Investigating absolute value: A real world application

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Making connections between various representations is important in mathematics. Take, for example, absolute value. In many of today's textbooks, the students are given $|x|$, where x is any real number, and asked to find the absolute value of x. In an ensuing course, they are given $y = |x - 1|$ and asked to find the value of y for an arbitrary value of x. At another point they are asked to graph a variety of these functions and at still another point, they are asked to find all of the values of y if the equation was transformed into an inequality. In this article, we will discuss the numeric, algebraic, and graphical representations of sums of absolute values of linear functions. The initial explanations are accessible to all students who have experience graphing and who understand that absolute value simply means "distance from". More specifically, $|x - h|$ means the distance that x is from a given point, "h."

Consider the following (see Figure 1):

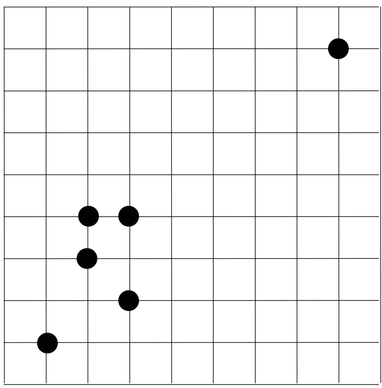

Figure 1. Our town.

The horizontal and vertical lines represent one way streets and the points represent houses which are located on corners only. Other simplifying assumptions we will make are that a fire breaks out at only one house at a time with equal probability at each house. A fire truck has to return to the fire house before going out to the next fire and at this time there is only one fire truck.

Using these assumptions, think about where the best place is to build a fire station to service these homes. The first thing that must be decided is what is meant by "best." This leads to some lively class discussions, including who lives in the outlier house — is it an orphanage or the home of a reclusive millionaire? Each decision can lead to a different mathematical method of ascertaining the "best" location for the fire station. For the purposes of this article, we will define the "best" placement as the one that minimises the total distance to all homes.

If there is only one house, the problem becomes trivial: the fire station should be across the street from the house. However, suppose there is one house which is located at position +1 on a number line. Where would you place a fire station so that it is two blocks away from this house?

Most students know the definition that $|x - 1|$ represents the distance from point x on a number line to the point +1 on the number line. So, we can see the distance from the house to a fire station placed at various points on the number line (see Figure 2).

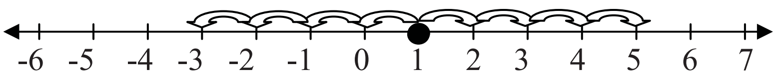

Figure 2. Representing the situation with a number line.

Another way to demonstrate this is with the coordinate plane. It can be used to graph the distance (y-axis) from the fire station to the house at +1 versus the location of the fire station (x-axis; see Figure 3).

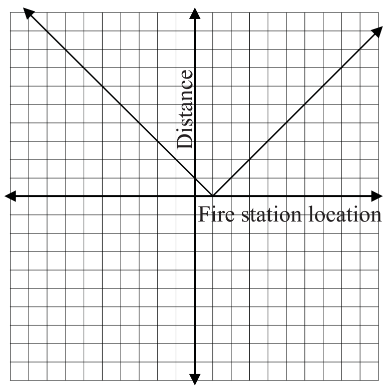

Figure 3. Representing the situation graphically.

We see that when $x = 3$ for example, $y = |3 - 1| = 2$ means when the fire station is at location 3, it is 2 units from the house located at 1. Similarly, when $x = -1$ we see that the fire station located at -1 is 2 units from the house at 1 as well ($y = |-1-1| = |-2| = 2$). Thus, a horizontal line through the point (0, 2) on our graph will intersect $y = |x - 1|$ at all points where x is 2 units from 1 as seen in Figure 4.

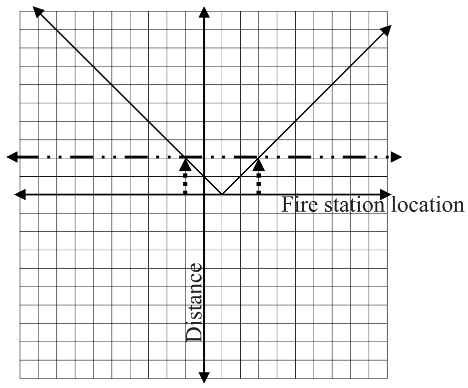

Figure 4. Two possible fire station locations, both 2 units from the house at (1, 0).

We could do the same for other house "locations." For example, $y = |x + 1| = |x - (-1)|$, represents the distance that the fire station located at x is from the house located at –1. Have students investigate other graphs in this context until they can conjecture that, in general, $y = x - k$ represents the distance that the fire station located at x is from the house located at k. They should also understand that these functions have two solutions for x for each value of y except at the vertex (when the fire station and the house are across the street from each other).

Next investigate what happens when there are two houses in our tiny city. Arbitrarily, we can say the Abbots live at –3 ($A = -3$) and the Brownes live at +5 ($B = +5$). What location for the fire station, x, will minimise the sums of the distances to both of the houses? As an entry point for all students, have them start summing the total distances with a table like the one in Table 1. Students can fill the table to determine the total distance to both houses.

Table 1. Calculations for $y = |x + 3| + |x - 5|$ for various values of x.

x	$\|x + 3\|$	$\|x - 5\|$	$y = \|x + 3\| + \|x - 5\|$
–10	7	15	22
–9	6	14	20
–8	5	13	18
–7	4	12	16
–6	3	11	14
–5	2	10	12
–4	1	9	10
–3	0	8	8
–2	1	7	8
–1	2	6	8
0	3	5	8
1	4	4	8
2	5	3	8
3	6	2	8
4	7	1	8
5	8	0	8
6	9	1	10
7	10	2	12
8	11	3	14
9	12	4	16
10	13	5	18

$y = 8$ when $A \le x \le B$

Experience has shown that when students fill the table down, they see that the numbers are changing by one. As they add the two distances, they do get confused at first when they see the total distance being repeated for nine locations. In fact, if they are not attentive they will think the pattern continues to decrease by two and miss the point of the table.

If possible, students should also graph the equation $y = |x + 3| + |x - 5|$. The x-axis represents the location of the fire station and the y-axis represents the total distance from the fire station to each of the houses. The graph will show the horizontal section at $y = 8$ when $-3 \le x \le 5$ as in Figure 5.

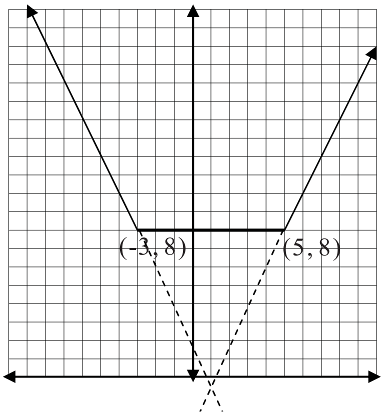

Figure 5. Graph of $y = |x + 3| + |x - 5|$.

We can now see that our table and graph reveal the solution of where the best placement of the fire station is. Students should see that in general $y = |x - A| + |x - B|$ has a minimum of $B - A$ when $A \le x \le B$.

Next, have students investigate what happens when there are three houses, A, B, and C. Where should the fire station (x) be built in order to minimise the sum of the distances from x to A, x to B, and x to C? Using a number line, arbitrarily place $A = -3$, $B = 2$ and $C = 5$ (see Figure 6), and have students make a table for various values of the fire station location x, as in Table 2.

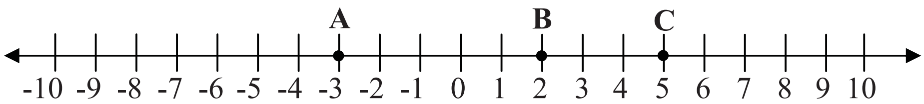

Figure 6. Situation with three houses.

This time there is only one place to build the fire station, at 2, which means the fire station should be built across the street at house B. This, too, makes sense because we know from the previous situation that C – A = 5 –(–3) = 8 is the minimum number of units from a fire station located between A and C and building the station anywhere else but at B will add the distance to B thereby increasing the total distance.

Students should use the data or a graphing calculator to graph $y = |x + 3| + |x - 2| + |x - 5|$ (see Figure 7). Although students have studied mean, mode and median, the connection to median may have to be explicitly pointed out if they do not recognise it.

Table 2. Calculations for $|x + 3| + |x - 2| + |x - 5|$ *for various values of x.*

Fire station located at x	$\lvert x + 3\rvert$	$\lvert x - 2\rvert$	$\lvert x - 5\rvert$	$\lvert x + 3\rvert + \lvert x - 2\rvert + \lvert x - 5\rvert$
–10	7	12	15	34
–9	6	11	14	31
–8	5	10	13	28
–7	4	9	12	25
–6	3	8	11	22
–5	2	7	10	19
–4	1	6	9	16
–3	0	5	8	13
–2	1	4	7	12
–1	2	3	6	11
0	3	2	5	10
1	4	1	4	9
2	5	0	3	8 minimum
3	6	1	2	9
4	7	2	1	10
5	8	3	0	11
6	9	4	1	14
7	10	5	2	17
8	11	6	3	20
9	12	7	4	23
10	13	8	5	26

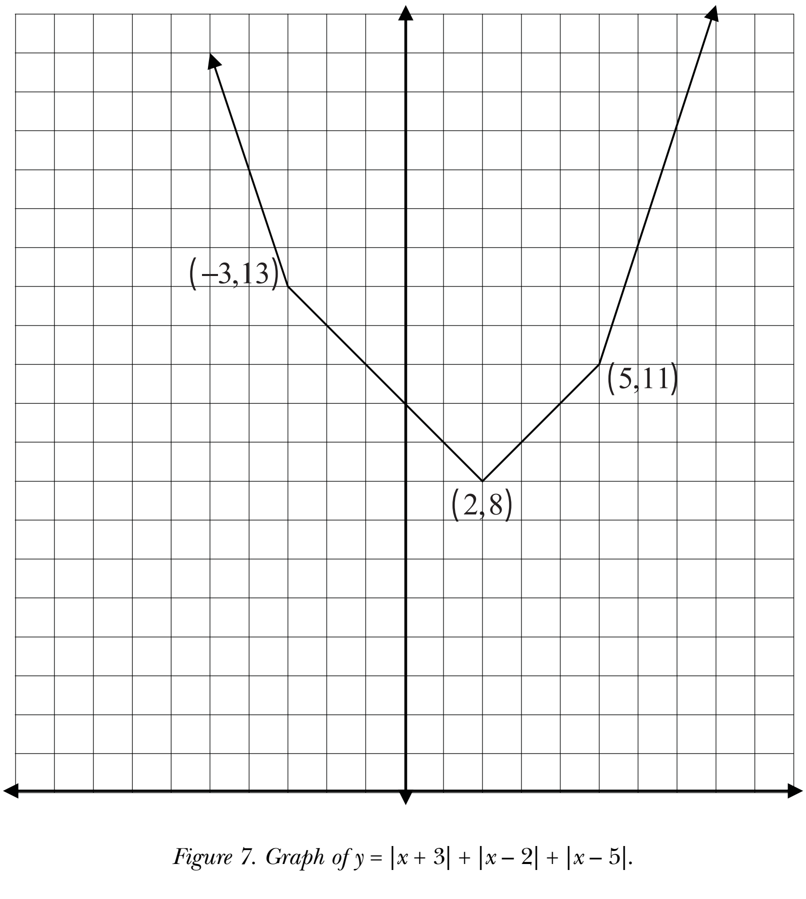

Figure 7. Graph of $y = |x + 3| + |x - 2| + |x - 5|$.

Does this generalise for $y = |x - A| + |x - B| + |x - C|$ where $A < B < C$? Let us continue our investigation to see if we can answer this.

Next, we talk about building a fire station for four houses, A, B, C, and D which lie along the same street. Using the number line (Figure 8), label $A = -3$, $B = 1$, $C = 2$ and $D = 5$ and have students make a table for various values of x as in Table 3.

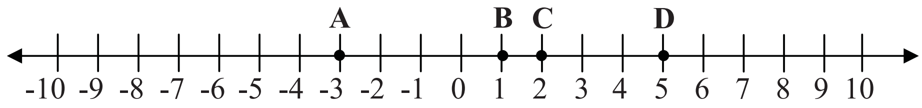

Figure 8. Situation with four houses.

Table 3. Calculations for $y = |x + 3| + |x - 1| + |x - 2| + |x - 5|$ for various values of x.

Fire station located at x	$\|x + 3\|$	$\|x - 1\|$	$\|x - 2\|$	$\|x - 5\|$	$y = \|x + 3\| + \|x - 1\| + \|x - 2\| + \|x - 5\|$
–10	7	11	12	15	45
–9	6	10	11	14	41
–8	5	9	10	13	37
–7	4	8	9	12	33
–6	3	7	8	11	29
–5	2	6	7	10	25
–4	1	5	6	9	21
–3	0	4	5	8	17
–2	1	3	4	7	15
–1	2	2	3	6	13
0	3	1	2	5	11
1	4	0	1	4	9 $y = 9$ when $1 \leq x \leq 2$
2	5	1	0	3	9
3	6	2	1	2	11
4	7	3	2	1	13
5	8	4	3	0	15
6	9	5	4	1	19
7	10	6	5	2	23
8	11	7	6	3	27
9	12	8	7	4	31
10	13	9	8	5	35

Graphing these results shows that once again, there is a range of values for x that will minimize the total distance, or median interval, from the fire station to the four houses. In our example, the minimum distance is 9 units when $1 \leq x \leq 2$, that is, when the fire station is between the two middle houses, B and C, inclusive (see Figure 9).

As a final example, have students investigate (on their own) a case using five points (house locations). The points might be $A = -3$, $B = -1$, $C = 2$, $D = 3$ and $E = 5$. Any values can be used; however reusing previous locations minimises the tediousness of filling in the table. Have them complete a table of sums of distances at x from the points and graph:

$$y = |x + 3| + |x + 1| + |x - 2| + |x - 3| + |x - 5|.$$

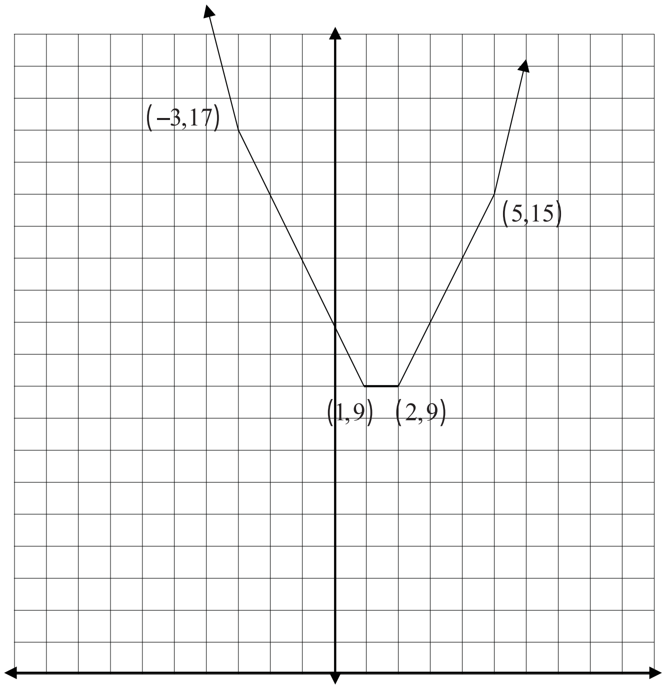

Figure 9. Graph of $y = |x + 3| + |x - 1| + |x - 2| + |x - 5|$.

At this point the students should be able to see the relationship between the four graphs and generalise the problem. If there is an even number of points (house locations) the minimum distance will occur within the median interval defined as the two middle points, inclusive. If there is an odd number of points the minimum distance will be found at the middle point (house), or the median point. This is just the beginning in terms of the rich mathematics that can be mined from this problem.

To find the equations of the line segments that make up the graphs, one can "go algebraic" and fall back on the algebraic definition of absolute value:

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Thus, $$|x-5| = \begin{cases} x-5 \text{ if } x-5 \geq 0 \ (\text{i.e., } x \geq 5) \\ -(x-5) \text{ if } x-5 < 0 \ (\text{i.e., } x < 5) \end{cases}$$

Using our previous equation, $y = |x + 3| + |x - 5|$, we can work backward from the definition of $|x|$. For $x \leq -3$, $|x + 3| = -(x + 3)$ and $|x - 5| = -(x - 5)$, therefore, $y = -x - 3 - x + 5 = -2x + 2$. For $-3 \leq x \leq 5$, $|x + 3| = x + 3$ and $|x - 5| = -(x - 5)$, therefore $y = x + 3 - x + 5 = 8$, as we saw in Table 1. Finally, for $x \geq 5$, $|x + 3| = x + 3$ and $|x - 5| = x - 5$, therefore $y = x + 3 + x - 5 = 2x - 2$.

Table 4 summarises the various linear equations for the graphs students investigated for 2, 3, 4, and 5 points (houses). Have students observe the pattern for the slopes of the various lines and reflect on why this is so.

We can connect this to our graph in Figure 10. Using the point-slope form of a linear equation, we can confirm that the equations of the line segments are the same. By doing so, students have different entry points for this concept. Those who prefer visuals are served while those who prefer symbolic

Table 4. Linear equations that make up equations of sums of absolute values.

Equation	Domain for x	Lines
$y = \|x+3\| + \|x-5\|$	$x \le -3$ $-3 \le x \le 5$ $x \ge 5$	$y = -2x + 2$ $y = 8$ $y = 2x - 2$
$y = \|x+3\| + \|x-2\| + \|x-5\|$	$x \le -3$ $-3 \le x \le 2$ $2 \le x \le 5$ $x \ge 5$	$y = -3x + 4$ $y = -x + 10$ $y = x + 6$ $y = 3x - 4$
$y = \|x+3\| + \|x-1\| + \|x-2\| + \|x-5\|$	$x \le -3$ $-3 \le x \le 1$ $1 \le x \le 2$ $2 \le x \le 5$ $x \ge 5$	$y = -4x + 5$ $y = -2x + 11$ $y = 9$ $y = 2x + 5$ $y = 4x - 5$
$y = \|x+3\| + \|x+1\| + \|x-1\| + \|x-3\| + \|x-5\|$	$x \le -3$ $-3 \le x \le -1$ $-1 \le x \le 1$ $1 \le x \le 3$ $3 \le x \le 5$ $x \ge 5$	$y = -5x + 5$ $y = -3x + 11$ $y = -x + 13$ $y = x + 11$ $y = 3x + 5$ $y = 5x - 5$

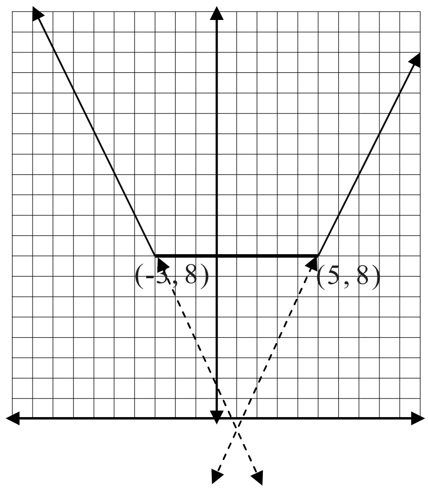

Figure 10. Graph of $y = |x + 3| + |x - 5|$

manipulation can become aware of the connection between the equations, numeric tables and graphs.

Through this investigation students examine the definition of $|x|$ for different intervals of x and quickly discover that for an even number of reference points, the slopes of the lines that make up the original equation follow the pattern, … –4, –2, 0, 2, 4, …; for an odd number of reference points, the slopes of the lines follow the pattern –5, –3, –1, 1, 3, 5. Engaging in this investigation will also reveal why the patterns are there.

Essentially, when there is an even number of points for x values to the left of the leftmost point the definition $|x| = x$ is applied to all points, resulting in a negative slope for an even coefficient of x. As we move to the next point to the right we begin using $|x| = x$ for some points and $|x| = -x$ for the remaining points, thus cancelling out two "x-s" at a time which reduces the coefficients (still negative) by two. When we get to half the points using $|x| = -x$ and the other half using $|x| = x$, the result is a coefficient of 0 for x which is where the constant value for y is the minimum distance within the median interval for the points. As we continue to move to the right, the number of terms using $|x| = x$ is greater than those using $|x| = -x$ and we increase the coefficient of x (now positive) by two at a time. We do this until we are at the right of the rightmost point where we use $|x| = x$ for all points, resulting in a positive slope for an even coefficient of x.

When there are an odd number of points, for x values to the left of the left-most point the definition $|x| = -x$ is applied to all points, resulting in a negative slope for an odd coefficient of x. As we move to the next point to the right we begin using $|x| = x$ for some points and $|x| = -x$ for the remaining points, thus cancelling out two "x-s" at a time which reduces the coefficients (still negative) by two. When we get to the middle point, or median, the left points use $|x| = x$ and all points to the right use $|x| = -x$, thus canceling all x-s. The minimum distance at the median point is then the sum of the absolute values of the magnitudes of the points (houses) to the left and right of the median point. Thus, in the case of an odd number of points we have the V-shape above the middle point and a single minimum distance for the sum of absolute values. As we continue to move to the right the number of terms using $|x| = x$ is greater than those using $|x| = -x$ and we increase the coefficient of x (now positive) by two at a time until we are at the right of the rightmost point where we use $|x| = x$ for all points, resulting in a positive slope for an odd coefficient of x.

Of course we selected the points for our examples and one might ask, can we generalise for any points A, B, C, …, etc.? We start with any two points A and B where $A < B$ and find the sum $y = |x - A| + |x - B|$.

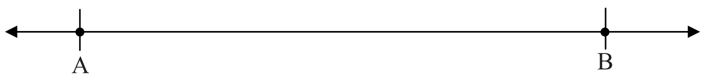

Figure 11. Generalising to any two points.

First, we make a table with the critical points below.

Table 5. Calculations for $y = |x - A| + |x - B|$.

x	$\|x - A\|$	$\|x - B\|$	$y = \|x - A\| + \|x - B\|$
A	0	$B - A$	$B - A$ (minimum)
B	$B - A$	0	$B - A$ (minimum)

The graph of $y = |x - A| + |x - B|$ (see Figure 12) resembles our graph of $y = |x + 3| + |x - 5|$ (see Figure 10).

For $x \le A$, $y = -x + A - x + B = -2x + A + B$;
for $A \le x \le B$, $y = x - A - x + B = B - A$;
for $x \ge B$, $y = x - A + x - B = 2x - A - B$.

Substituting $A = -3$ and $B = 5$ verifies that this agrees with the entries in Table 4.

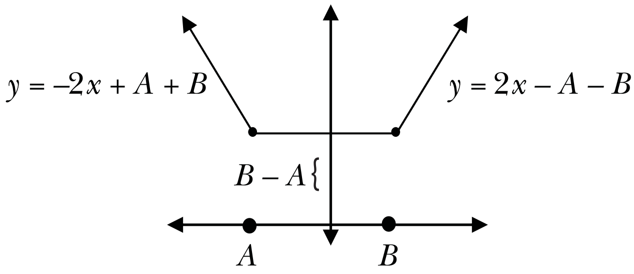

Figure 12. Graph of $y = |x - A| + |x - B|$.

Next, we generalise for any three points, A, B, C, where $A < B < C$.

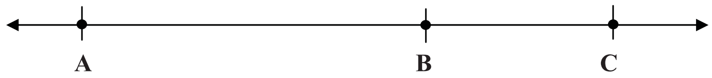

Figure 13. Generalising for any three points.

Table 6. Calculations for $y = |x - A| + |x - B| + |x - C|$.

x	$\lvert x - A\rvert$	$\lvert x - B\rvert$	$\lvert x - C\rvert$	$y = \lvert x - A\rvert + \lvert x - B\rvert + \lvert x - C\rvert$
A	0	$B - A$	$C - A$	$B + C - 2A$
B	$B - A$	0	$C - B$	$C - A$ (minimum)
C	$C - A$	$C - B$	0	$2C - A - B$

For $x \le A$, $y = -x + A - x + B - x + C = -3x + A + B + C$;
for $A \le x \le B$, $y = x - A - x + B - x + C = -x - A + B + C$;
for $B \le x \le C$, $y = x - A + x - B - x + C = x - A - B + C$;
for $x \ge C$, $y = x - A + x - B + x - C = 3x - A - B - C$.

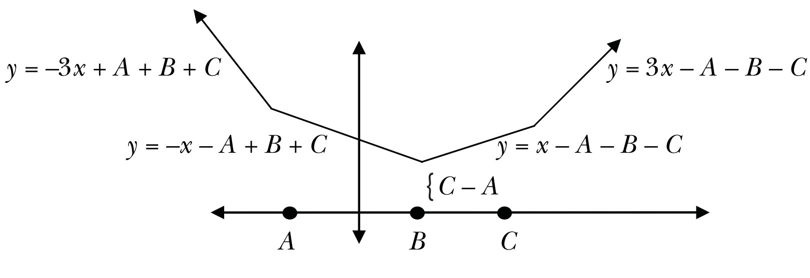

Figure 14. Graph of $y = |x - A| + |x - B| + |x - C|$.

Substituting $A = -3$, $B = 2$, and $C = 5$ from our earlier numerical example agrees with the result displayed in Table 4.

Students can also generalise the situation for four points and five points and verify that the formulas agree with the numerical and graphical examples. They will also find that the pattern for slopes of the line segments that make up each graph follow the pattern observed earlier.

Finally, if the students do not make the connection, they should be reminded of the relationship of the equation $y = |x - h| + k$ to the equation of a line $y = m(x - h) + k$ and parabola $y = (x - h)^2 + k$. Although the graphs of each of these equations are quite different, the basics are the same. Each parent equation, $y = x$, $y = x^2$, $y = |x|$, is translated (h, k). This can also be used to reinforce earlier lessons.

If the students tire of finding and graphing total distance, other mathematical adventures can be suggested. For example, how would one determine and graph the round trip distance from the fire station to the house? Compare and contrast this graph with that of the total distance. Investigate the same situations if the houses do not have to be placed on corners only (for example at 2.5 or 3.7). Does the generalisation from above still hold? Another twist could be to replace single family homes with apartment buildings which are weighted by the number of apartments in the building. How would you determine the minimum maximum distance instead of the minimum total distance? The possibilities are almost endless and well within the reach of all students.