Intersection of the exponential and logarithmic curves

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A calculus student is likely to think that the graphs of the exponential and logarithmic functions do not intersect. To a large extent, that is because calculus text books (e.g., Larson, Hostetler, & Edwards, 1998) usually show the graph of $y = e^x$ (which lies above the line $y = x$) and the graph of $y = \ln x$ (which lies below the line $y = x$) thus giving the impression that, regardless of the base, exponential and logarithmic curves do not meet. For the general exponential and logarithmic functions $y = a^x$ and $y = \log_a x$, where $a \in (0,1) \cup (1,+\infty)$ that is not true (see Couch, 2002) as can be easily demonstrated by having the two functions plotted on the same set of axes for various values of the base a using Mathematica, Matlab, Maple, or any other computer algebra package.

Classroom presentation outline

The study of the number of intersection points of $y = a^x$ and $y = \log_a x$ can be an interesting topic to present in a single-variable calculus class. Our presentation involves the basic algebra and the elementary calculus of the exponential and logarithmic functions. The proofs are given either in a "forward" manner or by contradiction. The presentation can be broken down into parts as follows:

1. Explain why intersection points (if any) of $y = a^x$ and $y = \log_a x$ lie on the line $y = x$. That can be done either by working directly with the exponential and logarithmic functions or by using the fact that they are inverse functions.
2. Study the monotonicity and concavity of the functions $y = a^x$ and $y = \log_a x$.
3. Show that the x-axis is a horizontal asymptote of the graph of $y = a^x$ and the y-axis is a vertical asymptote of the graph of $y = \log_a x$.
4. By the continuity of $y = a^x$ and $y = \log_a x$, conclude that their graphs can meet at zero, one, or two points for $a > 1$ while they will always meet at exactly one point for $0 < a < 1$ (see Figure 1). In particular, in proposi-

tion 2 we give a mathematical proof of the fact that there exists a base $a > 1$ for which the functions $y = a^x$ and $y = \log_a x$ have at least one common point (x_0, y_0) with $x_0 > 0$.

5. For $a > 1$ provide a geometric description by noticing that a point (x_0, y_0) of intersection of the graphs of $y = a^x$ and $y = \log_a x$ is a point of intersection of the graph of the natural logarithmic function $y = \ln x$ and the straight line $y = (\ln a)x$ (see Figure 2). In proposition 3 we provide a mathematical proof of the fact that, for $a > 1$ arbitrarily close to 1, the line $y = (\ln a)x$ intersects the graph of $y = \ln x$ at some point (x_0, y_0) with $x_0 > e$.

Mathematical proofs

In this section we present a detailed discussion and proofs of all parts of Section 2.

Proposition 1

(i) If (x_0, y_0) is a point of intersection of the curves $y = a^x$ and $y = x$ then (x_0, y_0) is also a point of intersection of $y = \log_a x$ and $y = x$.

(ii) If (x_0, y_0) is a point of intersection of the curves $y = \log_a x$ and $y = x$ then (x_0, y_0) is also a point of intersection of $y = a^x$ and $y = x$.

(iii) If (x_0, y_0) is a point of intersection of the curves $y = a^x$ and $y = \log_a x$ then $x_0 = y_0$, i.e., (x_0, y_0) lies on the line $y = x$.

Proof

(i) Since $y_0 = a^{x_0}$ and $y_0 = x_0$ it follows that $x_0 = a^{y_0}$ and so $y_0 = \log_a x_0$.

(ii) Since $y_0 = \log_a x_0$ and $y_0 = x_0$ it follows that $x_0 = \log_a y_0$ and so $y_0 = a^{x_0}$.

(iii) Since $(a^x)' = (\ln a)a^x$ and $(a^x)'' = (\ln a)^2 a^x$ it follows that the function $y = a^x$ is increasing and concave up for $a > 1$ and decreasing and concave up for $0 < a < 1$. Moreover, since $\lim_{x\to-\infty} a^x = 0$ for $a > 1$ and $\lim_{x\to+\infty} a^x$ for $0 < a < 1$, the x-axis is a horizontal asymptote of the graph of $y = a^x$. Similarly, since $(\log_a x)' = 1/(x\ln a)$ and $(\log_a x)'' = -1/(x^2 \ln a)$, it follows that the function $y = \log a_x$ is increasing and concave down for $a > 1$ and decreasing and concave up for $0 < a < 1$. Since $\lim_{x\to0+} \log_a x = -\infty$ for $a > 1$ and $\lim_{x\to0+} \log_a x = +\infty$ $0 < a < 1$, the y-axis is a vertical asymptote of the graph of $y = a^x$. Thus (see Figure 1), by the continuity of $y = a^x$ and $y = \log_a x$, their graphs can meet at zero, one, or two points for $a > 1$ while they will always meet at one point for $0 < a < 1$. If (x_0, y_0) is a point of intersection of $y = a^x$ and $y = \log_a x$ that is not on the line $y = x$ then, since $y = a^x$ and $y = \log_a x$ are inverse functions, (x_0, y_0) is also a point of intersection of $y = a^x$ and $y = \log_a x$ which lies on the opposite side of the line $y = a^x$ with respect to (x_0, y_0). By the continuity of $y = a^x$ and $y = \log a_x$ and the symmetry of their graphs with respect to the line $y = x$, there must be a third point of intersection of the graphs of $y = a^x$ and $y = \log a_x$ on the line $y = x$. But this is a contradiction to the fact that the maximum number of points of intersection is two. Thus $x_0 = y_0$.

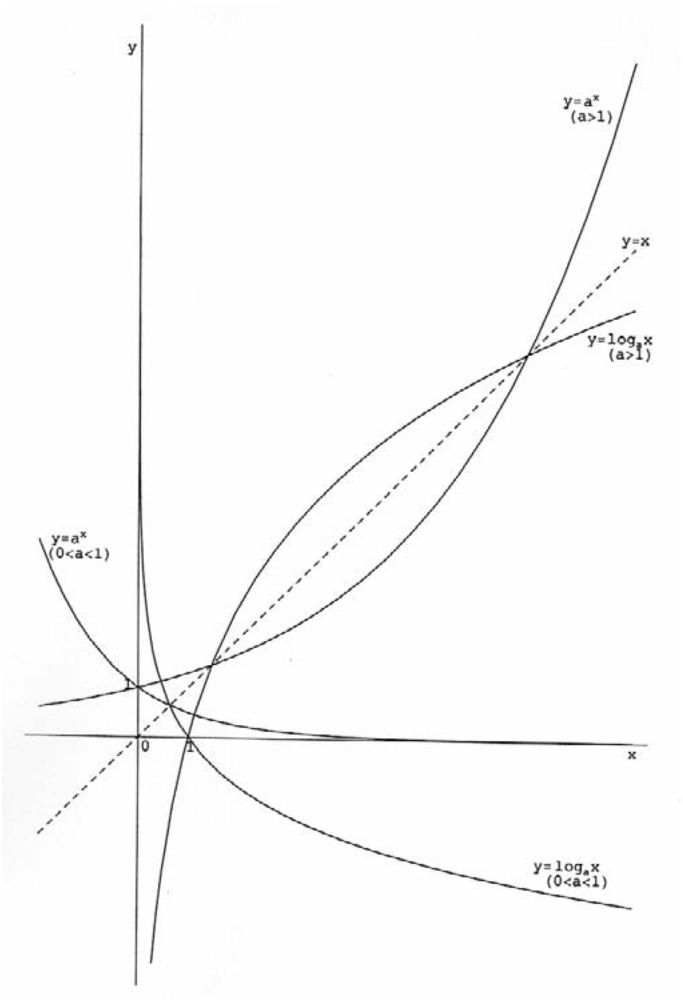

Figure 1

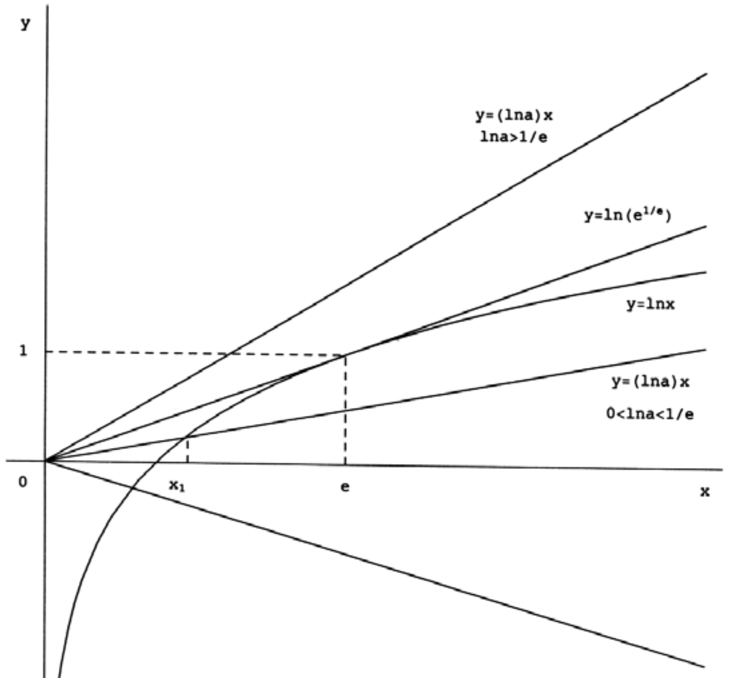

Figure 2

As discussed in the proof of Proposition 1, the graphs of $y = a^x$ and $y = \log_a x$ will always meet at exactly one point if $0 < a < 1$. The next proposition shows that the base $a > 1$ can always be chosen so that the graphs of $y = a^x$ and $y = \log_a x$ will meet at some point.

Proposition 2

There exists $a > 1$ for which $a^{x_0} = \log_a x_0$ for at least one $x_0 > 0$.

Proof

$a^x = \log_a x$ implies $e^{x\ln a} = \ln x/\ln a$ and so $(\ln a)\,e^{x\ln a} = \ln x$. If, for all $a > 1$, there is no $x_0 > 0$ for which $(\ln a)\,e^{x\ln a} = \ln x_0$, then $(\ln a)\,e^{x\ln a} > \ln x$ for all $a > 1$ and $x > 0$ (thus for $x > 1$ also). But then, taking the limit as $a \to 1^+$, we obtain $0 \geq \ln x$ which is not true for $x > 1$.

Proposition 3

Let $a > 1$ be arbitrarily close to 1. Then the line $y = (\ln a)\,x$ intersects the graph of $y = \ln x$ at some point (x_0, y_0) with $x_0 > e$.

Proof

Suppose that this is not true. Then, for all $x > e$, the slope of the tangent to the graph of $y_1 = \ln x$ at (x, y_1) is bigger or equal to the slope of $y_2 = (\ln a)\,x$ at (x, y_2). Thus $1/x \geq \ln a$ for all $x > e$. But then, taking the limit as $x \to +\infty$, we obtain $0 \geq \ln a$ which is a contradiction to $a > 1$.

Geometric description

Let (x_0, y_0) be a point of intersection of $y = a^x$ and $y = \log_a x$. By Proposition 1, (x_0, y_0) lies on the line $y = x$. Thus $a^{x_0} = x_0 \Rightarrow e^{(\ln a)\,x_0} = x_0 \Rightarrow (\ln a)\,x_0 = \ln x_0$. Therefore (x_0, y_0) is a point of intersection of the graph of the natural logarithmic function $y = \ln x$ and the straight line $y = (\ln a)\,x$. These two curves are tangent at the point (x_0, y_0) where their slopes are equal, i.e., where $\ln a = 1/x_0$. This is true for $x_0 = e$ and $a = e^{1/e}$. If $\ln a > 1/e$ the two graphs do not meet. If $0 < \ln a < 1/e$, in view of Proposition 3, they meet at two points. If $\ln a < 0$ then they meet at exactly one point. The situation is illustrated in Figure 2.

References

Couch, E. (2002). An overlooked calculus question. *The College Mathematics Journal, 33*(5), 399–400.

Larson, R., Hostetler, R. P. & Edwards, B. (1998). *Calculus with analytic geometry (6th ed).* Boston: Houghton Mifflin.