problem-solving would do well to alert students to their inner resources, found by attending to feeling in its deeper sense, just as Poincaré is reported to have done. This is not meant to suggest that students should not check these feelings of knowing and test their validity, but rather, that they should recognise that by attending to feeling, greater rationality and novel problem solving productivity, are more likely to be achieved.

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