The Mathematics Educator
2007, Vol. 17, No. 2, 7–14

A Problem With Problem Solving: Teaching Thinking Without Teaching Knowledge

Jamin Carson

Problem solving theory and practice suggest that thinking is more important to solving problems than knowledge and that it is possible to teach thinking in situations where little or no knowledge of the problem is needed. Such an assumption has led problem solving advocates to champion *content-less heuristics* as the primary element of problem solving while relegating the *knowledge base* and the *application of concepts* or *transfer* to secondary status. In the following theoretical analysis, it will be argued that the knowledge base and transfer of knowledge—not the content-less heuristic—are the most essential elements of problem solving.

Problem solving theory and practice suggest that thinking is more important in solving problems than knowledge and that it is possible to teach thinking in situations where little or no knowledge of the problem is needed. Such an assumption has led problem solving advocates to champion content-less heuristics as the primary element of problem solving while relegating the knowledge base and the transfer or application of conceptual knowledge to secondary status. Yet if one analyzes the meaning of problem solving, the knowledge base and the transfer of that knowledge are the most essential elements in solving problems.

Theoretical Framework

Problem solving is only one type of a larger category of thinking skills that teachers use to teach students how to think. Other means of developing thinking skills are problem-based learning, critical thinking skills, creative thinking skills, decision making, conceptualizing, and information processing (Ellis, 2005). Although scholars and practitioners often imply different meanings by each of these terms, most thinking skills programs share the same basic elements: (1) the definition of a problem, (2) the definition of problem solving, (3) algorithms, (4) heuristics, (5) the relationship between theory and practice, (6) teaching creativity, (7) a knowledge base, and (8) the transfer or the application of conceptual knowledge.

The Definition of a Problem

The first element of the theory of problem solving is to know the meaning of the term *problem*. This theoretical framework uses the definition of problem presented by Stephen Krulik and Jesse Rudnick (1980) in *Problem Solving: A Handbook for Teachers*. A problem is "a situation, quantitative or otherwise, that confronts an individual or group of individuals, that requires resolution, and for which the individual sees no apparent or obvious means or path to obtaining a solution" (p. 3).

The Definition of Problem Solving

Krulik and Rudnick (1980) also define problem solving as

> the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. The student must synthesize what he or she has learned, and apply it to a new and different situation. (p. 4)

This definition is similar to the definition of the eighth element of problem solving, transfer: "[w]hen learning in one situation facilitates learning or performance in another situation" (Ormrod, 1999, p. 348).

Problem Solving is Not an Algorithm

One of the primary elements of this framework is that problem solving is not an algorithm. For example, Krulik and Rudnick (1980) say,

> The existence of a problem implies that the individual is confronted by something he or she does not recognize, and to which he or she cannot merely apply a model. A problem will no longer be considered a problem once it can easily be solved by algorithms that have been previously learned. (p. 3)

Dr. Jamin Carson is an assistant professor of curriculum and instruction at East Carolina University. He teaches the theory and practice of instruction as well as classroom management and discipline. His primary research interest is the epistemology of curriculum and instruction.

Table 1

Types of Problem Solving

	John Dewey (1933)	George Polya (1988)	Stephen Krulik and Jesse Rudnick (1980)
Steps in Problem Solving	Confront Problem	Understanding the Problem	Read
	Diagnose or Define Problem	Devising a Plan	Explore
	Inventory Several Solutions	Carrying Out the Plan	Select a Strategy
	Conjecture Consequences of Solutions	Looking Back	Solve
	Test Consequences		Review and Extend

Additionally, advocates of problem solving imply that algorithms are inferior models of thinking because they do not require thought on a high level, nor do they require deep understanding of the concept or problem. Algorithms only require memory and routine application. Further, they are not useful for solving *new* problems (Krulik & Rudnick, 1980).

Problem Solving is a Heuristic

Advocates of problem solving argue that educators need to teach a method of thought that does not pertain to specific or pre-solved problems or to any specific content or knowledge. A heuristic is this kind of method. It is a process or a set of guidelines that a person applies to various situations. Heuristics do not guarantee success as an algorithm does (Krulik & Rudnick, 1980; Ormrod, 1999), but what is lost in effectiveness is gained in utility.

Three examples of a problem solving heuristic are presented in Table 1. The first belongs to John Dewey, who explicated a method of problem solving in *How We Think* (1933). The second is George Polya's, whose method is mostly associated with problem solving in mathematics. The last is a more contemporary version developed by Krulik and Rudnick, in which they explicate what should occur in each stage of problem solving. I will explain the last one because it is the more recently developed. However, the three are fundamentally the same.

The following is an example of how the heuristic is applied to a problem.

> **Problem:** Twelve couples have been invited to a party. The couples will be seated at a series of small square tables, placed end to end so as to form one large long table. How many of these small tables are needed to seat all 24 people? (Krulik & Rudnick, 1987, pp. 29–31)

The first step, *Read,* is when one identifies the problem. The problem solver does this by noting key words, asking oneself what is being asked in the problem, or restating the problem in language that he or she can understand more easily. The key words of the problem are *small square tables*, *twelve couples*, *one large table*, and *24 people*.

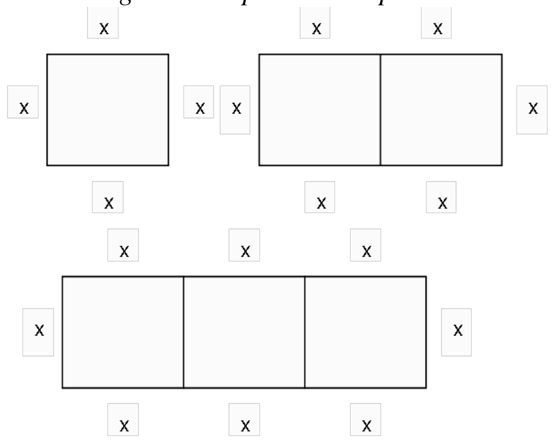
The second step, *Explore*, is when one looks for patterns or attempts to determine the concept or principle at play within the problem. This is essentially a higher form of step one in which the student identifies what the problem is and represents it in a way that is easier to understand. In this step, however, the student is really asking, "What is this problem *like*?" He or she is connecting the new problem to prior knowledge. The student might draw a picture of what the situation would look like for one table, two tables, three tables, and so on. After drawing the tables, the student would note patterns in a chart. (See below.)

The third step, *Select a Strategy,* is where one draws a conclusion or makes a hypothesis about how to solve the problem based on the what he or she found in steps one and two. One experiments, looks for a simpler problem, and then conjectures, guesses, forms a tentative hypothesis, and assumes a solution.

The fourth step is *Solve the Problem.* Once the method has been selected the student applies it to the problem. In this instance, one could simply continue the chart in step three until one reached 24 guests.

Step 2: Explore.

Draw a diagram to represent the problem.



Make a chart, record the data, and look for patterns.

Number of tables	1	2	3	4	.	.	.
Number of guests	4	6	8	10	.	.	.

Pattern: As we add a table, the number of guests that can be seated increases by 2.

Step 3: Select a Strategy.

Number of tables	1	2	3	4	5	6	7
Number of guests	4	6	8	10	12	14	16

Form a tentative hypothesis. Since the pattern seems to be holding true for 16 guests, we can continue to add 1 table for every additional guest until we reach 24. Therefore, we add 4 additional tables for the additional guests (16 + 8 = 24). Hypothesis: It will take 11 tables to accommodate 24 guests.

Step 4: Solve the Problem

Number of tables	1	2	3	4	5	6	7	8	9	10	11
Number of guests	4	6	8	10	12	14	16	18	20	22	24

The final step, *Review and Extend*, is where the student verifies his or her answer and looks for variations in the method of solving the problem; e.g., $t = \frac{n-2}{2}$, where represents the number of tables. Or we could ask for a formula to determine how many guests we can seat given the number of tables. For example, $n = 2t + 2$ or $n = 2(t + 1)$.

Problem Solving Connects Theory and Practice

A perennial charge brought against education is that it fails to prepare students for the real world. It teaches theory but not practice. Problem solving connects theory and practice. In a sense this element is the same as the definitions of problem solving and transfer, only it specifically relates to applying *abstract* school knowledge to *concrete* real world experiences (Krulik & Rudnick, 1980).

Problem Solving Teaches Creativity

Real world situations require creativity. However, it has often been claimed that traditional classrooms or teaching approaches do not focus on developing the creative faculty of students. Advocates of problem solving, by contrast, claim that problem solving develops the students' creative capacities (Frederiksen, 1984; Slavin, 1997).

Successful Problem Solvers Have a Complete and Organized Knowledge Base

A knowledge base consists of all of the specific knowledge a student has that he or she can use to solve a given problem. For example, in order to solve algebraic problems, one not only needs to know information about numbers and how to add, subtract, multiply, and divide, but one must also possess the knowledge that goes beyond basic arithmetic. A knowledge base is what must accompany the teaching of a heuristic for successful problem solving to occur.

Problem Solving Teaches Transfer or How to Apply Conceptual Knowledge

Transfer, or the application of conceptual knowledge, is the connecting of two or more real-life problems or situations together because they share the same concept or principle. Transfer or the application of conceptual knowledge helps students see similarities and patterns among seemingly different problems that are in fact the same, or similar, on the conceptual level.

Some research about problem solving claim that it is more effective than traditional instruction (Lunyk-Child, et al., 2001; Stepien, Gallagher, & Workman, 1993), that it results in better long-term retention than traditional instruction (Norman & Schmidt, 1992), and that it promotes the development of effective thinking skills (Gallagher, Stepien, & Rosenthal, 1994; Hmelo & Ferrari, 1997).

On the other hand, in *Research on Educational Innovations*, Arthur Ellis (2005) notes that the research base on problem solving lacks definition, possesses measurement validity problems and questionable causality, and it fails to answer the claim that successful problem solvers must have a wealth of content-specific knowledge. Ellis further notes that there is "no generally agreed-on set of definitions of terms" (p. 109), that thinking skills are notoriously difficult to measure, and that given these first two problems, it is impossible to trace cause back to any specific set of curricular instances. Ellis states,

> [t]he idea that thinking skills are content specific and cannot be taught generically must be seriously entertained until it is discredited. We don't think that will happen. And if this is so, how does one construct content-free tests to measure thinking skills? (pp. 109–110)

The conclusions of Ellis and other research studies I will cite later state that it would be impossible to reinvent solutions to every problem that develops without recourse to past knowledge. This recourse to past knowledge is evidence, in itself, that one must not completely construct reality. One must apply knowledge that has already been formed by others and understand that knowledge, or else not solve the problem. It is this critique that I will invoke in the following treatment of problem solving. What I hope to show is that the heuristic for problem solving cannot be successful if one holds strongly to the theoretical framework in which it is often situated. Rather, one must accept that already formed knowledge is essential to problem solving. In fact, the meanings of problem solving found in articles and textbooks often convey this contradiction. On the one hand, it is argued that problem solving is the antithesis of a content-centered curriculum. On the other hand, a successful problem solver must possess a strong knowledge base of specific information, not merely a generalizeable heuristic that can be applied across several different situations.

The Problem With Problem Solving

The main problem with problem solving lies in the fourth element listed above: problem solving is a heuristic. Recall that a heuristic is a guideline that may or may not yield success but, unlike an algorithm, it does not depend on knowledge of the problem to be successful. Heuristic is a method of thought that does not pertain to any specific problems or content. The element is problematic because it contradicts three other elements within the theory: the definition of problem solving, successful problem solving requires a knowledge base, and problem solving enables learners to transfer knowledge. Each of these three elements implies that previously learned knowledge of the problem is necessary to solving the problem, whereas use of a heuristic assumes no knowledge is necessary.

I argue, like Peikoff (1985), that there is no way to separate thinking or problem solving from knowledge. Just like instruction and curriculum, these concepts imply one another and cannot be discussed separately for long. Likewise, to acquire knowledge, one must think. This is not to say that students cannot construct knowledge as they solve a given problem, only to say that often the problems they are presented only require them to apply existing knowledge. From this perspective, it must be assumed that students do not construct all of the knowledge in a given curriculum.

Yet *problem solving as a heuristic* is the most cherished aspect of problem solving *because* it is content-less. For example, in the preface to *Mathematical Discovery*, George Polya (1962), one of the foremost thinkers on problem solving says,

> I wish to call *heuristic* the study that the present work attempts, the study of means and methods of problem solving. The term heuristic, which was used by some philosophers in the past, is half-forgotten and half-discredited nowadays, but I am not afraid to use it.
>
> In fact, most of the time the present work offers a down-to-earth practical aspect of heuristic. (p. vi)

Instructional textbooks sometimes play off this process versus content dichotomy: a teacher can either teach students to be critical thinkers and problem solvers or she can teach students more content knowledge. The authors of one textbook say,

> Too often children are taught in school as though the answers to all the important questions were in textbooks. In reality, most of the problems faced by individuals have no easy answers. There are no reference books in which one can find the solution to life's perplexing problems. (Gunter, Estes, & Schwab, 2003, pp. 128–129)

The dichotomy implies that thinking and knowledge are mutually exclusive, when in fact critical thinking and problem solving require a great deal of specific content knowledge.

Problem solving and heuristics cannot be content-less and still be effective. Critical thinking, problem solving, and heuristics must include a knowledge base (Fredricksen, 1984; Ormrod, 1999). Including the knowledge base enables the principle cognitive function of problem solving—the application of conceptual knowledge, or transfer—to occur (Peikoff, 1985). However, the degree to which Dewey and Polya actually believed that a heuristic could be completely content-less and still be effective is not clear. Further, many instructional textbooks actually stress the importance of content knowledge in solving problems (Henson, 2004; Kauchak & Eggen, 2007; Lang & Evans, 2006).

The Elements of Problem Solving Revised

Each of the above elements of problem solving will be reviewed again in light of the relationship between thinking and knowledge and the research base on problem solving. Element one, the *definition of a problem*, implies that one must have some knowledge of the problem to solve it. How can one solve a problem without first knowing what the problem *is*? In fact, identification of the problem is what is called for in the first two steps, *Read* and *Explore,* of the heuristic. In this step, the student first becomes *aware* of the problem and then seeks to *define* what it is or what the problem requires for its solution. Awareness and definition comprise the *knowledge* that is essential to solving the problem. Consider the effectiveness of students relative to their respective experiences with a given problem. The student more familiar with the problem will probably be better able to solve it. In contrast, the student new to the problem, who has only studied the heuristic, would have to re-invent the solution to the problem.

So the first two steps of the heuristic imply that one needs a great deal of knowledge about the problem to be an effective problem solver. In fact, if one wants to solve the problem for the long term, one would want to thoroughly study the problem until some kind of principles were developed with regard to it. The final outcome of such an inquiry, ironically, would yield the construction of an algorithm.

The second element, the *definition of problem solving*, also implies a connection between thinking and knowledge. It says that problem solving is essentially applying old knowledge to a new situation (Krulik & Rudnick, 1987). However, if knowledge or a problem is genuinely new, then the old knowledge would not apply to it in any way. Ormrod (1999) suggests that the so-called new situation is really the same as the old in principle. For example, the principle of addition a student would use to solve the problem $1 + 2 = 3$ is essentially the same principle one would apply to $1 + x = 3$. The form may be different but ultimately the same principle is used to solve both problems. If this is the case, then a more proper element of problem solving would be number eight, the *transfer of knowledge or application of conceptual knowledge*.

The third and fourth elements algorithms and heuristics are problematic. Krulik and Rudnick (1980) distinguish between algorithms and heuristics. Unlike employing an algorithm, using a heuristic requires the problem solver to think on the highest level and fully understand the problem. Krulik and Rudnick also prefer heuristics to algorithms because the latter only applies to specific situations, whereas a heuristic applies to many as yet undiscovered problems.

However, an algorithm requires more than mere memorization; it requires deep thinking too. First, in order to apply an algorithm, the student must have sufficient information about the problem to know which algorithm to apply. This would only be possible if the student possessed a conceptual understanding of the subject matter. Further, even if a student could somehow memorize *when* to apply certain algorithms, it does not follow that he or she would also be able to memorize *how* to apply it (Hu, 2006; Hundhausen & Brown, 2008; Johanning, 2006; Rusch, 2005).

Second, algorithms and problem solving are related to one another. Algorithms are the product of successful problem solving and to be a successful problem solver one often must have knowledge of algorithms (Hu, 2006; Hundhausen & Brown, 2008; Johanning, 2006; Rusch, 2005). Algorithms exist to eliminate needless thought, and in this sense, they actually are the end product of heuristics. The necessity to teach heuristics exists, but heuristics and algorithms should not be divided and set against one another. Rather, teachers should explain their relationship and how both are used in solving problems.

A secondary problem that results from this flawed dichotomy between algorithms and heuristics is that advocates of problem solving prefer heuristics because algorithms only apply to *specific* situations, whereas heuristics do not pertain to any specific knowledge. If one reflects upon the steps of problem solving listed above one will see that they require one to know the problem to be successful at solving it..

Consider the sample problem above to which the heuristic was applied. If one knows the heuristic process and possesses no background knowledge of similar problems, one would not be able to solve the problem. For example, in the first step of the heuristic one is supposed to *Read* the problem, identify the problem, and list key facts of the problem. Without a great deal of specific content knowledge how will the student know what the teacher means by "problem," "key facts," and so on? The teacher will probably have to engage the student in several problems. Without extensive knowledge of facts, how does the student know what mathematical facts are, and how they apply to word problems, for example?

In the second step, *Explore*, the problem solver looks for a pattern or identifies the principle or concept. Again, how can one identify the pattern, principle, or concept without already possessing several stored patterns, principles, and concepts? Indeed, to a student with very little mathematical knowledge, this problem would be extremely difficult to solve. The heuristic would be of little help.

The heuristic says to draw a diagram, presumably to make the problem more concrete and therefore more accessible to the student, but without already knowing what the concept the problem exhibits this would very difficult, if not impossible. Using the chart with the data as an example, it would require previous knowledge in mathematics to be able to construct it. It seems that the heuristic in this problem is in reality just another algorithm that the teacher will have to teach as directly and as repetitively until the students learn how and when to apply it, which is the very opposite of what advocates of problem solving want. The same is also true of step five, *Review and Extend*. Presumably if a student could represent this problem in algebraic form, he or she should also be able to solve the same problem without recourse to drawing diagrams, recording data, etc. One could simply solve the problem right after step one.

The sample problem illustrates what scientists have discovered about novices and experts. In studies that examined expert and novice chess players, researchers found that their respective memories were no different in relation to random arrangements of chess pieces. When the pieces were arranged in ways that would make sense in a chess game, the experts' memories were much better. The theory is that an expert chess player is not a better problem solver, he or she just has a more extensive knowledge base than a novice player. He or she is past the rudimentary hypothesis testing stage of learning, past the problem solving heuristic stage and is now simply applying algorithms to already-solved problems (Ross, 2006). The same could be said for students applying a heuristic to the above problem. The only ones who could solve it would be those who use an algorithm. Even if a teacher taught the heuristic to students, he or she would essentially be teaching an algorithm.

Advocates of problem solving are not solely to blame for the misconception between thinking and knowledge and between heuristics and algorithms. The misconception is likely due to teachers that have over-used algorithms and never shown students how they are formed, that they come from heuristics, and that one should have a conceptual understanding of when they should be used, not merely a memorized understanding of them.

The fundamentally flawed dichotomy within problem solving probably stems from thinking in terms of "either-ors." One side defines appropriate education as teaching algorithms by having students memorize when to use them but not why. The other side, by contrast, emphasizes that thinking for understanding is preferable to simply memorized knowledge. Perhaps what has happened in the shift from the former to the latter practices is the instructional emphasis has shifted from content to thinking so much that the knowledge base has been wiped out in the process. Ironically, eliminating knowledge from the equation also eliminates the effectiveness of problem solving.

The dichotomy between knowledge and thinking has also affected elements five and six. Number five states that problem solving *connects theory and practice*. At the core of this element is yet another flawed dichotomy. Many educators hold that education should prepare students for the real world by focusing less on theory and more on practice. However, dividing the two into separate cognitive domains that are mutually exclusive is not possible. Thinking is actually the integration of theory and practice, the abstract and the concrete, the conceptual and the particular. Theories are actually only general principles based on several practical instances. Likewise, abstract concepts are only general ideas based on several concrete particulars. Dividing the two is not possible because each implies the other (Lang & Evans, 2006).

Effective instruction combines both theory and practice in specific ways. When effective teachers introduce a new concept, they first present a perceptual, concrete example of it to the student. By presenting several concrete examples to the student, the concept is better understood because this is in fact the sequence of how humans form concepts (Bruner, Goodnow, & Austin, 1956; Cone 1969; Ormrod, 1999; Peikoff, 1993). They begin with two or more concrete particulars and abstract from them the essential defining characteristics into a concept. For example, after experiencing several actual tables a human eventually abstracts the concept *a piece of furniture with legs and a top* (Lang & Evans, 2006).

On the other hand, learning is not complete if one can only match the concept with the particular example of it that the teacher has supplied. A successful student is one who can match the concept to the as yet unseen examples or present an example that the teacher has not presented. Using the table as an example, the student would be able to generate an example of a new table that the teacher has not exhibited or discussed. This is an example of principle eight, the transfer of knowledge or applying conceptual knowledge.

The dichotomy between theory and practice also seems to stem from the dichotomous relationship between the teaching for content-knowledge and teaching for thinking. The former is typically characterized as teaching concepts out of context, without a particular concrete example to experience through the five senses. The latter, however, is often characterized as being too concrete. Effective instruction integrates both the concrete and abstract but in a specific sequence. First, new learning requires specific real problems. Second, from these concrete problems, the learner forms an abstract principle or concept. Finally, the student then attempts to apply that conceptual knowledge to a new, never before experienced problem (Bruner, Goodnow, & Austin, 1956; Cone, 1969; Ormrod, 1999; Peikoff, 1993).

The theory vs. practice debate is related to problem solving because problem solving is often marketed as the integration of theory and practice. I argue, however, it leaves out too much theory in its effort to be practical. That is, it leaves out the *application of conceptual knowledge* and its requisite *knowledge base*.

Element six, *problem solving teaches creativity*, is also problematic. To create is to generate the new, so one must ask how someone can teach another to generate something new. Are there specific processes within a human mind that lead to creative output that can also be taught? The answer would depend at least in part on the definition of *create*. When an artist creates, he or she is actually *re*-creating reality according to his or her philosophical viewpoint, but much, if not all, of what is included in the creation is not a creation at all but an integration or an arranging of already existing things or ideas. So in one sense, no one creates; one only integrates or applies previously learned knowledge. No idea is entirely new; it relates to other ideas or things. The theory of relativity, for example, changed the foundational assumptions of physics, but it was developed in concert with ideas that already existed. There may be no such thing as pure creativity, making something from nothing. What seems like creativity is more properly transfer or the application of concepts, recognizing that what appears like two different things are really the same thing in principle.

On the other hand, it is possible to provide an environment that is conducive to creativity. Many problem-solving theorists have argued correctly for the inclusion of such an atmosphere in classrooms (Christy & Lima, 2007; Krulik & Rudnick, 1980; Slavin, 1997; Sriraman, 2001). I only object to the claim that problem solving teaches creativity defined as creating the new. It can, however, teach creativity defined as the application of previously learned principles to new situations.

Element seven, *problem solving requires a knowledge base*, although not problematic is only neglected within the theory of problem solving. This is ironic given how important it is. Jeanne Ormrod (1999) says, “Successful (expert) problem solvers have a more complete and better organized knowledge base for the problems they solve” (p. 370). She also relates how one research inquiry that studied the practice of problem solving in a high school physics class observed that the high achievers had “better organized information about concepts related to electricity” (p. 370). Not only was it better organized, the students were also aware of “the particular relationships that different concepts had with one another” (Cochran, 1988, p. 101). Norman (1980) also says,

> I do not believe we yet know enough to make strong statements about what ought to be or ought not to be included in a course on general problem solving methods. Although there are some general methods that could be of use…I suspect that in most real situations it is…specific knowledge that is most important. (p. 101)

Finally, element eight, *problem solving is the application of concepts or transfer,* is also not problematic; it too is only neglected within the theory of problem solving. Norman Frederiksen (1984) says, for example, “the ability to formulate abstract concepts is an ability that underlies the acquisition of knowledge. [Teaching how to conceptualize] accounts for generality or transfer to new situations” (p. 379). According to this passage, it is the application of conceptual knowledge and not the heuristic alone that as Frederiksen says, “accounts for generality or transfer," (p. 379) which the advocates of problem solving so desire.

Conclusion

Problem solving would be more effective if the knowledge base and the application of that knowledge were the primary principles of the theory and practice. Currently, it seems that a content-less heuristic is the primary principle, which, as I have argued, is problematic because it dichotomizes thinking and knowledge into two mutually exclusive domains. In fact, in the course of solving any problem one will find themselves learning of all things not a heuristic, but an algorithm. In other words, teachers must not only teach students the heuristic and set their students free upon the problems of everyday life. Rather, teachers must, in addition to teaching students sound thinking skills, teach them what knowledge in the past has been successful at solving the problems and why.

References

Alexander, P. A., & Judy, J. E. (1988). The interaction of domain-specific and strategic knowledge in academic performance. *Review of Educational Research*, *58*, 375–404.

Bruner, J., Goodnow, J., & Austin, G.(1956). *A study of thinking*. New York: Wiley.

Christy, A. D., & Lima, M. (2007). Developing creativity and multidisciplinary approaches in teaching Engineering problem-solving. *International Journal of Engineering Education*, *23,* 636–644.

Cochran, K. (1988). *Cognitive structure representation in physics*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Cone, E. D. (1969). *Audiovisual methods in teaching* (3rd ed.). New York: Holt, Rinehart and Winston.

Dewey, J. (1933). *How we think*. Boston: D. C. Heath.

Ellis, A. K. (2005). *Research on educational innovations* (4th ed.). Larchmont, NY: Eye on Education.

Fredriksen, N. (1984). Implications of cognitive theory for instruction in problem solving. *Review of Educational Research*, *54*, 363–407.

Gallagher, S. A., Stepien, W. J., & Rosenthal, H. (1994). The effects of problem-based learning on problem solving. *Gifted Child Quarterly, 36*, 195–200.

Gunter, M. A., Estes, T. H., & Schwab, J. (2003). *Instruction: A models approach* (4th ed.). Boston: Pearson Education,

Henson, K. T. (2004). *Constructivist teaching strategies for diverse middle-level classrooms*. Boston: Pearson Education.

Hmelo, C. E., & Ferrari, M. (1997). The problem-based learning tutorial: Cultivating higher order thinking skills. *Journal for the Education of the Gifted, 20*, 401–422.

Hu, C. (2006). It's mathematical after all—the nature of learning computer programming. *Education & Information Technologies*, *11*, 83–92.

Hundhausen, C. D., & Brown, J. L. (2008). Designing, visualizing, and discussing algorithms within a CS 1 studio experience: An empirical study. *Computers & Education*, *50*, 301–326.

Johanning, D. (2006). Benchmarks and estimation: A critical element in supporting students as they develop fraction algorithms. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 384–386). Mérida, Mexico: Universidad Pedagógica Nacional.

Kauchak, D. P., & Eggen, P. D. (2007). *Learning and teaching: Research-based methods*. Boston: Allyn and Bacon.

Krulik, S., & Rudnick, J. A. (1987). *Problem solving: A handbook for teachers* (2nd ed.). Boston: Allyn and Bacon.

Lang, H. R., & Evans, D. N. (2006). *Models, strategies, and methods for effective teaching*. Boston: Pearson Education.

Lunyk-Child, O. L., Crooks, D., Ellis, P. J., Ofosu, C., O'Mara, L., & Rideout, E. (2001). Self-directed learning: Faculty and student perceptions. *Journal of Nursing Education, 40*, 116–123.

Norman, D. A. (1980). Cognitive engineering and education. In D. T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research* (pp. 97–107). Hillsdale, NJ: Erlbaum.

Norman, G. R., & Schmidt, H. G. (1992). The psychological basis of problem-based learning: A review of the evidence. *Academic Medicine, 67*, 557–565.

Ormrod, J. (1999). *Human learning* (3rd ed.). Upper Saddle River, NJ: Prentice Hall.

Peikoff, L. (1985). *The philosophy of* education [CD Lecture Series]. Irvine, CA: Second Renaissance.

Peikoff, L. (1993). *Objectivism: The philosophy of Ayn Rand*. New York: Meridian.

Polya, G. (1954). *Mathematics and plausible reasoning: Vol. 1. Induction and analogy in mathematics*. Princeton, NJ: Princeton University Press.

Polya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving, Vol. 1*. New York: John Wiley & Sons.

Polya, G. (1988). *How to solve it: A new aspect of mathematical method* (2nd ed.). Princeton, NJ: Princeton University Press.

Ross, P. E. (2006, August). The expert mind. *Scientific American*, 64–71.

Rusch, T. L. (2005). Step one for developing a great mathematics lesson plan: Understand the mathematics. *Ohio Journal of School Mathematics*, *51*, 25–34.

Slavin, R. E. (1997). *Educational psychology: Theory and practice* (5th ed.). Boston: Allyn and Bacon.

Sriraman, B. (2004). Understanding mathematical creativity: A framework for assessment in the high school classroom. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 350–352). Toronto, Canada: OISE/UT.

Stepien, W. J., Gallagher, S. A., & Workman, D. (1993). Problem-based learning for traditional and interdisciplinary classrooms. *Journal for the Education of the Gifted, 16*, 338–357.