Motivating Prospective Elementary School Teachers To Learn Mathematics by Focusing upon Children's Mathematical Thinking[1]

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Elementary school children in the United States are not developing acceptable levels of mathematical proficiency (National Center for Education Statistics, 1999), and a major concern of teacher educators is that teachers lack the depth and flexibility of mathematical understanding and the corresponding beliefs they need to teach for proficiency (National Research Council [NRC], 2001). Although teachers' mathematical content knowledge plays a critical role in their instruction (Fennema & Franke, 1992; Hill, Sleep, Lewis, & Ball, 2007), teachers need more than content knowledge to be effective. Beliefs about mathematics, teaching, and learning affect not only the ways teachers teach mathematics (Philipp, 2007; Thompson, 1992) but also the ways prospective teachers learn mathematics. In California, the development of the mathematical content knowledge of prospective teachers takes place in undergraduate courses and is separated from their consideration of issues of teaching and learning, which often does not occur until students attend mathematics methodology courses as college graduates in a credential program. This article is based upon my assumption that, for prospective elementary school teachers, separating the learning of mathematics from the consideration of issues of mathematics teaching and learning is counterproductive to their development of mathematical content knowledge and to the devel-

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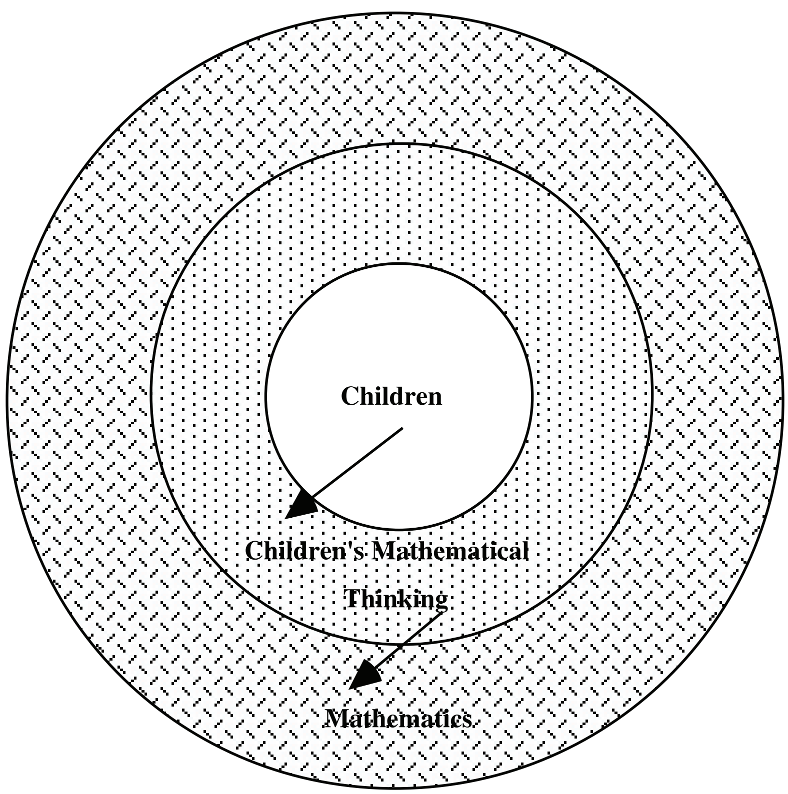
opment of their beliefs about mathematics teaching and learning. After presenting the theoretical underpinnings and summarizing the data in support of the claim that prospective elementary school teachers (PSTs) benefit by learning about children's mathematical thinking concurrently while learning mathematics, I describe four principles that serve as the focus of a mathematics laboratory developed and implemented for PSTs at San Diego State University and at local community colleges.

Why We Integrate Children's Mathematical Thinking into Mathematics Courses

Developing deep understanding of the mathematics of elementary school is far more difficult than was once thought (Ball, 1990; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). Furthermore, even when PSTs attend a thoughtfully planned course designed to engage them in rich mathematical thinking, many PSTs react to the course in a perfunctory manner. Most PSTs do not know what mathematics they need to know to teach effectively, and many are not open to approaching the content anew in a deeper and more conceptual way than they experienced in elementary school because they hold a self-perpetuating belief that "if I, a college student, do *not* know something, then children would not be expected to know it, and if I *do* know something, I certainly don't need to learn it again." Furthermore, many PSTs believe that mathematics is a fixed set of rules and procedures, and when combined with their belief that children and adults learn mathematics by being shown how to solve problems in a prescribed, step-by-step fashion, these beliefs clash with the more conceptual, meaning-making goals many mathematics-course designers hold for PSTs (NRC, 2001). The approach my colleagues and I have taken is based upon our belief that by providing PSTs opportunities to develop more nuanced beliefs about mathematics, teaching, and learning early in their undergraduate experiences, we might launch them on a different growth trajectory that may orient them toward learning mathematics from a relational or meaning-making, rather than an instrumental, perspective (Skemp, 1978).

When my colleagues and I approached the issue of teaching mathematics to PSTs, we asked ourselves what it is that PSTs care about in relation to mathematics teaching and learning. We decided that fundamentally, PSTs entered teaching because they cared deeply about children, and rather than try to get PSTs to care about mathematics for mathematics sake, we took the approach that we wanted PSTs to care about mathematics for the sake of the children they would one day teach. Our Circles of Caring model (see Figure 1) highlights how their thinking about children may lead to PSTs' learning mathematics. The innermost circle, *Children,* reflects PSTs' initial concern for children, which is to protect children and keep them comfortable, safe, and happy. Many PSTs initially associate their caring for children with the belief that they should avoid challenging children. However, when PSTs are supported so that they engage children in mathematical problem solving or when they observe carefully selected video of children solving problems, many of the PSTs' circles of caring begin to expand to include children's mathematical thinking. Furthermore, when they learn about children's mathematical thinking, many PSTs begin to redefine caring as including challenging children so that they grapple with meaningful mathematics. Finally, when PSTs are supported so that they begin

Figure 1.
Circles of Caring, a model of growth, by way of children's mathematical thinking, from PSTs' caring about children to their caring about mathematics.



to engage with details of children's mathematical thinking, many realize that, to be prepared to support children's learning, they must themselves grapple with the mathematics, and their circles of caring extend to learning mathematics. In other words, by having PSTs look at mathematics through the lens of children's mathematical thinking, we help them come to care about mathematics, not as mathematicians, but as teachers. Our approach is based upon an old idea. John Dewey (1902/1990) noted, more than 100 years ago, that every subject might be thought of as having two aspects, "one for the scientist as a scientist; the other for the teacher as a teacher" (p. 351). He wrote, "[The teacher] is concerned, not with the subject-matter as such, but with the subject-matter as a related factor in a total and growing experience [of the child]. Thus to see it is to psychologize it" (p. 352). Note that we view our approach as *a* way, not *the* way, to support PSTs' learning of mathematics. Although we recognize that PSTs can become excited about learning mathematics and that a mathematical approach may work for some, we have chosen to take a different starting point in our work with PSTs.

We tested our theory using a large-scale randomized experimental study, and the results showed that PSTs who studied children's mathematical thinking while learning mathematics developed more sophisticated beliefs about mathematics, teaching, and learning and improved their mathematical content knowledge more than those who did not (Philipp et al., 2007). I share two comments from students to highlight how learning about children's mathematical thinking supported PSTs in learning mathematics. The first is from Phil, a student who was relatively strong in the mathematics class but who came to recognize that he needed to go beyond simply learning the procedures; he needed to think deeply about the concepts.

> Phil: One thing I got out of 296 [The Children's Mathematical Thinking Experience]—if I hadn't taken 296, I probably would have gone through [the subsequent mathematics courses] focusing on the thing that I already knew, the algorithm that I already knew, and thinking, "All right, that's the best." But now I realize that I have to take it all in, everything that the class is teaching, not just what I think is the most important. Because all of this is important. I probably wouldn't have realized that if I hadn't taken 296.
>
> RP: Why is it important?
>
> Phil: Because people think in different ways, and not everyone thinks like me.

The other comment was made by a student on the last day of the

Children's Mathematical Thinking Experience when she stated what she might tell a friend she had learned from taking the class:

> For people who are going to take [the first mathematics class]—just because *a lot* of the times in class . . . people get *so* mad and *so* frustrated as to why they are learning what they are learning. And then you come [to the Children's Mathematical Thinking Experience], and you see a kid do exactly what you are learning in [the mathematics] class. And it just makes sense, and it eliminates that whole frustration of feeling like "Why am I learning this? Where am I going to ever use this?" So by taking this class, you *see* how … the children actually apply what you are learning, the different styles or the different methods for solving problems.

For this student, learning about children's mathematical thinking helped her recognize the importance of that which she was learning in her mathematics class. She had come to see that her content knowledge was insufficient for her to teach students mathematics for understanding. Learning about children's mathematical thinking provided the motivation for her to go beyond procedures and also learn the mathematical concepts.

As a result of this study, San Diego State University instituted a new course required for prospective elementary school teachers. This Children's Mathematical Thinking Course is a half-semester course designed to accompany a mathematics course on whole and rational number (including fractions, decimals, and percents) concepts and operations. This course, which might be thought of as a laboratory designed to accompany a mathematics course for PSTs, is neither a mathematics course nor a mathematics methodology course, although it combines aspects of both. This required course is also offered at the local community colleges, where several instructors have reported to me that they enjoy teaching this course more than any other because students find the content particularly pertinent. Instead of outlining this course, in this article I focus on support for mathematics instructors who may not be in a position to develop a new course but are willing to consider modest infusion of issues of children's mathematical thinking into their existing mathematics course for PSTs. I do so by highlighting four principles around which we focus our course and share examples that highlight some of the principles. Additional details about the Children's Mathematical Thinking Course are provided elsewhere (Philipp, Thanheiser, & Clement, 2002).

Four Principles of Mathematics and Mathematics Teaching and Learning

Figure 2 highlights four principles that serve as the backbone of our course; each principle could be addressed in any mathematics course for PSTs. I describe how we attempt to raise and address these principles, drawing upon examples to illustrate each.

Principle 1. *The way most students are learning mathematics in the United States is problematic. In particular, students learn to manipulate mathematical symbols without developing the underlying conceptual meanings for the symbols.*

With the persistent news stories in the popular press reporting students' difficulties with mathematics, one might assume that everyone in the United States is aware that we have a problem with our students' learning of mathematics. However, many undergraduate students are simply unaware of the depth of the problem, and until they understand the problem, they have little motivation for considering that the way they learned mathematics may not be the way they need to learn to teach mathematics. By focusing upon children's mathematical thinking, an instructor can highlight one aspect of the difficulty, namely that students often learn to manipulate symbols without understanding the underlying meanings of the symbols. I find three sources helpful for highlighting Principle 1. One source is data from reports such as the National Assessment for Educational Progress (NAEP). Figure 3 shows a fraction-estimation task and the results for 13-year-old students in the United States (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). When discussing this item, I find that PSTs are generally surprised to find that fewer than 25% of the students in middle school are able to estimate fraction size. I also find that PSTs do not understand how children reason about this problem, and many cannot explain why students selected the incorrect estimates of 19 or 21. I point out to my

Figure 2.
Four principles of mathematics and mathematics teaching and learning addressed by focusing upon children's mathematical thinking.

Principle 1	The way most students are learning mathematics in the United States is problematic. In particular, students learn to manipulate mathematical symbols without developing the underlying conceptual meanings for the symbols.
Principle 2	Learning concepts is more powerful and more generative than learning procedures.
Principle 3	Students' reasoning is varied and complex, and generally it is different from adults' thinking.
Principle 4	Elementary mathematics is not elementary.

Figure 3.
NAEP item and percentage of 13-year-old students selecting each response.

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$. You will not have time to solve the problem using paper and pencil.

(a) 1 (b) 2 (c) 19 (d) 21 (e) I don't know

Percentage of Students Selecting Each Response

(a) 7% (b) 24% (c) 28% (d) 27% (e) 14%

PSTs that when looking at these fractions, many children, instead of seeing two numbers, 12/13 and 7/8, see four numbers, 12, 13, 7, and 8. Furthermore, because students cannot remember which operations to perform on these four numbers, they often resort to adding the numerators or the denominators. Although I consider it important for PSTs to learn which tasks are difficult for children, it is even more important that they begin to understand how children reason about the tasks.

A second source for highlighting Principle 1 is to interview, or view video clips of, children. Interviews of intermediate-grades children are generally effective for demonstrating that many children lack conceptual understanding of fractions; however, PSTs can also come to see this problem by viewing and discussing carefully selected video clips. One video clip that I have used effectively with PSTs shows a girl named Ally, identified by her teacher as an average fifth grader from a class in a high-performing school; Ally struggles to make sense of fractions (VC #11, Philipp, Cabral, & Schappelle, 2005[2]). In the video clip, Ally explains her reasoning for comparing fractions, and within one minute, on three consecutive comparison tasks, Ally's explanations highlight three common fraction conceptions that often lead to students' incorrect reasoning. On one task she explains that 1 is bigger than 4/3 "because 1 is a whole number." On the next task she explains that 1/2 is greater than 3/6 because "if you change the denominator [of 1/2] to 1, just one digit lower, then it would equal to 1, and 1 is a whole number." On the subsequent task, she explains that 1/7 is greater than 2/7 because "I wasn't quite sure about this one, so I chose one seventh because I thought it was just the smallest number, and usually you go down to the smallest number to get to the biggest number [in fractions]."

Our class discussion highlights two issues, the first of which my students can usually extract on their own but the second of which I generally identify. First, the students grapple with how the student is reasoning, and after a discussion we converge upon three conceptions driving Ally's thinking in this 1-minute span of the video clip: (a) Fractions are less than 1; (b) fractions can be compared by comparing the differences between the numerators and the denominators; and (c) with fractions, the number that looks larger is smaller. We then discuss why students may hold these conceptions, and I help my students realize that all three were valid ways of reasoning at one point in students' mathematical experiences. Ally's first conception, that fractions are less than 1, has its roots in the initial introduction to fractions when students learn that fractions are parts of a whole and, hence, must be less than the whole. The fraction language commonly used reinforces this conception when, for example, students conceptualize 3/4 as "three out of four." Does it make sense, in this context, to consider "five out of four"?[3] The residue of conceptualizing fractions as less than 1 is evident in our commonly used language, when, for example, we say, "I completed only a fraction of the job." The technique of comparing fractions by determining the differences between numerators and denominators yields correct answers when constrained to positive fractions less than 1 with equal denominators (or numerators). For example, one could correctly conclude that 7/9 is greater than 4/9 because the difference between 9 and 7 is less than the difference between 9 and 4. The third conception, that larger numbers result in smaller values, yields correct answers in comparison of unit fractions. For many years students learned that 8 is larger than 6, but then, for fractions, 1/8 is smaller than 1/6. When PSTs begin to understand students' reasoning, they can develop more nuanced understandings and, for example, come to view the statement that "you go down to the smallest number to get to the biggest number" as one that makes sense in some contexts but not in others.

The third source for highlighting Principle 1 is to ask PSTs to reflect upon their own learning experiences. In any class of PSTs, I have found students willing to share their personal struggles learning mathematics, struggles they often continue to experience as adults; viewing video clips of children struggling to make sense raises personal recollections, often painful ones, for many PSTs. I return to this issue of PSTs' struggles when addressing Principle 4.

Principle 2. *Learning concepts is more powerful and more generative than learning procedures.*

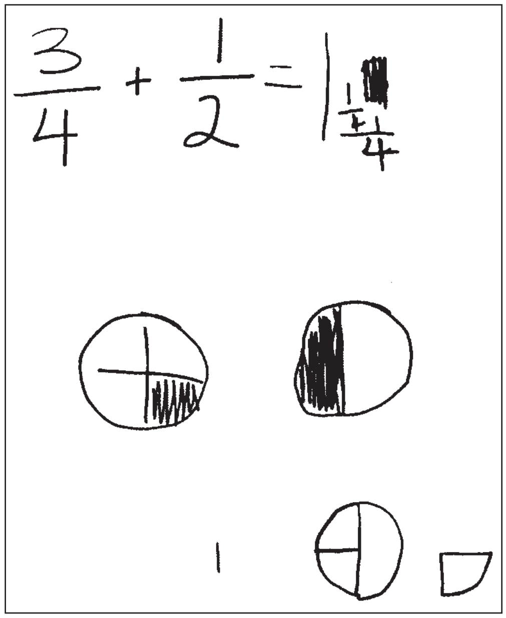
A corollary to Principle 1 is that students can learn mathematical procedures without understanding the underlying concepts, and related to this corollary is Principle 2: Learning concepts is more powerful and more generative than learning procedures. One source for highlighting

Principle 2 is to consider research about children's mathematical thinking. For example, PSTs (and many teachers) are surprised to discover that many primary-grades children are able to solve multiplication and division problems when the problems are embedded in real-life contexts. Consider the problem *"Tad had 15 guppies. He put 3 guppies in each jar. How many jars did Tad put guppies in?"* PSTs think that because they view this problem as division, primary-grades children cannot solve it because they have yet to learn about division. When PSTs learn that more than 70% of kindergarten children in one study were able to solve this problem at the end of the year (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993), they are surprised. Perhaps even more surprising to PSTs is children's success at solving the problem "*Nineteen children are taking a mini-bus to the zoo. They will have to sit either 2 or 3 to a seat. The bus has 7 seats. How many children will have to sit 3 to a seat, and how many can sit 2 to a seat?"* At the end of the year, among students who had been in classrooms in which the focus had been to teach mathematics meaningfully, more than half the kindergarten students correctly solved this problem. Note that this is a two-variable problem of the type often used in middle- or high-school algebra classes. Learning concepts had enabled these students to make sense of complex, even multistep, problems.

A second source for highlighting Principle 2 is to arrange for PSTs to interview or to observe video clips of primary-grades children, because young children often understand and approach mathematics in meaning-making ways. One video clip that effectively highlights Principle 2 shows a student, Felisha, who correctly adds 3/4 + 1/2 at the end of second grade although she had not yet learned any procedures for adding fractions. Felisha had spent seven mornings learning fractions with three other children of the same age from a teacher who approached the teaching of fractions by posing real-life, sharing situations. Because Felisha had developed rich understanding of partitioning and equivalence, she was able to flexibly approach fraction tasks. Figure 4 shows Felisha's written work, and a video clip of her solution is available on the IMAP Select CD (VC #15, Philipp et al., 2005) and may be viewed at http://www.sci.sdsu.edu/CRMSE/IMAP/vid_frac_add.html. This video clip demonstrates how a deep conceptual understanding is generative and enables children to extend and apply their thinking to new situations.

One poignant example of a video designed to address the relationship between conceptual and procedural knowledge shows Rachel, a fifth grader whose teacher normally teaches mathematics with a focus on understanding but who agreed to teach one lesson on converting between mixed numbers and improper fractions using a procedures-only lesson.

Figure 4.
A second-grade student's written solution for 3/4 + 1/2.



In an interview after the lesson, Rachel was asked to solve a problem like those she had correctly solved during the lesson. She could not remember the procedure, and she drew a causal link back to the way the procedure-only instruction impeded her understanding. She said that she "didn't figure it out for herself." She went on to explain,

> So when I figure that out, it's easier, and, um, once *I* figure it out, it's, it *stays* there 'cause I was the one who brought it there. So, and it is just easier to do when you figure it out yourself, instead of having teachers telling you.

Five weeks after the procedural lesson, Rachel's teacher presented a conceptual lesson on the same topic. After the conceptual lesson, Rachel was again interviewed, and when asked to convert a mixed number to an improper fraction, she incorrectly applied a procedure before she corrected herself by drawing a picture. When reflecting upon her approach, Rachel said that the reason she applied an incorrect procedure instead of first relying upon her conceptual understanding was because she learned the procedure first. She also stated that she would have preferred instruction that focused on helping her understand the concepts before instruction that showed her a particular procedure. (A video paper of

Rachel [Philipp & Vincent, 2003] is available on the NCTM Website at http://my.nctm.org/eresources/view_article.asp?article_id=6430 and a draft is permanently available at http://www.sci.sdsu.edu/CRMSE/IMAP/pubs/Reflections_on_Fractions.pdf). Other research aligned with this example was published by Pesek and Kirshner (2000) who found that exposure to procedures inhibited students' subsequent conceptual learning.

The third source for highlighting Principle 2 is to ask PSTs to reflect upon their own learning experiences as they relate to children's thinking. An anecdote that I have found effective is to describe my older son's confusion, during elementary school, about the procedure for converting mixed numbers into improper fractions (e.g., 4 2/3 to 14/3). He knew that he was supposed to multiply two numbers and add a third number but could not remember the procedure. He drew a picture of 4 wholes, as I requested, and immediately knew, when asked, that there are 3 thirds in one whole. I then asked how many thirds would be in 2 wholes, or 3 wholes, or 100 wholes, or a googol of wholes; while responding to this succession of questions, he generalized the procedure and noted that the product of the whole number and the denominator determines how many fractions of that size are in that number of wholes. He immediately saw why he needed to add the 2 to the 12—because he had yet to account for the 2/3. (At the conclusion of our discussion my son, who knew that occasionally I taught lessons in his class and who also knew that he was one of many confused students, said, "Dad, I think you need to come in and teach this to the class.") When PSTs draw their own pictures, they too understand the mathematics underlying this conversion procedure, and they are able to explain the rule for the inverse procedure of converting from improper fractions to mixed numbers. This experience provides them with a personal experience supporting the notion that learning concepts is more powerful and more generative than learning procedures.

Principle 3. *Students' reasoning is varied and complex, and it is generally different from adults' reasoning.*

One source for highlighting Principle 3 is to consider research about children's mathematical thinking. For example, many PSTs think that the problem "*Maria has 5 shells. How many more shells does Maria need to collect so that she'll have 11 shells altogether?*" is a subtraction problem because subtraction is the operation they generally associate with this problem. However, children in first or second grade tend to approach the problem as a joining problem because the action in the problem involves obtaining more (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Some children solve this problem by counting out 5 counters, then counting up to 11 counters, and then counting the 6 that they added. A more sophisticated solution involves children's starting with the first quantity, 5, "in their heads," counting on to 11 on their fingers, and noting that the answer is the number of extended fingers. In a still more sophisticated strategy, one that does not involve counting, a child uses a fact she knows, 5 + 5 = 10, to infer that because 11 is 1 more than 10, the answer must be 6, 1 more than 5.

A second source for highlighting Principle 3 is to arrange for PSTs to interview children or observe video clips of children solving problems, because when PSTs watch young children solve problems, they see multiple approaches to solving a problem, and they also observe that the approaches children use are different from the approaches that they, the PSTs, would have used. A video clip that effectively highlights Principle 3 is of Javier, a fifth-grade student, determining how many eggs are in six dozen. Javier reasons that 5 times 12 is 60, and 12 more is 72, so there are 72 eggs in six dozen. When asked how he knows that 5 times 12 is 60, Javier, a recent immigrant to the United States from Mexico, responds in his second language, "Because 12 times 10 equals 120. If I take the [*sic*] half of 120, that would be 60." Figure 5 provides a formal analysis of the mathematics underlying Javier's reasoning. Note, I neither suggest that Javier is aware of the names of these properties nor that he would have analyzed his reasoning as was done in Figure 5; instead, I suggest that Javier's reasoning provides an example of how a child's sense-making solution invoked sophisticated properties in mathematically appropriate ways. A video clip of Javier solving this problem is available on the IMAP Select CD (VC #6, Philipp et al., 2005) and may be viewed at http://www.sci.sdsu.edu/CRMSE/IMAP/vid_mult.html

Javier's solution raises another important point related to Principle 3. If teachers teach in ways that support children's reasoning, then the teachers are likely to find that they have children in their classes who are better mathematicians than they, the teachers, are. This statement is not to imply that the children know more mathematics than the teachers; they have yet to study formal algebra, or geometry, or other

Figure 5.
A formal analysis of the mathematics underlying Javier's reasoning.

6 × 12
= (5 × 12) + (1 × 12) (Distributive prop. of x over +)
= [(1/2 × 10) × 12] + 12 (Substitution property)
= [1/2 × (10 × 12)] + 12 (Associative property of x)
= [1/2 × (120)] + 12
= 60 + 12
= 72

high school mathematics topics. But the children approach mathematics with more innovation, creativity, and confidence than their teachers, all signs of rich mathematical proficiency (NRC, 2001). How are teachers to respond to these children? One hopes that they will welcome, appreciate, encourage, and even embrace these innovative problem-solving approaches, and to do so, PSTs must learn to delve deeply into their students' solution strategies and, hence, into the mathematics. In the process, both the teacher (Franke, Carpenter, Levi, & Fennema, 2001) and other students in the class will learn more mathematics.

A third source for highlighting Principle 3 is to consider a corollary of Principle 3: Adults' reasoning is also varied and diverse. One way to quickly highlight this corollary is to pose a simple task that tends to elicit a diverse set of responses from adults. Two useful tasks are to solve each of the following without using pencil and paper: (a) 99 + 98 and (b) Find 15% of $42.

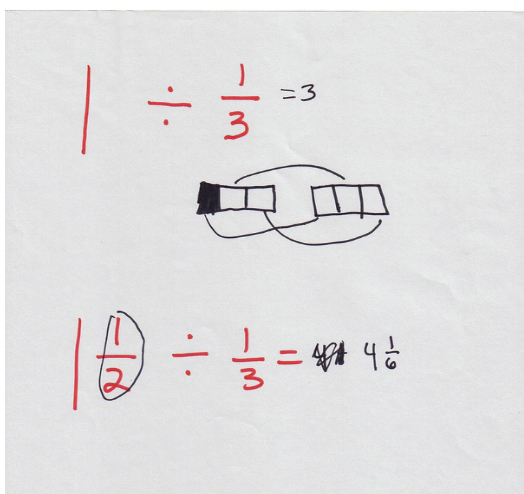
Principle 4. *Elementary mathematics is not elementary.*

A common belief among lay people is that the content of elementary school mathematics is simple; however, when PSTs learn more about children's mathematical thinking, they begin to realize that engaging deeply with issues of mathematics and mathematics teaching and learning, at any grade level, is complex. For example, although adults may remember the procedures we use to divide fractions, few people, even few mathematics majors, can explain why, when we divide fractions, we invert and multiply. Teachers need two types of mathematical knowledge: They need to know and understand the content that they will teach to children, but they also need to hold a deeper understanding if they are to attend to their students' ways of reasoning. A distinction drawn in the literature is between *common content knowledge,* the mathematical knowledge teachers are responsible for developing in students, and *specialized content knowledge,* the mathematical knowledge that is used in teaching but not directly taught to students (Hill et al., 2007). For example, the procedure used for dividing 1 1/2 by 1/3 is common content knowledge taught to students, but a deeper understanding of the mathematical issues raised by trying to understand how students make sense of fraction division is part of the specialized content knowledge teachers need to teach mathematics to children (for a view of this specialized knowledge of fraction division, see Philipp, 2005).

One source for highlighting Principle 4 is to arrange for PSTs to interview children or observe video clips of children solving problems. A video clip that effectively highlights Principle 4 shows Elliot, a sixth-grade student, solving two division tasks. Elliot solves the first task, 1 ÷ 1/3, by correctly reasoning that three 1/3s make one whole. Elliot is then asked to solve 1 1/2 ÷ 1/3. He again correctly reasons that because three 1/3s make 1 and 1/2 has another 1/3, there are four 1/3s in 1 1/2. He also correctly determines that after removing four 1/3s from 1 1/2, 1/6 remains. However, Elliot's final answer is incorrect because instead of reconceptualizing the 1/6 as 1/2 of 1/3 and answering that there are 4 1/2 one-thirds in 1 1/2, he leaves the 1/6 as a remainder but treats it as a quotient when incorrectly answering 4 1/6. Elliot's written work is displayed in Figure 6, and a video clip that shows Elliot solving these two problems is available on the IMAP Select CD (VC #16, Philipp et al., 2005) and may be viewed at http://www.sci.sdsu.edu/CRMSE/IMAP/vid_frac_div.html

This video clip highlights that mathematical understanding is seldom either complete or nonexistent but is, instead, held in degrees; although students usually understand some aspects of a concept (in this case, that $a \div b$ may be thought of as "How many *b*s are in *a*?"), they may be struggling with other aspects of the concept (for Elliot, that the remainder may be reconceptualized as part of the quotient). This principle, curiously, applies to all learners, and when we discuss this as a principle of learning, PSTs begin to see that instead of viewing their conceptual holes as weaknesses, they may view them as a natural part of engaging with a rich domain and that, hence, they need to be open to continuing to learn throughout their lives, including from their students.

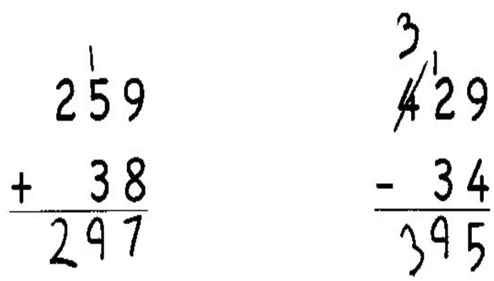
Figure 6.
Elliot's work for two fraction-division tasks.



A second source for highlighting Principle 4 is drawn from a content domain generally considered relatively easy for adults: whole number place value. Below is a whole number place-value task my colleagues and I have used to assess PSTs' and teachers' mathematical content knowledge for teaching (Philipp, Schappelle, Siegfried, Jacobs, & Lamb, 2008). Figure 7 shows two algorithms commonly used in the United States, the multidigit addition and subtraction algorithms, and two questions designed to assess one's understanding of the algorithms.
We found that although every PST in our study understood how to apply these algorithms, 31 of 36 PSTs, upon entering their first mathematics course for prospective teachers, scored 0 (on a 0–4 scale) on this task. That is, almost none of the students explained that the regrouped 1 in the addition problem represents one group of ten (or 10 groups of 1), whereas the regrouped 1 in the subtraction problem represents one group of one hundred (or 10 groups of ten). Thus, at this stage of their professional development, most of the PSTs incorrectly explained the mathematics underlying these algorithms; if they were teaching, their only recourse would be to present the procedures without helping their students understand the underlying place-value concepts.

Figure 7.
Ones Task, designed to assess PSTs' and teachers' content knowledge of whole number algorithms and place value.

Problem A | Problem B



Part 1
Does the 1 in each of these problems represent the same amount? Please explain your answer.

Part 2
Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in Problem B) 10 is added to the 2.

A Proposed Sequence for Using Written Student Work and Video

Over the many years I have used written student work and video, I have developed a sequence that guides me in thinking about how to orient my PSTs to reflect on issues of mathematics, teaching, and learning. Although I state the steps generally so that they might be applied to many examples of artifacts, I modify each step to fit the particular written student work or video. Figure 8 shows the sequence, as applied to the first problem solved by Javier (see Figure 5). Although I present six steps, I do not always use all six. However, I strongly recommend including Steps 1 and 2 because PSTs who think about a problem, both from their own perspectives and from a student's perspective, before they see the problem solved in the video clip are poised to attend to details of the video that they would otherwise miss. When PSTs initially solve the problem two ways, they generally first select a procedural approach, but then they often consider a more conceptual approach for their second solution. Instructors will modify Step 6 to match the goals they hold for sharing student work or showing a video, so that, for example, a mathematics instructor might choose to focus more on issues of mathematics

Figure 8.
A sequence for using video with PSTs, applied to the video of Javier.

(1) PSTs solve the problem in two ways:
How many eggs would you have if you had six dozen eggs?

(2) PSTs consider children's thinking:
How might a child solve this problem without using the standard multiplication algorithm?

(3) If the child's written work is available, PSTs are shown the work and are asked to analyze it.

(4) If video is available, PSTs view the video and explain the child's reasoning.

(5) PSTs compare their own solutions with the child's solution.

(6) PSTs consider implications for mathematics, teaching, and learning:
- Describe how this student used the distributive and associative properties.
- What would a teacher need to know to understand this student's thinking?
- Describe the interviewer's role in revealing the student's thinking?
- What problem would you pose next to this child? Why would you choose that problem?
- What do we as teachers do when faced with a child who is more mathematically creative and innovative than we are?

content and less on issues of pedagogy than a mathematics methods instructor would.

Final Reflections

I have met many mathematics instructors committed to providing rich mathematics courses for prospective elementary school teachers. As effective as these courses are for enhancing the mathematical content knowledge of prospective elementary school teachers, I believe that they can be even more powerful if instructors help their PSTs see how the mathematics content of the course applies directly to the world of teaching. In this article, I present one model for motivating PSTs to learn the mathematics. I share four principles that emerge when PSTs engage with children's mathematical thinking, and I present examples of how these principles support PSTs' motivation for learning.

Now I highlight an issue of professionalism that is dear to me. The work in which my colleagues and I have engaged over many years with children and teachers, visiting their classes, videotaping children, and showing these clips to others, requires a degree of trust on the part of those agreeing to be videotaped. I ask my students to view the video clips *respectfully,* explaining what I mean: When viewing a child who seems confused, please do not focus blame on the child or, for that matter, on his or her teacher. Rather, try to understand what sense the child is, or is not, making, and try to understand the circumstances that may have led to this child's becoming confused. Respectful viewing is important not only for those who have agreed to be videotaped but for all teachers and students. If my students begin to investigate what underlies Ally's confusion and understand where her conceptions are correctly applied, they can not only better understand the teaching/learning process but also become more open to facing, and even accepting, their own mathematical histories. These students are more willing to share their confusions with their peers, thereby creating a learning community in which we may support one another and grow together.

I end with a fifth principle: We best help a learner by starting where he or she is and building upon his or her current understanding. This fifth principle applies not only to elementary school children but also to prospective elementary school teachers, to experienced teachers, and even to university faculty who teach courses for PSTs. If the reader wishes to see my syllabus or other materials I use when teaching a children's mathematical thinking course, please contact me at RPhilipp@mail.sdsu.edu and I will e-mail you some materials.

Notes

[1] Preparation of this article was supported by a grant from the National Science Foundation (ESI-0455785). The views expressed are those of the author and do not necessarily reflect the views of NSF.

[2] My colleagues and I developed the *IMAP Select Videos of Children's Reasoning* CD (Philipp et al., 2005) for use in our courses. The CD, which contains 25 video clips of elementary school children engaged in mathematical thinking, runs on PC and Mac platforms and comes with an interface that includes the transcript (full and synchronized) and background information for each clip. Also included on the CD is a video guide containing questions for students to consider before and after viewing each video clip, interviews that teachers or prospective teachers can use when working with children, and other resources.

[3] The language "three out of four" is associated with another difficulty for students. Students think of "3 out of 4" as literally meaning that 3 parts of the 4 have been removed, and these students conclude, for example, that 1/7 is greater than 2/7 because they are removing only 1 of 7 from 1/7 whereas they are removing 2 of 7 from 2/7. Perhaps this is the reasoning that Ally applied when concluding, although she "wasn't quite sure," that 1/7 was greater than 2/7. Note that given the "out of" conception, this reasoning, though mathematically incorrect, is understandable and consistent.

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