A literature review of pedagogical research on mathematical induction

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Introduction

Many students experience considerable difficulties when they learn and then attempt to construct and communicate proofs of conjectures using the principle of mathematical induction (e.g., Avital & Libeskind, 1978; Blumfiel, 1974; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Ernest, 1984; Ron & Dreyfus, 2004). Although research on the pedagogy of mathematical induction has gained only occasional attention since the 1970s (e.g., Avital & Libeskind, 1978; Blumfiel, 1974; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Ernest, 1984), there has been an increasing interest in this field of study in the past 10 years (e.g., Baker, 1996; Harel, 2001; Ron & Dreyfus, 2004). In order to facilitate pedagogical research on the principle of mathematical induction and its place in the curriculum, Avital and Libeskind (1978) organised the difficulties with mathematical induction into three categories: technical, mathematical and conceptual. Accordingly, this literature review uses these categories to discuss the pedagogical issues with respect to mathematical induction.

Mathematical induction is a formal procedure designed to prove propositions "under special circumstances" (Simpson, Posetti & Saccoia, 2003, p. 246). These circumstances normally involve the establishment of structures and patterns through the use of the set of natural numbers {1, 2, 3, …}. There are two steps in proofs by mathematical induction:

1. *Basis step:* show that $P(m)$ is true whereby P is the identity to be proven and m is the first natural number for which the identity is true; and,
2. *Induction step:* supposing that $P(k)$ is true for any $k \geq m$, show that $P(k+1)$ is true.

Although mathematical induction is often used to prove identities of finite sums, in-equations, divisibility, and Fibonacci's sequence, most textbook exercises on mathematical induction require the student to prove the summation of a finite series (Ernest, 1984), that is, to show that

$$\sum_{r=m}^{n} P(r) = Q(n)$$

where m is the first natural number for which the identity is true. For this reason, let us demonstrate a proof by mathematical induction by showing that

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

for all integers $n \geq 1$.

1. Basis step

Let $n = 1$.

Then, the left-hand side $LHS = \sum_{r=1}^{1} r = \sum 1 = 1$

and the right-hand side $RHS = \frac{1}{2}(1+1) = 1$

Since $LHS = RHS$, we have shown that

$$\sum_{r=1}^{1} P(r) = Q(1)$$

2. Induction step

Suppose that $P(k)$ is true for any $k \geq m$. Then,

$$\sum_{r=1}^{k} r = \frac{k}{2}(k+1) \text{ for all integers } k \geq m. \qquad (1)$$

Given this assumption, we now want to show that $P(k + 1)$ is true for all integers $k \geq m$. That is, we want to show that

$$\begin{aligned}\sum_{r=1}^{k+1} r &= \frac{k+1}{2}((k+1)+1) \\ &= \frac{k+1}{2}(k+2) \\ &= \frac{(k+1)(k+2)}{2} \qquad (2) \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{k^2}{2} + \frac{3k}{2} + 1\end{aligned}$$

Since $\sum_{r=1}^{k+1} r = (k+1) + \sum_{r=1}^{k} r$, we use (1) to obtain:

$$\begin{aligned}\sum_{r=1}^{k+1} r &= (k+1) + \frac{k}{2}(k+1) \\ &= k+1+\frac{k^2}{2}+\frac{k}{2} \qquad (3) \\ &= \frac{k^2}{2}+\frac{3k}{2}+1\end{aligned}$$

Now that (2) = (3), we have satisfied the induction step of this proof.

Therefore, we can now conclude that $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$ for all integers $n \geq 1$.

Technical difficulties with mathematical induction

The technical problems with mathematical induction encompass students who are unable to work through the steps required to develop the proof. In a study of both high school and undergraduate university students, Baker (1996) claimed that an "insufficient formal mathematics background" (p. 16) and "a lack of mathematical content knowledge" (p. 15) are the fundamental factors that contribute most significantly to students' inability to construct a proof by mathematical induction. He recognised that some students do not know how to use the summation notation correctly and are unable to realise basic algebraic arguments, such as $2^{k+1} = 2 \times 2^k$. Avital and Libeskind (1978) also commented more generally on the problems that students experienced in the algebraic manipulations of the induction step of proof by mathematical education. Likewise, Ernest (1984) noted that substituting $k + 1$ for n in the induction step is a problem for many students. For example, consider the induction step of the proof by mathematical induction that

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

After establishing the supposition that

$$\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$$

for any $k \geq 1$, we must now show that

$$\sum_{r=1}^{k+1} r^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

The placement of brackets around each $k + 1$ is required in the first and third factors (and provided in the second factor for consistency and emphasis only). Without the brackets, students will overlook the requirement to use the distributive property of algebra to multiply $k + 1$ properly. As many of these problems are attributed to the level of mathematical competency of the student, they do not provide much insight into the cognitive difficulties of the principle of mathematical induction in itself.

Mathematical difficulties with mathematical induction

Unlike the technical difficulties, the mathematical difficulties with mathematical induction are not attributed to an inadequate ability to work with mathematical expressions. Instead, students who make mathematical mistakes are usually sufficiently competent mathematically but have made incorrect interpretations in the mathematical application of the principle of mathematical induction. Usually, mathematical difficulties occur in one of the two steps of mathematical induction. An example of a mathematical mistake in the basis step was provided by Avital and Libeskind (1978). Since $n^2 < 2^n$ is true for $n = 1$, but not for $n = 2$, 3 and 4, it is possible for the student to establish the basis step for all $n \geq 1$ and then complete the induction step correctly without recognising that the basis step should have been established for all $n \geq 5$. There are other such examples in which failure does not occur until a very large value of n is reached. For instance, let us consider the conjecture that $\sqrt{1141n^2 + 1}$ is not an integer for all $n \geq 1$. Mathematical induction can be used to show that this conjecture is true for all $n \geq 1$, even though the conjecture is false when $n = 30\ 693\ 385\ 322\ 765\ 657\ 197\ 397\ 208$ (Davis, 1981).

The second area of mathematical difficulties in mathematical induction comprises the induction step itself. When asked to prove a statement true for a subset of the natural numbers, students do not always recognise that the induction step is not to prove that $P(k + 1)$ is true. For example, if working with even numbers, then the induction step requires students to instead prove that $P(k + 2)$ is true, after assuming that $P(k)$ is true, of course (Avital & Libeskind, 1978). This common oversight contributes to the mathematical difficulties of the principle of mathematical induction.

Conceptual difficulties with mathematical induction

Conceptual difficulties with mathematical induction cover a range of factors. First, approximately 70% of students do not thoroughly understand the difference between deductive and inductive reasoning (Williams, 1979), which is manifested by a general "lack of understanding" that prevents students from recognising mathematical induction as a valid form of proof (Ernest, 1984, p. 183). For many students, induction means, "Take an equation involving n and add something to both sides so as to produce a similar equation with $n + 1$ in place of n" (Woodall, 1981, p. 100). As a result, "mathematical induction turns out to be a mechanical procedure triggered by the statement, 'Prove that for all n…' and it is only a mechanical procedure" (Lowenthal & Eisenberg, 1992, p. 238). For this reason, students perceive mathematical induction as a "technique of drawing a general conclusion from a number of individual cases" (Harel, 2001, p. 11) thereby demonstrating only an ability to follow an example and not a greater understanding of the inherent relationships involved.

Another conceptual difficulty with mathematical induction is that many students believe that the basis step is "not really essential" (Ernest, 1984, p. 182) and will often omit it from their proof altogether (Dubinsky, 1989). They do not know about or are otherwise not exposed to examples in which the induction step can be proven when the basis step fails. With regards to the induction step in which $P(k + 1)$ is to be proven true under the assumption that $P(k)$ is true, students often exhibit a degree of "uneasiness" (Avital &

Libeskind, 1978, p. 430) with this *modus ponens* argument form of reasoning (also, Baker, 1996). This argument form has two premises. The first premise is that, if *P*, then *Q*. The second premise is that *P* is true. Based on these two premises, therefore, it can be logically concluded that *Q* must also be true. One high school student, in particular, expressed his disapproval for this form of reasoning by saying, "When I do [mathematical] induction, I don't believe it's true" (Harel, 2001, p. 10). Therefore, a common mistake in writing a proof by mathematical induction is in the creation of the hypothesis of the induction step (Dubinsky, 1989). Similarly, "the problem of induction may be formulated as the problem of how to make inferences form observed to unobserved cases, especially to future cases" (Vetter, 1969, p. 56). Baker (1996) has already reported that a conceptual understanding of mathematical induction has been a topic largely neglected by previous research. In fact, his study concluded that none of his subject high school students and only 23% of his subject university students were able to demonstrate a conceptual understanding of the principle of mathematical induction.

Conclusion

Although there have been some attempts to ascertain the pedagogical difficulties of proof by mathematical induction, much of the published research has only confirmed general observations. Students who do not have a strong technical mathematical background are likely to have problems with mathematical induction, because they will routinely make technical and mathematical errors that contribute to their failure to construct a proof by mathematical induction. However, once the technical and mathematical difficulties are removed, are students really developing the conceptual knowledge required to fully comprehend the principle of mathematical induction? Or, are these students merely following a prescribed script for writing proofs by mathematical induction without actually understanding what they are doing? These are questions for further research.

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