Numerical integration

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Senior secondary students cover numerical integration techniques in their mathematics courses. In particular, students would be familiar with the *midpoint rule*, the elementary *trapezoidal rule* and *Simpson's rule*. The following paper derives these techniques by methods which secondary students may not be familiar with and an approach that undergraduate students should be familiar with. Secondary students will also find interesting the *two-point Gauss rule*, which is an extension of the trapezoidal rule. There are many applications of integral calculus and developing a deeper understanding of some of the numerical methods will increase understanding of the techniques. The methods chosen in this paper have been investigated as secondary students will be familiar with their applications. However, secondary text books and teachers may not use the techniques covered in this paper, and this alternate approach may increase the understanding of the importance and applications of the techniques, as well as increase an appreciation of the beauty of mathematics in general. The paper also provides a detailed summary of the techniques that will be beneficial for undergraduate students.

Numerical integration enables approximations to be found for

$$\int_a^b f(x)dx$$

where the integral for $f(x)$ cannot be written in terms of elementary functions. A use of the definite integral is to determine the area between a curve and the horizontal axis (see Figure 1).

In this article the midpoint rule, the elementary trapezoidal rule, the two-point Gauss rule and Simpson's elementary rule will be developed.

The general form of a numerical integration rule — also known as the quadrature rule — is

$$\sum_{i=1}^{N} w_i f(x_i)$$

In this form, N is a natural number, w_i are called the *weights* or *coefficients* and

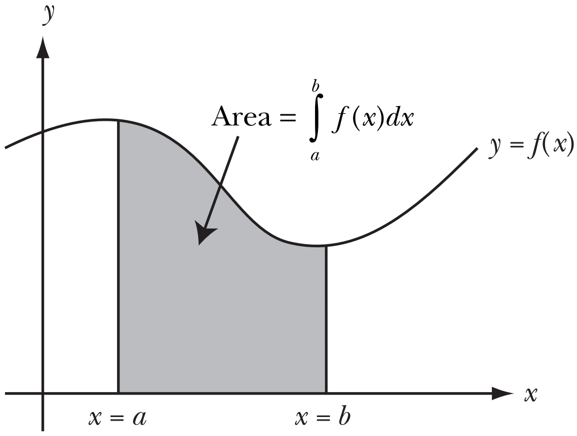

Figure 1

the values of $f(x)$ are called the *ordinates*. The quadrature rule is not exact for every $f(x)$. However, the rule can be exact for some simple functions such as 1, x, x^2. Say we wish to integrate

$$\int_0^1 f(x)dx$$

by choosing an appropriate w_1 and $f(x_1)$ and $N = 1$, i.e.:

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{N} w_1 f(x_1)$$

If we consider $f(x) = 1$, then $f(x_1) = 1$.

Now, $$\int_0^1 f(x)dx = \int_0^1 1\,dx = 1$$

So, $$w_1.f(x_1) = w_1.1 = w_1$$

By equating $\int_0^1 f(x)dx$ and $w_1.f(x_1)$, then $w_1 = 1$.

Say we now consider $f(x) = x$, then $f(x_1) = x_1$.

Now, $$\int_0^1 f(x)dx = \int_0^1 x\,dx = \frac{1}{2}$$

So, $$w_1.f(x_1) = w_1.x_1.$$

By equating $\int_0^1 f(x)dx$ and $w_1.f(x_1)$, then: $w_1.x_1 = \frac{1}{2}$. But since $w_1 = 1$, then $x_1 = \frac{1}{2}$.

Therefore $\int_0^1 f(x)dx \approx f\left(\frac{1}{2}\right)$ which is the midpoint rule on [0, 1].

Now, consider the midpoint rule on $[a, b]$.

Say, $$\int_a^b f(x)dx \approx w_1 f(x_1)$$

If $f(x) = 1$, then $\int_a^b 1\, dx = b - a.$

Also, $w_1.f(x_1) = w_1.1 = w_1$

Therefore, $w_1 = b - a.$

Consider $f(x) = x$, then

$$\int_a^b x\, dx = \frac{1}{2}\left(b^2 - a^2\right)$$

Now, $w_1.f(x_1) = w_1.x_1$; but, $w_1 = b - a$, so,

$$\left(b-a\right).x_1 = \frac{1}{2}(b^2 - a^2)$$

$$\left(b-a\right).x_1 = \frac{1}{2}(b-a)\,(b+a)$$

$$x_1 = \frac{b+a}{2}$$

Therefore,

$$\int_a^b f(x)\, dx \approx \left(b-a\right) f\left(\frac{a+b}{2}\right)$$

which is the midpoint rule on $[a,b]$. Note that the midpoint rule will integrate linear functions exactly.

In using the endpoints of the interval when using the simple quadrature rule,

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i)$$

we can commence with $x_1 = 0$, $x_2 = 1$ where $N = 2$; i.e.:

$$\int_0^1 f(x)dx = w_1 f(0) + w_2 f(1)$$

As there are two unknowns, w_1 and w_2 are chosen such that 1 and x can be exactly integrated.

Now, when $f(x) = 1$ then $\int_0^1 f(x)dx = 1$

That is, $w_1 f(0) + w_2 f(1) = w_1.1 + w_2.1 = w_1.w_2$

as $f(0) = f(1) = 1$.
So, $w_1 = w_2 = 1$.

Letting $f(x) = x$ $\int_0^1 f(x)dx = \frac{1}{2}$

Then, $w_1.f(0) + w_2.f(1) = w_1.0 + w_2.1 = w_2$

as $f(0) = 0$ and $f(1) = 1$.

Therefore, $w_2 = \frac{1}{2}$, which leads to $w_1 = \frac{1}{2}$.

Thus $$\int_0^1 f(x)dx = \frac{1}{2}\left[f(0)+f(1)\right]$$

Now, consider the interval $[a,b]$ where $x_1 = a$ and $x_2 = b$.

When $f(x) = 1$, $$\int_a^b f(x)dx = b-a$$

Also, $$w_1 f(a) + w_2 f(b) = w_1 + w_2$$

Therefore, $$w_1 + w_2 = b - a.$$

Now, when $f(x) = x$, $$\int_a^b f(x)dx = \frac{1}{2}\left(b^2-a^2\right)$$

Also, $$w_1 f(a) + w_2 f(b) = w_1 a + w_2 b.$$

$$\therefore w_1 a + w_2 b = \frac{1}{2}\left(b^2-a^2\right)$$

Now, $$w_1 + w_2 = b - a \qquad \ldots(1)$$

$$w_1 a + w_2 b = \frac{1}{2}(b-a)(b+a) \qquad \ldots(2)$$

Multiply Equation (1) by a and subtract from Equation (2):

$$w_2 = \frac{b-a}{2} \text{ and } w_1 = \frac{b-a}{2}$$

Therefore, the general form of the elementary trapezoidal rule on $[a,b]$ is:

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

This Rule will integrate linear functions exactly.

The two-point Gauss rule is an extension of the trapezoidal rule. $N = 2$ is now considered. This quadrature rule is of the form:

$$\int_0^1 f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

and unlike the trapezoidal rule in which x_1 and x_2 are fixed at the ends of the interval, x_1 and x_2 are not predetermined.

As there are four unknowns, w_1, w_2, x_1 and x_2 are chosen such that 1, x, x^2 and x^3 can be exactly integrated.

When $f(x) = 1$:

$$\int_0^1 1\,dx = 1$$

$$= w_1 + w_2$$

When $f(x) = x$:

$$\int_0^1 x\,dx = \frac{1}{2}$$
$$= w_1.x_1 + w_2.x_2$$

When $f(x) = x^2$:

$$\int_0^1 x^2 dx = \frac{1}{3}$$
$$= w_1.x_1{}^2 + w_2.x_2{}^2$$

When $f(x) = x^3$:

$$\int_0^1 x^3 dx = \frac{1}{4}$$
$$= w_1.x_1{}^3 + w_2.x_2{}^3$$

In solving these four equations in four unknowns:

$$w_1 = w_2 = \frac{1}{2}$$
$$x_1 = \frac{1}{2} - \frac{1}{2\sqrt{3}}$$
$$x_2 = \frac{1}{2} + \frac{1}{2\sqrt{3}}$$

This gives the two-point Gauss rule:

$$\int_0^1 f(x)\,dx \approx \frac{1}{2} f\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)$$

The technique to solve the above system of equations is difficult and beyond secondary school methods. An appropriate method can be found in Kelly (1967, p. 57).

When considering the general integral:

$$\int_a^b f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

the Two-Point Gauss Rule can be derived similarly by considering the following integrals:

For $f(x) = 1$:

$$\int_a^b 1\,dx = b - a$$
$$= w_1 + w_2$$
$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$$
$$= w_1 x_1 + w_2 x_2$$

$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$$
$$= w_1{x_1}^2 + w_2{x_2}^2$$
$$\int_a^b x^3\,dx = \frac{b^4-a^4}{4}$$
$$= w_1{x_1}^3 + w_2{x_2}^3$$

The solutions to the simultaneous equations are:

$$w_1 = w_2 = \frac{b-a}{2}$$
$$x_1 = \left(\frac{b-a}{2}\right)\left(\frac{-1}{\sqrt{3}}\right) + \frac{b+a}{2}$$
$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

Hence,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

The technique to solve the above systems of equations is difficult, and an appropriate method can be found in Kelly (1967, p. 57).

Simpson's elementary rule will now be considered, with $N = 3$. Say on the interval [0,1] we choose

$$x_1 = 0,\; x_2 = \frac{1}{2} \text{ and } x_3 = 1$$

for $$\int_0^1 f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

Then $$\int_0^1 f(x)\,dx = w_1 f(0) + w_2 f\left(\frac{1}{2}\right) + w_3 f(1)$$

Values need to be chosen for w_1, w_2 and w_3 such that 1, x and x^2 can be exactly integrated.

For $f(x) = 1$:

$$\int_0^1 f(x)dx = 1$$
$$= w_1 + w_2 + w_3$$

For $f(x) = x$:

$$\int_0^1 f(x)dx = \frac{1}{2}$$
$$= \frac{1}{2}w_2 + w_3$$

For $f(x) = x^2$:

$$\int_0^1 f(x)\,dx = \frac{1}{3}$$

$$= \frac{1}{4}w_2 + w_3$$

Solving the three simultaneous equations:

$$w_1 = w_3 = \frac{1}{6} \text{ and } w_2 = \frac{2}{3}$$

Hence, the elementary Simpson's rule is given by:

$$\int_0^1 f(x)dx \approx \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

Simpson's rule integrates quadratics as well as cubics exactly.

If the interval $[a,b]$ is taken it can be shown that the elementary Simpson's rule is given by:

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

A parabola is taken that passes through the following points of the function in the interval $[a,b]$: $(a,f(a))$, $(b,f(b))$ and $(a+b,f(a+b))$. The area under the parabola then estimates the area under the function. Following is a proof.

Let $F(x) = Ax^2 + Bx + C$ be the equation of a parabola, then:

$$\int_a^b \left(Ax^2 + Bx + C\right)dx$$

$$= \left|\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right|_a^b$$

$$= \frac{A\left(b^3 - a^3\right)}{3} + \frac{B\left(b^2 - a^2\right)}{2} + C\left(b - a\right)$$

$$= \frac{b-a}{6}\left(2A\left(a^2 + ab + b^2\right) + 3B\left(a + b\right) + 6C\right)$$

$$= \frac{b-a}{6}\left(2Aa^2 + 2Aab + 2Ab^2 + 3Ba + 3Bb + 6C\right)$$

$$= \frac{b-a}{6}\left(Aa^2 + Ba + C + Aa^2 + 2Aab + Ab^2 + 2Ba + 2Bb + 4C + Ab^2 + Bb + C\right)$$

$$= \frac{b-a}{6}\left(\left(Aa^2 + Ba + C\right) + 4\left(A\left(\frac{a+b}{2}\right)^2 + b\left(\frac{a+b}{2}\right) + C\right) + \left(Ab^2 + Bb + C\right)\right)$$

$$= \frac{b-a}{6}\left(F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right)$$

To improve the accuracy of applications of the discussed rules, the number of sample points can be increased by deriving more complicated rules or by dividing the range into many sub-intervals.

As can be seen, the numerical integration methods are able to approximate a value for a definite integral using the values of the function at points within the interval of the integrand.

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Reference

Kelly, L. G. (1967). *Handbook of numerical methods and applications.* Reading, MA: Addison-Wesley.