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The Paradigm Shift in Mathematics Education: Explanations and Implications of Reforming Conceptions of Teaching and Learning

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In this article, we argue that the debates about mathematics education that have arisen in the United States over the past decade are the result of a major shift in how we conceptualize mathematical knowledge and mathematics learning. First, we examine past efforts to change mathematics education and argue they are grounded in a common traditional paradigm. Next, we describe the emergence of a new paradigm that has grown out of a coalescence of theories from cognitive psychology, an awareness of the importance of culture to learning, and the belief that all students can and should learn meaningful mathematics. Reforms grounded in the new paradigm have the potential to dramatically alter the way in which students—as well as *which* students—experience success in school mathematics. We discuss some implications of these reforms related to how mathematics educators might work with teachers of mathematics.

An examination of articles and reports about what should be done to improve mathematics education in the United States can be alarming. One finds calls for more skill building and less use of calculators while at the same time there is a push to teach for understanding through the use of technology; there are textbooks deemed "superior" that require teachers to take students individually in lock-step fashion through a vast collection of skills and procedures, while another set of "excellent" texts position teachers as guides for cooperative learning activities through which students construct an understanding of a few key concepts (American Association for the Advancement of Science, 2000; Clopton, McKeown, McKeown, & Clopton, 1999). Some want more standardized testing of mathematical skills, while others want more authentic assessments that are based on mathematical standards. One perspective holds tracking in mathematics to be inevitable while the other sees it as inequitable. Yet, everyone claims to want what is "best" for all students. How can this be?

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In this article, we develop a critical view of efforts to improve mathematics education over the past century which allows us to demonstrate that the current debates can be understood as the result of a fundamental conflict of paradigms. We argue that up until the 1980s, efforts at change came out of a common perspective or paradigm in which mathematics was viewed as a disembodied set of objective truths to be communicated to students, most of whom would then struggle to internalize them. Thorndike's Stimulus-Response Bond theory, the progressive movement's efforts to make learning vocationally relevant and schooling more efficient, the curricular changes associated with New Math, and the emphasis on rote facts, skills, and procedures of the back-to-basics movement all developed within this common paradigm. We will discuss these efforts at improving mathematics education and provide some perspective on their failure to positively impact the inequitable outcomes in mathematics learning that have persisted since the early twentieth century.

A History of Revisions in Mathematics Education

Revision can be characterized as a renewal effort that captures educators' attention for a short period of time but fails to address critical issues that are at the root of students' difficulties with mathematics. Revision perpetuates a "quick fix" approach whereby new components are adapted to fit within the bounds of the accepted paradigm. Thus, revision in mathematics education leads to surface level modifications but does little to substantively alter deeply held beliefs about the nature of mathematics, how it is to be taught, the sort of learning that is valued, and how success is determined. In contrast, *reform* is transformative and leads to a redefining of the epistemological position toward the field. Reform raises questions about the core beliefs of mathematics education, moving to restructure thinking about the nature of mathematics, how it is taught, how it is learned, and, ultimately, what constitutes success in learning it.

Throughout the last century, mathematics education in the United States has been a revolving door for revisions—under the guise of so-called reform movements—that failed to question traditional assumptions and beliefs about mathematics teaching and learning and, therefore, failed to change significantly the face of the mathematically successful student. Consequently, speculation exists about whether the current "reform" movement will promote mathematical equity and excellence or turn out to be another trend that fails to significantly alter the status quo in mathematics education (Martin, 2003). We argue that the work currently underway is fundamentally different from past movements. In order to build a case, we will examine the motives and intentions of past revisions in mathematics education—taking care to distinguish them from actual reform. We will then examine current NCTM standards-based efforts to reform mathematics education in contrast to these past movements. A model of shifting paradigms will be offered as a way to understand what makes the current efforts distinct and why they have garnered such sharp criticism from some quarters.

Thorndike's Stimulus-Response Bond Theory

The opening of the twentieth century saw much change in the character of the United States. It was during this time that Edward L. Thorndike, president of the nascent American Psychological Association, led a new class of educational psychologists whose work was aimed at making schools more efficient and effective in educating and stratifying the masses of children who had recently come to populate public schools (Gould, 1996; Henriques, Hollway, Urwin, Venn, & Walkerdine, 1998; Oakes, 1985; Stoskopf, 2002; Thayer, 1928) In particular, Thorndike's Stimulus-Response Bond theory (Thorndike, 1923) had a profound influence on the teaching and learning of mathematics (English & Halford, 1995; Willoughby, 2000).

Thorndike and his colleagues contended that mathematics is best learned in a drill and practice manner and viewed mathematics as a "hierarchy of mental habits or connections" (Thorndike, 1923, p. 52) that must be carefully sequenced, explicitly taught, and then practiced with much repetition in order for learning to occur. In Thorndike's work there was an explicit denial of the ability of students to reason about mathematical concepts, as in this example where he explained how students solve 23 + 53:

> Surely in our schools at present children add the 3 of 23 to the 3 of 53 and the 2 of 23 to the 5 of 53 at the start, in nine cases out of ten because they see the teacher do so and are told to do so. They are protected from adding 3 + 3 + 2 + 5 not by any deduction of any sort but…because they have been taught the habit of adding figures that stand one above the other, or with a + between them; and because they are shown or told what they are to do…In nine cases out of ten they do not even think of the possibility of adding in any other way than the '3+3, 1+5' way, much less do they select that way on account of the facts that 53=50+3 and 23=20+3, that 50+20=70, that 3+3=6…[n]or, I am tempted to add, would most of them by any sort of teaching whatever. (Thorndike, 1923, pp. 68–9)

Thorndike and his colleagues used "scientific" evidence to persuasively argue that mathematics is best learned in a drill and practice manner, leading a large portion of the education community to embrace this view and influencing the teaching of mathematics throughout the twentieth-century (English & Halford, 1995; Glaser, 1984; Willoughby, 2000). Notably, Thorndike's view of mathematics learning fails to address the nature of mathematical thinking students must apply in problem-solving situations (Wertheimer, 1959). Thorndike's psychology situates mathematics as *a priori* knowledge, based on objective reason alone, without taking into account the experiences students bring to mathematics or the meaning they make of what is learned (Brownell, 1935; Resnick & Ford, 1981; Thayer, 1928). This, then, allows students' mathematics achievement to be discretely measured, quantified, and stratified.

The Progressive Movement

The Progressive Education Association (PEA) in the 1920s constructed its movement, in part as a reaction against the highly structured rote schooling practices supported by Thorndike's theories. Influenced by John Dewey's (1899) thinking about schooling and society that emphasized the need to harness and provide direction to the child's natural impulse toward activities of learning, early progressive educators theorized that learning occurs best when it is connected to students' experiences and interests (English & Halford, 1995). Among their guiding principles, the PEA asserted that: a) children should have the freedom to develop naturally; b) interest should be the motivation for all work; and c) the teacher is a guide and not a taskmaster (Kliebard, 1987). The early phases of the progressive movement were perceived by many educators to be too radical in their refusal to allow any sort of authority for the teacher and its disregard for organized subject matter. By the time Dewey (1938) chastised such excesses in *Experience and Education* the PEA was a relatively minor organization in terms of its impact on schooling practices.

The primary influence of the progressive movement on mathematics education came from a later off-shoot, the social efficiency movement, that emphasized maintaining social order through the differentiation of course placement and instruction. Although influenced by progressivism's concern for the learner as an individual, those in the social efficiency movement argued that students "could be guided to expressive self-realization and social integration through scientific educational practices as they were evaluated and trained by experts according to their natural inclinations and abilities" (Holt, 1994). Social efficiency progressives questioned the importance of mathematics for all students in the secondary curriculum, arguing that for many such coursework was unnecessary (Tyack, Lowe, & Hansot, 1984). They turned toward science in the form of standardized testing which offered proof that certain children were more "able" than others for advanced coursework and, ironically, embraced Thorndike's work in their development of scientific methods for teaching basic mathematics to the masses of students.

The revisions pressed for by these later progressives proposed that the teaching and learning of mathematics have a utilitarian focus while the study of rigorous mathematical subjects was appropriate for a small elite (see, for example, Bonser, 1932). Studies were undertaken to determine what mathematics content would be of most utility to students outside of school and which students were most likely to succeed in higher-level coursework (Kilpatrick, 1992). The general belief behind these efforts was that the study of advanced mathematics was best suited for those who had a perceived future need for the subject matter—primarily white, middle class males (Willoughby, 2000). Columbia professor William H. Kilpatrick, a former student of Dewey, summed up this sentiment well when he said, "We have in the past taught algebra and geometry to too many, not too few" (in Tenenbaum, 1951, p. 109).

The combined effects of Throndike's structured "science" of teaching mathematics and the social efficiency movement's sorting of students into mathematics courses suited to their perceived future needs meant that by the 1940s tracking in mathematics had become standard practice, with most students steered into vocational, consumer, and industrial mathematics courses. This is reflected in the sharp decline in the percentage of high school students taking algebra—from 57% in 1905 to about 25% in the late 1940s and early 1950s (Jones & Coxford, 1970)—that accompanied the dramatic rise in high school enrollment during the same period.

New Math

By the mid-twentieth century, a new rationale for the study of advanced mathematics was found in national security and people saw a more rigorous mathematics curriculum as a necessity—at least for some. Congress created the National Science Foundation (NSF) in 1950 in order to develop a national strategy for promoting education in the sciences (Jones & Coxford, 1970). Several projects aimed at overhauling mathematics education received funding from the NSF. At first, "a variety of approaches were being taken to improve mathematical education" (Hayden, 1983, p. 3). As examples of the early "pioneers" of this era, Hayden cites the following: Henry Van Engen and Maurice Hartung developed texts guided by principles of the early progressives; Catherine Stern incorporated the use of manipulatives to deepen conceptual understanding of arithmetic; Robert Davis taught algebra to inner city junior high school students; and Max Beberman and colleagues at the University of Illinois designed a high school mathematics program built around discovery learning that required teachers to attend extensive, intensive workshops before and during implementation. These efforts, while invaluable to the next generation of mathematics educators and their approach to reform, failed to achieve widespread influence in their day.

One group that did obtain a national spotlight for its ideas was the College Entrance Examination Board (CEEB). They established Advanced Placement testing in 1955 with calculus being the first exam offered. Four years later the Commission on Mathematics of the CEEB issued the final draft of a report, written by a group of fourteen mathematicians and mathematics educators (with just one of the early 1950s reformers included), that recommended changes in the content of the college preparatory mathematics curriculum designed to reflect developments in mathematics such as set theory and Boolean algebra—commonly referred to as modern mathematics (Commission on Mathematics, 1959; Hayden, 1983).

With Russia's launch of Sputnik in 1957, the United States government became impatient with the slow pace of reform efforts and used NSF funding to create the School Mathematics Study Group (SMSG). Utilizing the CEEB commission's report as a guide for its task of modernizing course content, the SMSG swiftly produced and distributed textbooks that "reflected the content and viewpoint of modern mathematics much more completely and accurately than they reflected the pedagogical innovations" (Hayden, 1983, p. 5) of the earlier "pioneer" programs. These textbooks were sent to schools nationwide in a massive dissemination effort that, for the most part, failed miserably, ultimately leading to calls for a return to more familiar mathematical fare—basic skills. Critics of the SMSG New Math textbooks, who were numerous (see, for example, Ahlfors et al., 1962; Kline, 1973), claimed that the content was too abstract and not related to real world problems, the language used was unknown to most educated adults, and that more harm than good was the result. The early goals of New Math having to do with issues of pedagogy and access were largely forgotten, leading Willoughby (2000) to conclude that during the New Math era "this apparent recognition of the need for better mathematics education tended to be applied specifically to males of European extraction" (p. 3). Tate (2000) adds to this the insight that most of the federally funded programs now identified as New Math or modern math were developed by mathematicians who believed higher-level mathematics should be limited to college capable students.

Back-to-Basics

In the early 1970s, the "back-to-basics" call was sounded in response to the perceived shortcomings of New Math (Burrill, 2001; National Institute of Education, 1975). This movement called for decontextualized and compartmentalized skills-oriented mathematics instruction and was closely connected to the minimum competency testing movement used extensively by states in the 1970s and 1980s (Resnick, 1980; Tate, 2000). This basic skills mentality dominated textbook publishing through the early 1980s, leading to another generation of Thorndike-like mathematics textbooks (English & Halford, 1995). Although the emphasis on skills did result in slightly improved standardized test scores for students traditionally underserved by school mathematics, it was criticized for not adequately preparing these students for mathematics coursework requiring higher levels of cognition and understanding (United States Congress Office of Technology Assessment, 1992; Tate, 2000).

Failed Revisions

Despite a century of "reform" efforts, school mathematics practices in the late twentieth century remained stubbornly similar to what Florian Cajori (1974/1890) described one hundred years earlier in his study of mathematics classrooms across the United States: "[There were] no explanations of processes either by master or pupil…the problems were solved, the answers obtained, the solutions copied" (p. 10). Research in mathematics classrooms during the latter years of the twentieth century found that in the United States teachers were still the center of authority, first disseminating rote skills and procedural knowledge to their students who then worked individually on sets of problems in order to internalize this knowledge (Cobb, Wood, Yackel, & McNeal, 1992; Fey, 1979; Price & Ball, 1997; Stodolsky, 1988). Likewise, the century of "reforms" did not significantly alter historical assessment methods and patterns of learning outcomes: timed measures consisting of primarily procedural problems showed disproportionate levels of mathematics achievement and attainment across groups of students (Martin, 2003; Schoenfeld, 2002; Gonzales et al., 2004; United States Congress Office of Technology Assessment, 1992).

The behaviorist science of Thorndike's psychological models, the vocational focus of the social efficiency progressives, the curricular elitism of the New Math program, and the shallow content of the back-to-basics movement have all been referred to as efforts to reform mathematics education but, for the most part, have resulted in superficial revisions to standard practices and outcomes. None of these promoted true reform in mathematics education because they were trapped within an inherently inequitable system of thought toward mathematical knowledge and the teaching and learning of mathematics. The revisions of the past century situated many learners in an *a priori* deficit position relative to disembodied mathematical knowledge—meaning learning mathematics was taken to be harder for certain groups of students due to their backgrounds and/or innate abilities—and failed to acknowledge the importance of mathematics for all students. Excellence, as defined by these models, meant either remembering rules and procedures with little concern for the connection of mathematics to students' lived experiences or, in the case of the progressives, focusing on the child's perceived interests or needs to the exclusion of being concerned with the learning of critical mathematical concepts. In the end, these revisions failed to substantively challenge educators' thinking about equity and excellence in mathematics education (Martin, 2003; Rousseau & Tate, 2003).

Shifting Paradigms in Mathematics Education

The traditional models for mathematics education found within the revisions of the past century have been formulated within a perspective we are calling the *procedural-formalist paradigm* (PFP). The PFP holds that mathematics is an objective set of logically organized facts, skills, and procedures that have been optimized over centuries. This body of knowledge exists apart from human experience, thus making it inherently difficult to learn.

Thinking within the Traditional Paradigm

Positioning oneself within the frame of the PFP, one might reasonably believe the goal of school mathematics education should be for students to internalize a fundamental body of basic mathematical knowledge. In order to facilitate such learning, teachers must deliver carefully sequenced bits of mathematics to students through explanation and demonstration. Students repetitively practice these facts, skills, and procedures in an effort to memorize them and are then tested to discern what has been "learned." Learning and assessment are structured around the notion that there is a unique, mathematically correct way to solve a problem. This set of assumptions guided the work of Thorndike and the back-to-basics advocates.

Alternatively, still operating within the PFP, one might try to show students forthright the logical structure of mathematics and hope they catch on. While many may not, they might at least catch a glimpse of the inherent beauty of mathematics. The students who do catch on to this structure will be well positioned to succeed in higher-level mathematics. This was the position taken by many of the New Math advocates.

The later social efficiency progressives espoused yet another approach to school mathematics, but one still grounded in the PFP. Since most mathematics is outside of human experience, it is not relevant for students to learn. Any necessary practical mathematical skills can be learned within the context in which they might be used. This thinking led to the creation of consumer and vocational mathematics courses that offered the average student an "escape" from the rigors of such formal topics as algebra and trigonometry (reserving such classes for the mathematically elite).

Importantly, the claim being posited here is *not* that there were no other possibilities being offered for reform during the past century. Rather, it is that the paths taken—those "reforms" that received strong support and were widely implemented (and the ways in which they were implemented)—were reflective of the PFP. Ideas that fell outside the bounds of this paradigm, though having localized effects, failed to gain wide acceptance, recognition, and/or support. However, that such efforts were not insignificant is acknowledged for these small-scale deviations from the paradigmatic ways of operating demonstrated to those involved that there was a fundamentally different way to conceptualize mathematics education. In other words, new possibilities were envisioned and experienced, providing the seeds for later efforts at true reform. It would be profitable for further analyses to be brought to the specifics of the ways in which these "fringe" programs had rather important effects on the field.

The Makings of a Paradigm Shift

Several strands of thought came together in the 1980s, ultimately leading to what we are characterizing as (the start of) a paradigm shift in mathematics education. Beginning in the mid– to late–1980s several documents were published expressing a concern that, particularly given the trend toward a more technological world, the poor performance of American students in mathematics needed to be addressed in a different way (Leitzel, 1991; National Commission on Excellence in Education, 1983; National Research Council, 1989; Romberg, 1989). We know from examining the past century of mathematics education that such a concern was not new. However, this time there was a co-incidence with the renewed public interest in mathematics education. A new generation of mathematics educators—many of whom came out of K-12 teaching and had been touched by the early professional development efforts of New Math such as Robert Davis' Big Cities Project (Wilson, 2003)—were looking to a wider body of research to inform their thinking about how best to improve the quality of mathematics learning for all students. Examining not only the content of school mathematics, these educators also focused attention upon the question of how students learn mathematics. Ideas from mathematical theory were married with theories of learning from cognitive psychology, initially those of Piaget (Rosskopf, Steffe, & Tabeck, 1971) and Bruner (1971), directing attention to the learner's active role in developing mathematical knowledge. And, most importantly, the idea that students from diverse backgrounds should all come to understand important mathematical concepts was explicitly endorsed (MSEB, 1990; NCTM, 1989 & 1991; NRC, 1989).

The coalescence of these strands of thought precipitated the release of the National Council of Teachers of Mathematics groundbreaking *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). While some involved in the authoring of the *Standards* described them as "more evolutionary than revolutionary" (McLeod, Stake, Schappelle, Mellissinos, & Gierl, 1996, p. 46), the vision they offered re-directed the profession and generated new ways to conceptualize teaching and learning. Advocates of the *Standards* realized that what had been missing from American mathematics education was a focus on how students come to form meaningful understandings of and connections between mathematical concepts (Gelman, 1994; Fuson, 1988; Schoenfeld, 1988; Steffe & Cobb, 1988). Students' engagement with mathematical thinking was given primacy over rote procedural manipulations. Learning came to mean more than memorization and repetition. Some of the new goals of mathematics education included: making sense of conceptual connections; articulating original insights; explaining and justifying mathematical arguments; and applying knowledge to new situations (Carpenter & Lerher, 1999).

The publication of the NCTM standards was accompanied by a growing awareness of and interest in research examining the significance of culture, specifically the interaction of student culture and classroom culture, to the construction of meaningful mathematical understanding (Cobb, Wood, & Yackel, 1990; Ladson-Billings, 1997; Lave, 1988; Malloy & Malloy, 1998; Saxe, 1991). There was broad recognition that teaching mathematics to all students required something other than the transmission of objective, disembodied content (see, for example, Boaler, 2000). Rather, the ways in which students experienced mathematical ideas and concepts and how this was connected to their own lived experiences came to be seen as critical to the learning process.

Brought together, ideas about the importance of cognition and culture to learning have led to the development of standards, curriculum materials, and pedagogical strategies intended to promote the use of thoughtfully structured experiences and interactions that afford all students opportunities to develop an understanding of the relevance and logic of mathematics.

A New Paradigm Emerges

It is the integration of new thinking about cognition and the greater acknowledgement of culture that has enabled mathematics educators to frame questions and conceptualize solutions in ways that were unlikely to develop from within the procedural-formalist paradigm. The unique perspective toward mathematics education that has come from the blending of cognitive psychological and (socio)cultural research has been made possible by the emergence of what we are calling the *cognitive-cultural paradigm* (CCP). The CCP takes mathematics to be a set of logically organized and interconnected concepts that come out of human experience, thought, and interaction—and that are, therefore, accessible to all students if learned in a cognitively connected and culturally relevant way.

The fundamentals of the cognitive-cultural paradigm lead to a radically different view of mathematics education than that of the procedural-formalist. Many of the core beliefs of traditional paradigm are challenged. Emphasis is shifted from seeing mathematics as *apart from* human experience to mathematics as *a part of* human experience and interaction. This is not to imply that students must reinvent mathematics in order to learn it. Rather, for students to really understand mathematics they need opportunities to both a) share common experiences with and around mathematics that allow them to meaningfully communicate about and form connections between important mathematical concepts and ideas, and b) engage in critical thinking about the ways in which mathematics may be used to understand relevant aspects of their everyday lives. The challenge is no longer how to get *mathematics into students*, but instead how to get *students into mathematics* (Philipp, 2001). This implies a need for flexibility in how teaching is approached and how learning is evaluated, a move away from more static and deterministic models of the PFP.

Conflict Caused by the Paradigm Shift

The controversy that has erupted in recent years over how best to approach mathematics education (see, for example, Kilpatrick, 1997; Loveless, 2001; Wu,

1997) is evidence of the tension caused by a paradigm shift. In his essay, "What are scientific revolutions?" Thomas Kuhn (1987) explains that a revolutionary change in a paradigm results in a re-conceptualization of the entire set of assumptions and generalizations within a field and necessitates the development of new vocabulary with which to describe the new ideas. Looking at mathematics education today, while we find tacit agreement that students' poor performance in mathematics is problematic, the ideas being proposed to address these woes often seem (and often are) incommensurate.

Viewed from the perspective of the traditional PFP, the ideas for reform coming out of the CCP may appear largely irrelevant and wholly misguided. This explains the competing lists of "excellent" curricula and the contradictory views of assessment in mathematics that have appeared in recent years. If mathematics is a disembodied set of facts, skills, and procedures, it makes little sense to ask students to work collaboratively to construct understanding. Where would this knowledge come from? If mathematics is objective, it makes no sense to be concerned with learners' cultures and lived experiences. If mathematical achievement can be accurately and fairly measured with standardized tests of routinized items, it makes no sense to develop more "subjective" assessments of mathematical understanding. And if mathematics is inherently too difficult for many to master, it makes no sense to try to teach all students rigorous aspects of the discipline. (See Sfard, 2003 for a thorough and thoughtful discussion of these tensions.)

Beyond the Conflict—From Rhetoric to Reality

Disputes aside, curricula designed and implemented in ways aligned with the perspective of the cognitive-cultural paradigm have resulted in what has previously been denied—students of diverse backgrounds, including those from historically underserved populations, have had success in learning not only to solve mathematical problems but also to communicate meaningfully about and value mathematical thinking (see, for example, Boaler, 1997; Campbell, 1996; Knapp et al., 1995; Silver & Stein, 1996; Van Haneghan, Pruet, & Bamberger, 2004). This research has demonstrated that students learning mathematics in reform-oriented classrooms outperform peers in traditionally structured classrooms on not only standard measures of mathematical skill but, more importantly, on measures of mathematical application and understanding. What is more, students in reform-oriented classes report stronger interest in and motivation toward mathematics (Boaler, 1997; Knapp et al., 1995), important predictors for future course-taking in mathematics. These efforts have the potential to generate true reform by valuing students' abilities to make sense of mathematics through meaningful learning experiences including the discussion of mathematical ideas, engagement with non-trivial problem solving tasks, and constructive support from teachers who themselves understand mathematics as more than rote facts, skills, and procedures.

The most salient question for those involved with the CCP-oriented reforms in mathematics education is no longer simply, "Do these reforms work?" While this is still an important query, much effort has moved to thinking about how to prepare teachers—the majority of whom have learned to think about mathematics from within the procedural-formalist paradigm—to understand and implement instructional strategies reflective of the CCP. But what might such work entail?

Working Toward Reform with Teachers of Mathematics

A large component of reforming mathematics education in the United States requires asking teachers to think differently about mathematics and to strengthen their own conceptual understanding of mathematics (CBMS, 2001), leading many to reconstruct knowledge that had heretofore seemed disembodied and absolute. Recommendations for instructional strategies situated in the cognitive-cultural paradigm, while eschewing prescriptive formulations of practice, generally call for teachers to structure learning environments that allow for mathematical discourse and the connection of mathematical ideas. This requires a different and more comprehensive knowledge of mathematics, stretching beyond rote facts, skills, and procedures to include important mathematical ideas and the interconnections among these (Ma, 1999; Sfard, 2003). Such a dramatic shift will not be easy and will not be quick; it will take much time and reflective thinking in order to move one's practice to a model of teaching oriented so differently than one's prior learning experiences (Lampert, 2001). Teachers need to see themselves as perpetual learners and be given opportunities to reform their own personal understandings of mathematics (Ellis, 2003; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998). These experiences must be supported by mathematics educators who not only understand but are willing to take up the challenge of reflecting on one's instructional practices and critically examining the sorts of opportunities that are being created for students to develop mathematical understanding.

Equally important (and under-researched), teaching successfully within the CCP requires that teachers develop understandings of multiple cultures—their own as well as those of their students—and how these are situated within both the local communities of schools and the larger society (Gutiérrez, 2002; Ladson-Billings, 1995; Remillard, 2000). Admittedly, we do not yet understand well the implications this has for our work with teachers. In her explication of what it might take to enable teachers of mathematics to move toward a more culturally cognizant view of their work, Gutiérrez (2002) argues that professional development must explicitly acknowledge teachers and students as

> members of communities that contribute to and are always in the process of remaking mathematics. That is, teacher practice aligns with the everyday dilemmas that teachers face, the power that they wield, the influence of local contexts, and the relationships between humans. (p. 175)

She explains that this does not indicate a disavowal of the "traditional" mathematics content but, rather, that such material is to be taken out of its "objective" context and placed it into the real lives and communities of teachers and students. This pushes mathematics beyond being a disembodied set of truths (as conceptualized within the PFP) and challenges us all to examine critically the role mathematics education has played in maintaining and justifying social inequities.

Conclusion

By critically examining the past century of "reform" movements in mathematics education in the United States, we have found them to have developed within a common perspective toward mathematical knowledge and mathematical learning that led to inherently inequitable practices and outcomes. An argument was made for conceptualizing the current reform efforts as fundamentally different in that they have come about within a perspective that acknowledges mathematics as culturally situated and views learning as tied to processes of cognition and interaction. This on-going paradigm shift holds the potential to change the look and outlook of successful learners of mathematics from what have historically been relatively small numbers of disproportionately "white" and middle class students, whose learning was focused on solving routinized problems by mastering procedural manipulations, to what should be large crowds reflective of the diversity found within the nation's public schools who learn to flexibly apply mathematical thinking to investigations of meaningful queries.

However, such an outcome is by no means certain or guaranteed. As we confront the challenges related to teacher preparation outlined above, we must retain a healthy degree of skepticism toward the work we do in order to guard against the simplistic promulgation of yet another set of absolute truths against which students are inequitably provided differential access to opportunities to learn. Indeed, Thomas Popkewitz (2004) has recently offered a critique aimed at what he perceives to be a possible extreme to which the ideas of the NCTM standards (and, by extension, ideas generated within the cognitive-cultural paradigm) may be taken, one in which strictly defined developmental models lead teachers to evaluate students' learning, ostensibly by criteria such as problem solving skills and the ability to participate in a mathematical community but that end up, in actual practice, promoting the (continued) stratification of students by markers of race and class.

It is in order to steer clear of such dangers that many involved with the reforms of the cognitive-cultural paradigm deliberately refrain from issuing rigid prescriptions for classroom practices and static lists of criteria for measuring learning, opting instead to share varied descriptions of learning environments and multiple examples of the sorts of outcomes to be expected (see, for example, Stein, Smith, Henningson, & Silver, 1999). Though this approach is at times criticized for its failure to provide concrete directives and quantifiable "objective" indicators, this may be taken as yet one more sign of the shifting paradigms in our midst.

It is clear that the work of reform requires large investments of time and energy in order to enact critical change in mathematics education. What sustains us as we engage in this work is thinking about what the future may hold when students of all backgrounds develop meaningful and powerful understandings of mathematics. While not claiming to know exactly what such a future may mean for society, we are hopeful it will at least be less inequitable than that of today.

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