Mathematics Education Research Journal 2008, Vol. 20, No. 3, 6-31

Learning From "Didactikids": An Impetus for Revisiting the Empty Number Line

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This article discusses students' perceptions as a source for understanding education. It addresses what didactically experienced children, called "didactikids", taught us about the empty number line as a didactical model for teaching whole number calculations. The article mainly reports on a student consultancy study carried out in the Netherlands. The findings are similar to what was revealed in an Australian study. Both studies explain what can go wrong when the number line is applied rigidly and wrongly implemented.

In this article I reflect upon the empty number line (a blank number line with no numbers or markers) as a didactical model for teaching calculations with whole numbers in the early grades of primary school. Children whom I call "didactikids" prompted this reflection. These are didactical experts who have an inclination to understand teaching, perhaps having developed this talent because a parent or relative works in education. Every child, though, has the potential to become a didactikid when given the opportunity to reflect upon education. As will be shown in this paper, teachers, teacher educators, and researchers and developers of mathematics education can learn a great deal from didactikids.

Based on what such didactical experts brought to the fore about their experience with the empty number line, I will discuss some critical issues to do with the use of the number line and what can go wrong when this didactical model is not implemented in the way it was originally conceptualised. In other words, this article has two foci: the issue of learning from children, and the empty number line.

In current thinking about mathematics education, children's contributions to the development of mathematics teaching are highly valued. In the Netherlands, it was Freudenthal who laid the foundation for this appreciation with inspiring observations of his children and grandchildren. His goal was to build knowledge about children's cognitive

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[1] This paper is an extended version of a paper written in Dutch and published in a joint special issue of two Dutch journals (see Van den Heuvel-Panhuizen, 2005) produced to celebrate the centenary of Freudenthal's birth.

development and generate guidelines for education through observing them.

Creating didactical knowledge was also the aim of a *student consultancy study* that I carried out recently. However, in this study, the intended guidelines for how to teach mathematics were acquired in a different way. Instead of observing the children's development, I was questioning them directly about their didactical expertise. The main part of this article consists of the report on this study, where I interviewed 2 children on their thoughts on particular teaching issues and on the approach they would choose if teaching children.

One of the topics I discussed with them was the empty number line. For the children the number line seemed to be a restraining factor, and the interview clarified why this is the case. After further questioning, it was discovered that the way in which the number line was taught to these children might have contributed towards their negative ideas about the usefulness of the number line. Later in the interview, these "didactikids" revised their opinion, coming up with a useful proposal.

After conducting the student consultancy study in the Netherlands, I learned about a similar study carried out in Australia in which Janette Bobis interviewed her daughter about the empty number line (Bobis & Bobis, 2005). The findings of both studies are strikingly similar. Both studies support the conclusion that the way in which children are asked to work with the empty number line can be detrimental to the development of their proficiency in operating with numbers. With the experiences of these studies as a starting point, I reviewed other research findings and approaches to the empty number line in order to come to a better understanding of the key aspects of the use of the number line as a didactical model for teaching calculations with whole numbers. This article summarises my research journey.

Children's Perceptions as a Source of Understanding Education

Since the inception of the reform project in 1968, *Wiskobas*, that later became the Realistic Mathematics Education [RME] (see for a concise overview Van den Heuvel-Panhuizen, 2001a), there has always been a great deal of input from children in the development of Dutch mathematics education. According to La Bastide-Van Gemert (2006), observing children might be considered as the most innovative aspect of the developmental work of *Wiskobas*.

The foundation for this approach was laid by Freudenthal's observations of his children and his grandchildren. He set a good example to others. There was not just Freudenthal's grandson Bastiaan (see Freudenthal, 1979; Van der Velden, 2000), who showed how high the clouds are in the sky on a sunny day in comparison with clouds that are precursors of rain, but there was also Streefland's son Coen (see Van den Brink & Streefland, 1979), who thought the whale on the poster was depicted too big. The observations undertaken of Bastiaan and Coen opened our eyes to the qualitative entries of the concept of ratio, the relationship between ratio and measurement, and the visual roots of ratio. All these issues are very important for teaching ratio and were didactical discoveries in the days of mechanistic mathematics education.

As well as setting a good example for recording "incidental observations" (Freudenthal, 1984a, p. 101) Freudenthal also indicated the crucial point of these observations: "What counts [in real learning processes], are the discontinuities, the jumps" (p. 103). That is, it is children's own discoveries that mark the jumps in their development. Freudenthal believed we have to look at the individual child to track these discoveries.

> If you take the average of a number of children, you flatten the jumps. The average child does indeed show a continuous development. But if you look closer, at the individual, you will see the jumps, and as far as I am concerned, they are the only thing that matters. (p. 103)[2]

In connection with this, Freudenthal, against the ideas of the time, exposed the strength of the evidence of qualitative small case study research (see Freudenthal, 1979). He also found confirmation for this in science: "Only one Foucault pendulum was enough to demonstrate the rotation of the Earth" (Freudenthal, 1984a, p. 101). Elsewhere he emphasised: "One good observation can be worth more than hundreds of tests or interviews" (Freudenthal, 1984b, p. 19). On the occasion of Freudenthal's seventy-fifth birthday Van den Brink (1980, p. 6) referred to the children who were observed as "children who teach".

Freudenthal's research mainly consisted of incidental observations. He rarely performed intentional experiments (Freudenthal, 1984a) and he did not like opinion polls and questionnaires for teachers and students (Freudenthal, 1978, 1988). According to him, these only yielded useless, unreliable, or predictable reactions. Though I can see his point and share his skepticism, I do think that Freudenthal would have judged differently if he had interviewed children *about* education. Despite all the pleas in favour of making use of the knowledge of students, investigating students' perceptions is not a well-established research practice (Keitel, 2003). However, as will be shown in the following, the studies that have been carried out revealed that the children's perspectives of education could bring in new knowledge about classrooms.

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[2] Some quotations in this article are the author's translation of the original Dutch text.

Some General Findings about Students' Perceptions of Education

This direct appeal to children's expertise in the area of education is what I had in mind with my student consultancy study. But before describing this study, I would like to explore somewhat more what is known about students' perceptions of education. My special lens in this review is whether students' perceptions are used as suggestions about how to teach. I discovered studies that investigate how students perceive mathematics education to be rather scarce, so the review I present now includes studies into subjects other than mathematics. My review comprises all student levels from pre-school and kindergarten through to university.

To begin with, it is good to realize that there are different interpretations of student perceptions. For example, Fraser (1998) makes a distinction between perceptions of actual or experienced teaching and perceptions of preferred or ideal teaching, while McRobbie, Fisher, and Wong (1998) distinguish between student perceptions of teaching with respect to the whole class in contrast to perceptions of teaching with respect to the students' own personal roles or subgroups in a learning situation.

What these studies all have in common is that they report that investigations into students' interpretations of what happens in classrooms can provide practitioners as well as researchers with valuable information. For instance, Dahlberg, Moss, and Pence (1999), who investigated the quality of early childhood care, say "children have a voice of their own, and should be listened to as a means of taking them seriously, involving them in democratic dialogue and decision-making and understanding childhood" (p. 49). Spratt (1999) underlined the role that students can play in the design of materials and syllabi.

The main pattern that emerges is that students' perceptions clearly differ from teachers' perceptions. Often, teachers are surprised when they hear about their students' thoughts and feelings (Barkhuisen, 1998). In some cases, especially when evaluations of educational innovations are concerned, these discrepancies are alarming. For example, in a study carried out at high school level, Hagborg (1994) found notable differences between teachers and students when teachers' instructional methods were examined. Frequency ratings by students and teachers revealed that students viewed the instruction methods as limited in scope and teacher-centred (e.g., lectures and seat work) while the teachers rated their own instructional methods as including a broad range of methods and requiring active student participation (e.g., small group work).

Another demonstration of a key difference between perceptions of teachers and students was found in a Japanese study carried out within the framework of the *Learner's Perspective Study* [LPS] (see www.edfac.unimelb.edu.au/DSME/lps/). As Clarke (2005) reported, the distinctive structure of the LPS research was the videotaping of a sequence of ten lessons followed by post-lesson interviews in which the students and teachers watched the tapes individually and were asked to respond to the lessons. The research subjects were asked to fast forward to parts of the lesson that seemed important to them, explaining what was notable. In Shimizu's (2002) study that used this procedure, data were collected at three public junior high schools in Tokyo, and many of the students were fast-forwarding through what was seen by their teachers as the essence (the "Yamaba") of the lesson.

An issue that might play a role in the discrepancies between teachers' and students' perceptions is different preferences regarding classroom activities. Spratt (1999) observed that teachers do not have a good picture of their students' activity preferences. When investigating students learning English at tertiary level in Hong Kong, Spratt (1999) found that teachers were able to gauge their learners' preferences with accuracy for only approximately 50% of the activities. According to Spratt (1999), this finding has far-reaching consequences for informed decision making in education by teachers, and for those who are involved in educational design. This in turn might have considerable implications for the effectiveness of instruction. As Kumaravadivelu (1991, p. 98) stated, "the narrower the gap between teacher intention and learner interpretation, the greater are the chances of achieving desired learning outcomes". Spratt (1999) also raised the issue of learners' preference for more "traditional" classroom work and mentioned three other studies that came to this conclusion. In the area of mathematics education, this conservatism of students prevents teachers from changing their teaching practice, as was also shown by Desforges and Cockburn (1987).

Young children are also very capable of giving a good description of what happens in classrooms. Wing (1995) studied a kindergarten and a Grade 1–2 classroom for one school year. During that time she interviewed 14 children from each classroom. The study showed that "in contrast to the early childhood maxim 'play is the work of a child', in children's minds, play is not work" (p. 227). The children had a clear view on work and play and were not fooled by work activities disguised as play. Play and work "represented different experiences for children, in spite of the fact that their teachers consciously attempted to make work play-like by incorporating hands-on materials, giving children choices, and encouraging exploration and discovery" (p. 227). In a further study by Wiltz and Klein (2001), children's perceptions of their experiences in care centres were under investigation. Wilz and Klein found that 4-year-olds demonstrated awareness and understanding of procedures and activities, and verbalized this information accurately in sequential and descriptive ways.

The general impression from the studies mentioned above is that children's knowledge of what is happening in the classroom might be a valuable resource to understand teaching and find out how we can best teach mathematics to children. To study the complex processes within classrooms, researchers have developed increasingly complicated research designs and methods of data collection and analysis, but enhanced understanding might be found closer to home. This approach draws on the children's didactical experience and expertise. Giving children the role of educational consultant is rare in research of mathematics education. To address this shortcoming I undertook the study that is described in the next section.

The Student Consultancy Study

The aim of the student consultancy study I conducted was to ask 2 students for detailed advice about a range of topics that play a part in mathematics education. A case study approach was taken, with two students being questioned about their opinions on, and preferences in, various aspects of mathematics education. The children are 11-year-old identical twin sisters, Ylja and Joni, who attend the same school. Both are in Grade 5, though they are not in the same class.

Figure 1. Ylja and Joni point out which problems they like and which ones they do not like.

Ylja and Joni were asked to participate in this student consultancy study. Their participation in earlier research had involved them in trialing a number of problems intended to investigate how well children who are good at mathematics perform in solving puzzle-like mathematics problems (see Van den Heuvel-Panhuizen & Bodin-Baarends, 2004). Ylja and Joni not only enjoyed solving these problems and explaining their solution strategies, they also spontaneously provided many suggestions for the best way to teach gifted students.

Ylja and Joni are not average students, but that does not mean they are inappropriate research participants. The goal of this student consultation was not to draw valid, general conclusions on what students think about mathematics education, but to understand better the learning and teaching of mathematics by listening to what children have to say about teaching practice.

For this student consultancy study, Ylja and Joni were interviewed twice: an exploratory interview and a more extensive interview. The interview issues included several recent research results and news items related to education.

First Interview

The first interview lasted an hour and a half and was intended to be exploratory. Here Ylja and Joni could talk freely about their experiences with mathematics education. As an introduction I told them that I wanted to find out the best way to teach children mathematics and how to improve mathematics education. The response of Ylja and Joni to this introduction was remarkable, but fully in accordance with the way I addressed them. They were not embarrassed. They did not giggle. I treated them like experts and they reacted accordingly.

The first interview was not recorded audio or video: I took some photographs and made notes. These notes are not a complete word for word account.

A central topic during this first interview was whether Ylja and Joni thought mathematics was a fun subject or not and why they thought so. The following is an impression of what was said on this topic.

Like/dislike for mathematics. Although Ylja and Joni are good at mathematics, neither of them expressed a liking for mathematics.

Ylja: What I do not like is that you always have to do the same problems such as 84×62. Another thing that is not fun either is when children in classroom do not understand particular problems and the teacher starts explaining these problems again to the group as a whole. Then you often hear the children moaning. There are always children who have difficulties in understanding problems. That is not that bad. Everybody has particular things that are hard, difficulties.

Joni: It would be good if there would be a particular arrangement [such as] who got it, can do something on their own; who does not get it, can get further explanation. The rule now is that everybody gets further explanation.

Another reason Ylja and Joni gave for disliking mathematics was that it is not challenging enough. A thing they disliked especially was writing a lot when solving mathematics problems.

Ylja: I am very happy that abbreviations *[sic]* exist.

Moreover, they disliked doing a lot of problems. Problems should not take long. However, they liked thinking about problems. Expanding on this, Joni said:

Joni: The teacher is always explaining how to solve a problem, but not why this is true.

Ylja: You cannot prove everything; for example 1+1=2. I have read about it in "The Number Devil."

Joni: However, you can show (visualise) it.

Ylja: The children in the classroom often ask each other why a particular strategy is not allowed or why it is correct, as well.

Ylja made it clear that she does not know why the children ask each other this question and why they do not ask the teacher.

Ylja: That is a difficult question.

Because Ylja and Joni were very eager to tell me what they think about the textbook series they use in class it was decided to take the use of the textbook as one of the issues to discuss further in the second interview.

Second Interview

The second interview was a structured interview in which the children were asked to react to a number of research findings or news items on mathematics education or education in general. The children were consulted on ten issues in total:

1. What is the best way to teach weak learners mathematics?
2. Is it possible to learn something while you are sleeping?
3. Does it help to chew gum during a mathematics test?
4. What are your ideas about a "holiday school"?
5. How should a teacher use the textbook series *Pluspunt*?
6. Assume you were to start a school yourself, what would your school look like?
7. Is mathematics a fancy school subject? Why?

8. What are your ideas about practising?
9. Does an empty number line help to solve number problems?
10. What do you think about mathematics lessons? What would you do differently?

The issues were presented by means of a PowerPoint presentation. On each issue, the children were given some information first, after which they were invited to give their opinion. For instance, in the case of Question 1, a newspaper clipping was shown about research carried out in special education. Then the main research result was explained, namely, that teaching one particular strategy for solving number problems turned out to be the best strategy for weak learners. After that Ylja and Joni were asked what they thought about this finding. The complete interview was videotaped and took about two hours, and the video sound recording was transcribed verbatim.

Ylja and Joni's Views on the Empty Number Line

Out of the ten issues that were presented to Ylja and Joni, only one is discussed in this paper: the use of the empty number line.

The Empty Number Line as a Didactical Model – a Brief Reflection in Advance

Within *Realistic Mathematics Education*, the empty number line is one of the most important didactical models for teaching calculations with numbers up to 100 and 1000 (see, for example, Treffers, 1991a, 1991b; Van den Heuvel-Panhuizen, 2001b). This model has been accepted widely in the Netherlands, with teachers, curriculum developers, and researchers all emphasising its importance. Its intended use is as a flexible mental model to support adding and subtracting, rather than a measuring line from which the exact results of operations can be read. Number lines can also be found in all new versions of the mathematics textbooks, though the way the number line is used is not always how it was intended.

Interview Question 9

To introduce the topic, the following problem was presented to the children: 386 + 298 = . The problem was given using a horizontal representation. Without any further instructions on how to solve the problem, Ylja and Joni were invited to find the answer. Without hesitation, they both began by carrying out the addition algorithm shown in Figures 2a and 2b. Both students used an algorithmic method of ciphering.

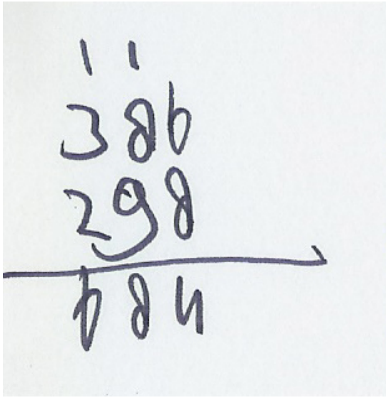

Figure 2a. Ylja's work.

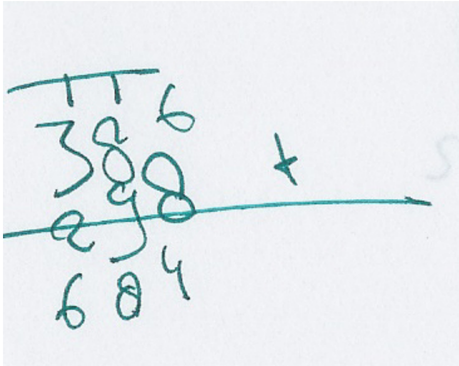

Figure 2b. Joni's work.

When I asked them whether there was another way to do it, they mentioned the strategy of column calculation. This strategy is more or less standard in the Netherlands as the first stage of carrying out a written ciphering strategy (see Van den Heuvel-Panhuizen, 2001b). In this strategy, numbers are processed as whole numbers rather than as individual digits (see Figure 3). Ylja and Joni considered this an easier solution than carrying out a ciphering algorithm.

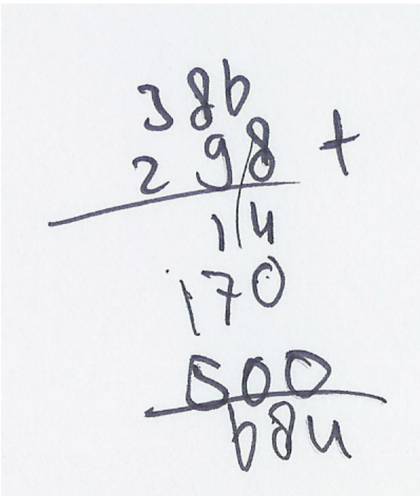

Figure 3. Ylja's demonstration of the column calculation strategy.

When the children were asked whether this problem could have been solved using the number line, their frustration emerged.

Joni: It can help, if you don't understand all the other ways that can be used for [solving these problems].

Ylja: Actually, of all years, for maths, I think the dumbest year is Grade 2, because it keeps going on about the number line ...

Joni: You have to do just about everything with the number line.

Ylja: You aren't allowed to use any other way. You have to use the number line.

Joni: And as far as I know, the number line is meant for the children who aren't so good at mathematics.

Ylja: See, for them it's very easy, a number line, to calculate something step by step.

Joni: … for some children the number line is faster than calculating underneath each other, simply because they don't understand how to calculate underneath each other …

I then attempted to prompt them to consider the option of applying a smart calculation strategy in which 386 plus 298 is changed into 386 plus 300 minus 2, which did not work:

I: But look at the two numbers.

Joni and Ylja started explaining their perception of the longitudinal learning pathway:

Joni: [...] we learn, like, the number line first. In Grade 3 we get between bars. [This means working with position lines that divide the ones, tens, and hundreds]. In Grade 4 we learn underneath each other like this [column calculation]. In Grade 5 we learn like this underneath each other [she means the algorithmic calculation method of ciphering]. For me, this could go faster [...].

Ylja: In Grade 3 you must calculate between bars. I don't really know anymore how that works. So I can't show it. Or with a number line [...].

Yet another attempt was made to get them to consider the smart strategy.

I: Look at the two numbers. What is special about them? If you had to make an estimation of the answer of this addition.

Joni: Then I would do three hundred plus three hundred ...

I: But you can also keep the 386 intact, and then say like ...

Joni: I do that, I do that, but then it suddenly gets a lot easier, then it's just 686 minus 2.

Joni: [...] in Grade 4, when children want to do tricks with zeros, for example with addition, no, with multiplication or division, [the teacher says] "now you will learn how it should be done" [...] and in Grade 5 you learn tricks to get it right.

Joni: And this, for example, is one of those tricks.

I: Is it a trick?

Ylja: Yes, well, it's actually easier to calculate it with a trick.

Joni: [It is] A simpler calculation trick. Except we're learning – some children already have worked out this trick, they're already using it in their head – but we're learning other things first.

Ylja: We aren't really learning this in school.

I: You're not learning this?

Ylja: The, the children, many children are using it, but it's never explained. So the children who don't think of it, can't use this quick and simple way.

I: So when I ask you "calculate this problem" you're almost automatically going to put the numbers underneath each other and calculate them?

Ylja: Yes, but I hadn't really looked well at the problem. But usually I look, "Oh wait, it's quicker that way."

Joni: Often it says [in the textbook] ... Sometimes it says "Calculate" and sometimes it says "Calculate smartly."

Ylja: And then you switch. If it says, "Calculate smartly" you immediately try to figure out how you can do it in the easiest way.

But Joni still was unconvinced that using the number line could be a smart way.

Joni: It may help, it may even help very well, but for the children who can do other things well too a number line is only time-wasting. It's clumsy.

Then I showed a solution using the number line (see Figure 4).

Figure 4. Solution on the empty number line.

I: [...] you make a big jump of three hundred, and two back.

Ylja: Yes, but that is not how we learned the number line. We really learned that you first add two hundred, then ninety and then eight.

Joni: We didn't ...

Ylja: This way ...

Joni: [This we did not really used a lot.] Very occasionally maybe.

Ylja: By the time that we, like, got a grip on this, we were already doing these steps by heart.

I returned to Ylja's and Joni's frustrating experiences with the number line.

I: But you don't have good memories of it?

Both children: No ...

I: Could you explain to me why?

Ylja: Because you always had to use it, always [...]. Then you had a problem in front of you and they would say: "You <u>have</u> to use the number line."

Joni: And there was so little variation. It's almost, always when we had an addition or a subtraction problem, we always had to.

I: So you would have done that differently?

Ylja: Yes. [...] Because the number line itself is very useful. The number line is very clear for some children, [...] but as soon as you know another way to calculate those numbers, that kind of problems, you no longer need the number line. Because there are so many different ways to do that ...

Joni: … and which are faster.

I: [...] fast too, when you no longer need to draw the number line, but that you have it in your head as it were.

Both children: Yes! ...

Joni: But then you're still using it.

Ylja: So, like, first learn the number line. Then learn it by heart, and then, like, being able to explain what you did in your head. It's actually the same as the number line. And as soon as children understand that, they switch, "Hey, I can do the number line in my head", and as soon as they know "Hey, I can do the number line in my head", they use that number line in their head.

Looking Back on Interview Question 9

The first thing that stood out with the two girls' interview responses is how professional Ylja and Joni were in discussing the didactics of mathematics, sounding at times like qualified teachers. With little effort, they outlined the recommended learning pathway for calculating with whole numbers—something teachers are not always able to do. In the most recent National Assessment of Educational Achievement [PPON], carried out in the Netherlands, about 20% of teachers of Grades 2 and 3 indicated they did not know when column calculation is to be taught (see Kraemer et al., 2005).

The answer to interview Question 9 is also revealing about the application of smart calculation strategies like 386 plus 300 minus 2. Such strategies may fall into the category of "tricks" and it seemed as if they are not taught in school but only applied when asked for. To have children use a smart calculation strategy requires putting children on the right track first.

What makes this situation more worrying is that in this case the interviewees were two children who are good at mathematics, who have number knowledge and skills that allow them to use smart calculation strategies, but who apparently have not developed this skill in school. This fits with the results from the earlier mentioned study that investigated how high achieving fourth graders solved puzzle-like mathematics problems. Although the participating students had been identified as high achievers in mathematics, they were poor at solving the puzzle-like problems (Van den Heuvel-Panhuizen & Bodin-Baarends, 2004) and did not do as well as expected on all aspects of problem solving. We do not know the role that teaching plays when children do not spontaneously apply their knowledge about numbers and properties of operations, but Ylja and Joni's thoughts about the number line does give a clue.

Clearly, Ylja and Joni found the empty number line constraining. This is concerning because the empty number line is intended as a flexible model that should give students a lot of freedom, and this includes both flexibility in the ways of recording results and flexibility in the jumps students make to solve problems. However, Ylja and Joni did not really experience this freedom. In Grade 2 they had no option other than to use the number line in a prescribed way, which they disliked. For them, working with the number line meant doing step-by-step calculations, and shortcuts were not allowed. It is also striking that Ylja and Joni did not realise that such shortcuts are possible on the number line. They chose the algorithmic calculation method of operating with digits because that was faster for them. They had not considered an easy mental calculation. Here again we can wonder about the influence teaching has.

Since the introduction of the number line in Dutch mathematics education at the end of the eighties, the model has been taught both in a prescriptive and a non-prescriptive way (see Menne, 2001). Treffers (1991a, following Whitney, 1985) argues for the latter, which is also the approach supported by most mathematics didacticians in the Netherlands. In spite of that, flexible use of an empty number line is not always reflected in textbooks. Moreover, not all textbooks use the empty number line as the first model to guide operations up to 100. For instance, the first edition of the textbook series *Pluspunt* (see Menne, 2001) emphasised decimal splitting (in tens and ones) and using manipulatives to lay out and process numbers. In this approach we can recognise exactly the ciphering procedure Ylja and Joni chose to solve the problem. On the other hand, in the new version of *Pluspunt* the didactical position of the empty number line has been strengthened (Menne, 2001). However, this has not been a great help for Ylja and Joni, who were taught with this version.

Experiences of an Australian Didactikid

Of course, The Netherlands is not the only country that has its didactikids; Australia has them too. Janette Bobis told me about the discussion she had with her 9 year old daughter, Emily, about teaching mathematics. The issue that they discussed happened to be the empty number line, and in particular the instructional pitfalls connected to this model. These reservations are made clear in their jointly written and presented paper (see Bobis & Bobis, 2005). Like Ylja and Joni, Emily spoke of her negative experiences when working with the empty number line. In Year 3, comparable with the Dutch Grade 2 where Ylja and Joni had their unfortunate experiences, Emily was expected to use pre-structured number lines in which the zero and tens were indicated. This approach made Emily think that her teacher wanted her always to start with zero, the first number that was marked on the line (see Figure 5).

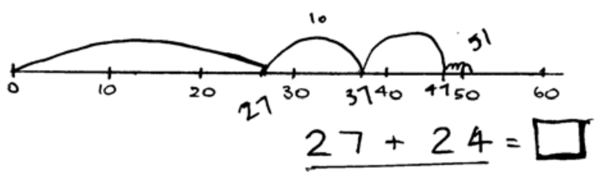

Figure 5. Starting from zero (from Bobis & Bobis, 2005, p. 71).

In Year 4, Emily had another confusing experience. A pre-structured number line with intervals of one, pictured in the textbook, made Emily revert back to a strategy of counting by ones (see Figure 6).

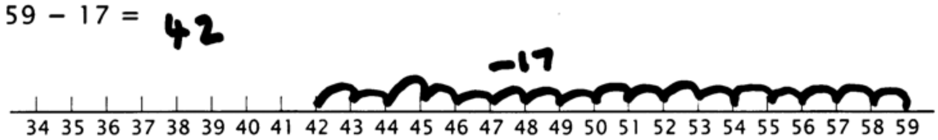

Figure 6. A pre-structured number line eliciting a counting strategy (from Bobis & Bobis, 2005, p. 71).

The conclusion of Bobis and Bobis was that if the number line is applied too rigidly or is wrongly understood as a didactical model, it can be detrimental to the children's understanding of and proficiency in mental calculation.

A Reflection

The use of didactical models in education is a sensitive issue. Incorrect use may have a harmful effect and be "anti-didactical". Freudenthal (1973) used this word about 30 years ago, when he commented on the tendency to take the scientific structure of the discipline as the guiding principle and present children with ready-made mathematics, rather then giving them the chance to develop mathematical concepts and methods themselves. However, anti-didactical reversals do not just threaten mathematics education at a macro level: they do so that at a micro level. By way of example, a textbook series first "teaches" decimal splitting as the basis for adding and subtracting, and then puts children to work on the number line. Such a directive goes against the didactical structure of the domain of calculations up to 100. These numbers do not lend themselves to digit-based algorithmic calculations, but are more suitable for applying a whole number strategy supported by the didactical model of the empty number line. Such applications are reflected in the learning-teaching trajectory for whole numbers developed in the TAL[3] project (see Van den Heuvel-Panhuizen, 2001b). The defining characteristics for the didactical structure are three basic strategies: *stringing*, *splitting* and *varying*.

In the case of stringing, a problem such as 34+27 is solved like this: 34+10→44+10→54+6→60+1→61 or, even quicker, 34+20→54+7→61. Stringing has its roots in counting and is related to the ordinal aspect of number. Using a stringing strategy in addition and subtraction problems means keeping the first number intact and only splitting the second number into tens and ones. The components of the second number are then added to or subtracted from the first number in parts. Line models such as the string of beads (see Figure 7) or the empty number line (see Figure 8) are suitable models to support this strategy.

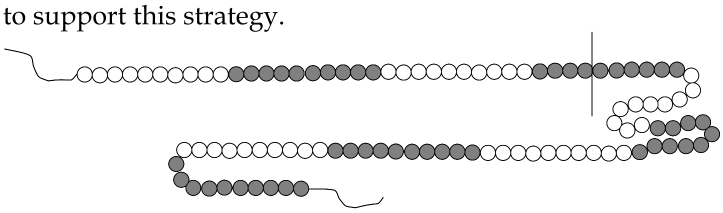

Figure 7. String of beads.

3 The acronym TAL stands for Intermediate Attainment Targets in Learning-Teaching Trajectories. The aim of the TAL project is to developed learning-teaching trajectories for mathematics in primary school. The development of the trajectory for whole numbers (see Van den Heuvel-Panhuizen, 2001b) was a joint enterprise of the Freudenthal Institute [FI] and the National Institute for Curriculum Development [SLO], and it reflects the principles of RME.

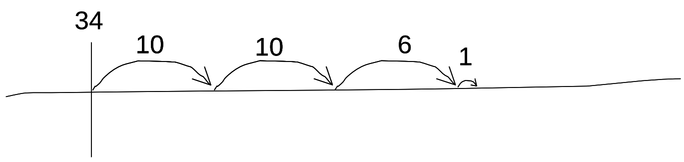

Figure 8. Empty number line.

In the case of splitting, a problem such as 34+27 is solved like this: 30+20=50 and 4+7=11 and 50+11=61. The numbers are broken down in tens and ones and processed separately when the operations are carried out. Group models consisting of groups of tallies, sticks, blocks (see Figure 10), counters, or coins are the number models that support this strategy. This splitting strategy strongly builds on the understanding of place value and is a predecessor of the written algorithm of ciphering.

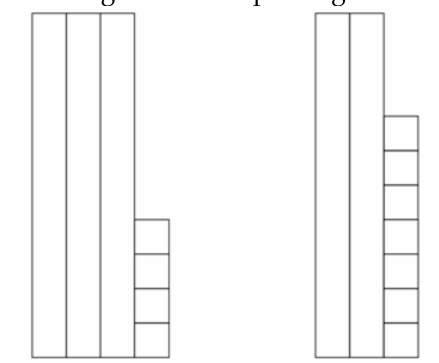

Figure 10. Blocks.

The essential difference between the two types of models is the way in which they represent numbers, which consequently also influences the way in which the operations are carried out. The model types have each their affinity for a particular strategy for operating with numbers.

The third strategy, varying, is related to the children's knowledge of number relationships and properties of operations. It involves all kinds of smart calculation strategies, such as using 6+6 for solving 6+7. It covers strategies in which children tinker with numbers. Examples of varying strategies are reordering (3+69 becomes 69+3), regrouping based on the associative property and making use of "easy" numbers such as 25 (26+27 becomes [25+1]+[25+2] which becomes 25+25+3), using inverse relationships (the answer of 52–49 is found by counting on from 49), and compensating (74–38 becomes 74–40+2= or becomes 76–40). Here, both line and group models and combinations thereof can be used as a support, but mostly when the students are able to tinker with numbers they have reached a stage in which a concrete model is no longer necessary. At that point, a mental image of such a model will suffice, or the problems may be solved by students without a supportive model.

Matching Models with Strategies

Whether the empty number line is a suitable model to solve number problems depends strongly on the strategy that is applied. Counting and skip counting strategies are best supported by a line model, and strategies in which the numbers are split in tens and ones are best supported by a group model such as arithmetic blocks consisting of rods depicting tens and units. The efficiency of strategies applied depends on the numbers involved in the operation. The number knowledge the child has available is also critical. Sequencing in the strategies is also evident, with stringing emerging from counting, before splitting and the use of smart compensation strategies.

Textbooks and other teaching resources often do not make clear that didactical models are closely connected to different strategies and that a particular model can elicit a particular strategy. For example, the guideline given in the New Zealand Number Framework (Ministry of Education, 2005) for calculating 43+35 might be very confusing for children (see Figure 10).

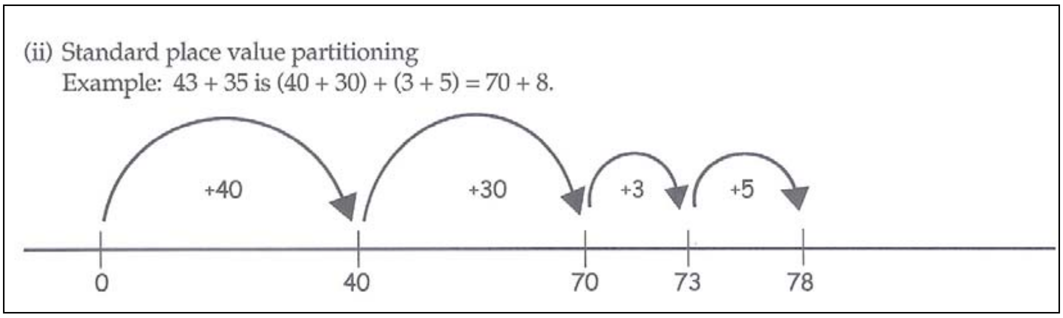

Figure 10. Guideline from the New Zealand Number Framework (Ministry of Education, 2005, Book 1, p. 4).

When the children are supposed to apply a splitting strategy (e.g., in the case of 43+35 resulting in 40+30 and 3+5; 70+8=78), it would be better if a group model such as the one pictured in Figure 11 were used.

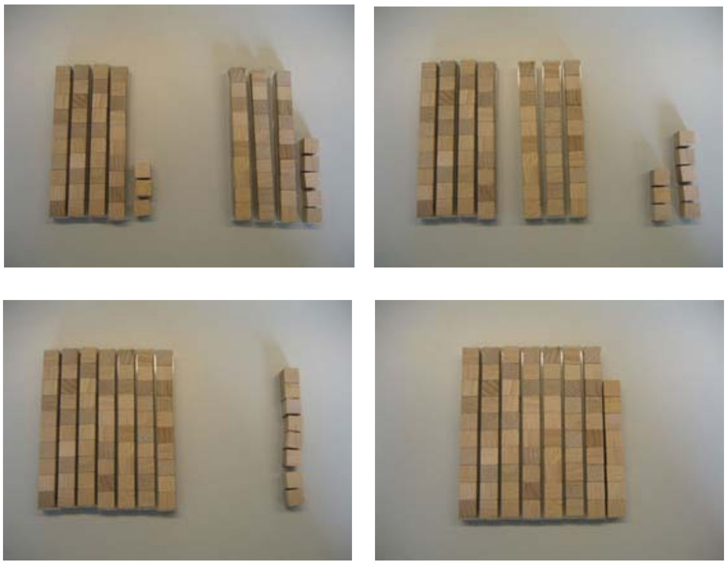

Figure 11. Group model made of arithmetic blocks that supports the splitting strategy.

Granted, in the case of 43+35 the empty number line could also be used. However, this should not suggest that the calculation starts with zero because the idea of the empty number line is that the students only mark the numbers they need for their calculation (see Figure 12).

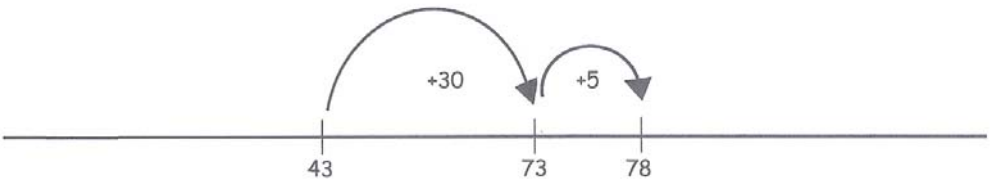

Figure 12. Using the empty number line to find the answer of 43+35.

I will not address here the further developments in later grades of the empty number into a double number line and a fraction or percentage bar (see Van den Heuvel-Panhuizen, 2002). Neither will I discuss the possibilities of zooming in on the empty number line to obtain more refined units and arriving at decimals, but it is clear that the number line is a very rich model that can have different manifestations. The danger is that these different aspects might confuse students. For instance, a student in a study by Stacey, Helme, and Steinle (2001, p. 223) said: "I get my number lines mixed up." In my view this confusion is very likely to happen if the nature of the number line model is not understood and the constituting aspects of its nature are not clearly recognized. The mismatch between model and intended calculation strategy may reflect such a confusion.

There is also a possibility of confusion when the empty number line that is meant as a counting line (referring to discrete quantities) is used as a measuring line (referring to continuous quantities). In the latter, the number line generally has a zero as a starting point and the numbers are placed at equal intervals. Doing a calculation based on such a line means "reading off" the number at which you arrive after carrying out the operation, while the empty number line is meant for structuring the consecutive calculation steps and recording them.

That the empty number line refers to discrete quantities was clearly expressed by Whitney (1985) when he used toothpicks to indicate the numbers or, more correctly, the amount of beads (see Figure 14).

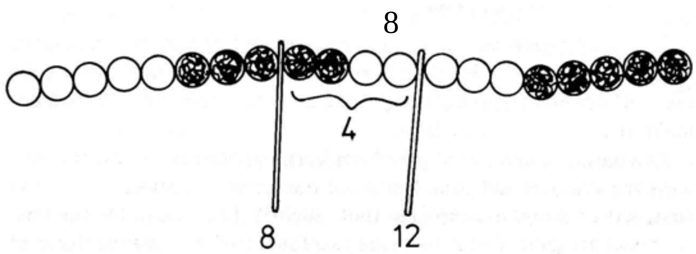

Figure 14. Whitney's chain of beads with toothpicks (Whitney, 1985, p. 134).

By using toothpicks, Whitney (1985) combined the two types of numbers (quantity numbers and measuring numbers) in one model. More importantly, this model clarified the difference between these two types of numbers. The "measuring" eight (at the end of the first toothpick) indicates that there are eight beads to the left of it. However, at the same time the model makes clear that this measuring eight does not coincide with the "quantity" eight, the interval after the eighth bead. This could solve the difficulties that up to that point had obstructed the use of number lines (Treffers, 1991a). It was often unclear for both children and teachers what should be counted: the beads or the intervals. Whitney's (1985) toothpicks clarified the difference between the two, and at the same time indicated their connection. By introducing the children to this chain of beads that used toothpicks or pegs to mark certain amounts, the foundation was created for the empty number line as a supportive model for calculation with whole numbers.

In the learning-teaching trajectory developed for this mathematical domain (Treffers & De Moor, 1990; Van den Heuvel-Panhuizen, 2001b), the chain of beads (mostly structured in groups of ten beads) is used in the main for counting and structuring activities and not for carrying out operations. The students are asked where particular numbers are and how they can jump to those numbers in different ways (see, for example, the programme developed by Menne, 2001). After they build up familiarity with the number line and the relationship between numbers, they do the same on an empty number line (without the beads). The next step is to use this empty number line as a supportive model to carry out additions and subtractions. The markers on the empty number line then stand for a particular number (of beads). One important characteristic of this empty number line model is that it can support and record the calculation steps in a flexible way. In no way are the children asked to put the numbers on the empty number line in a way that is proportionally correct. As noted before, the empty number line is not a measuring line.

The many number lines with equal intervals and starting at zero that are found in curriculum documents and research literature clearly demonstrate a different interpretation. A literature review of mathematics education by Owens and Perry (2001) make the observation that a debate about the use of number lines in early primary school has been going on for some years. "One difficulty of the use of the number line is that length is representing the size of the number but only the order of the numbers is transparent with the distance to zero or other point is not obvious" (Ownes & Perry, 2001, p. 72). In other words, they reported a concern that the number line often does not meet the requirements of a measuring line. My answer to this concern would be that we should use the empty number line as a didactical model to support adding and subtracting with numbers up to 100 and beyond, and should not treat the empty number as a measuring line.

To conclude this reflection on the empty number line, prompted by what I discovered when interviewing the two didactikids, I would like to mention the danger of *instrumentation*. Unfamiliarity with the number line's nature can initiate prescriptive use of it. Such teaching may prescribe what numbers to include, and may prescribe how to draw the jumps and all other symbols that should accompany the number line activities. Teaching the students this 'didactical ballast' (Van den Heuvel-Panhuizen, 1986) is not only very time-consuming, but also anti-didactical, because it takes away from the children any opportunity to mathematise: to find their own strategies, including shortcuts, and to come up with their own notations.

Conclusion

Although other researchers have concluded previously that children's expertise should be taken more seriously in education research and development, the findings of this small student consultancy study were surprising. Through their ideas on the number line as a didactical model and their thoughts on other educational questions that were discussed in other parts of the interview, Ylja and Joni illustrated why we should make use of the didactical qualities of children more often. Children who observe carefully how a teacher explains something, structures the curriculum, organises the class, handles differences, assists weaker students, and makes use of the textbook can contribute to researchers' and curriculum developers' understandings. The perspective offered by these children can enhance our insight into what is happening in the classroom. More research is necessary to find out how we can explore such knowledge and how we can identify these didactikids.

What Freudenthal (1984a) showed us about observing the development of children, is, in my opinion, true for consulting with children. The knowledge we gain here can also "profit" us in developing mathematics education. Furthermore, and here I also draw the parallel with Freudenthal observing,

> … it is not something we want to reserve exclusively for the developer and the researcher. We propagate this [consulting children] to others, to teachers, to teacher trainers and to those being trained, and we offer them material to promote this mentality. (Freudenthal, 1984a, p. 106)

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