A historical perspective on teaching and learning calculus

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Introduction

Calculus is one of those topics in mathematics where the algorithmic manipulation of symbols is easier than understanding the underlying concepts. Around 1680 Leibniz invented a symbol system for calculus that codifies and simplifies the essential elements of reasoning. The calculus of Leibniz brings within the reach of an ordinary student problems that once required the ingenuity of an Archimedes or a Newton. One can mechanically 'ride' the syntax of the notation without needing to think through the semantics (Edwards, 1979; Kaput, 1994). Calculus education typically has a strong routine aspect, focusing on methods for differentiation and integration without justifying these methods, since current teaching practice barely has time to discuss the underlying concepts.

A question for the design of a teaching trajectory that focuses on ways to support the understanding of the underlying concepts is: How can students invent this? It is useful to look at the history of a topic to gain insight into this issue, to investigate concept development, and to analyse how and why people tried to organise certain phenomena without having any notion yet about the basic principles of calculus (Freudenthal, 1991).

In addition to conceptual arguments, history can support the didactical repertoire of the teacher and illuminate the nature of mathematics as a developing discipline and the role of mathematics and mathematicians in society (Gulikers & Blom, 2001). A look at the source of algorithms and notations can help teachers and students to evaluate standards, to step away for a while from exercising to thinking and speaking about what they are doing (Van Maanen, 1997). We will return to this claim in our final section.

We will first review some highlights in the history of calculus. This review will lead into recommendations for an instructional sequence on calculus. We conclude with a plea for historical reflections in mathematics education as a method for changing routine-oriented practices.

A historical review

We will focus this historical review on the period from Aristotle up to the first calculus textbook by l'Hôpital (1696), during which time the basic concepts and models for calculus were shaped. The description covers a period of 2000 years. This could give an impression of a development by fits and starts, but one should realise that it was a long and gradual process, in which the breakthroughs can be localised in the work of a few, brilliant scientists.

Free fall

Questions about falling objects were essential for the development of calculus, and one could say that it emerged from modeling forced motion and free fall. Aristotle (c. 350 BC) was one of the first who formulated laws on motion. He based his laws on everyday experiences and common sense reasoning. Whether and with what (constant) speed an object falls to earth, or floats, depends on its properties.

Aristotle's ideas remained almost unchanged until the late Middle Ages. In the thirteenth century, scholars were convinced that a falling object increased its speed. They tried to improve on Aristotle's theory and developed the impetus theory of motion. It is in the nature of an object to have a propensity, or impetus, to move towards its natural place. The striving towards its natural place gives an object an impetus that determines its velocity in the first time-interval of free fall. After that moment, the object has both an impetus (its striving towards its natural place) and a velocity. This causes an increase in the object's velocity in the second time interval, etcetera; which explains the increasing velocity of a falling object.

Until Galileo's time, the impetus theory could be recognised in explanations of the trajectory of an object thrown into the air. The motion of a thrown object decelerates until the impetus, which it received from the throw, has decreased to zero. After that moment, the object's striving for the ground will cause the object to fall to earth vertically with an increasing velocity.

Modelling velocity

Scientists started using variables and proportionalities in the fourteenth century. This was the time of the so-called *Calculatores* at Merton College (Oxford, UK). They theorised about methods for describing changing qualities like temperature, size, and even a human quality like charity. Thomas Bradwardine, for example, tried to describe the velocity of an object when the proportion between a 'force' f that causes motion and the resistance r is changing. He based his description on the theory of proportions, which states that the addition of proportions equals the multiplication of the corresponding fractions (e.g., when proportion $a : b$ equals 1 : 2 and $b : c$ equals 1 : 4, then $a : c$ equals 1 : 8), and the multiplication of a proportion by a parameter n equals the corresponding fraction to the power n (three times the proportion 1 : 3 equals a proportion of 1 : 27).

The Calculatores tried to find mathematical laws in nature. Hence, their view on the role of mathematics differed from that of Aristotle, and it shows what kind of difficulties had to be solved in order to describe phenomena in a mathematical language. These difficulties not only originated from problematic physical assumptions, but also from limitations in the mathematical language available. The Calculatores did not describe velocity as a proportion of distance to time, because then they would have had a fraction of two different types of quantities. They still followed the Euclidean tradition and worked only with proportions between variables with similar dimension. Three important results were achieved at Merton College:

1. A definition of the notion of instantaneous velocity. The velocity at a certain moment in time can be described by the distance that would be travelled if the object would move on with that very velocity, unchanging during a certain time interval. This is a circular definition, because when you ask what that very velocity is, you can only say “the velocity at the fixed moment” which is still to be defined. However, it should be noted that the idea of a potential distance travelled in a certain time interval represents the instantaneous velocity of the object. This interpretation can also be seen in current everyday language. Driving at a speed of 70 is interpreted as: if you were to continue at this very speed for one hour, you would have travelled 70 miles.
2. A description of the notion of a uniform accelerating velocity: the velocity increases by equal parts in equal time intervals (and not in equal distances travelled!).
3. The Merton rule: if the velocity of an object is uniform accelerating from zero to a velocity v in a time interval t, then the distance travelled is equal to half the distance travelled by an object that moves with constant velocity $\frac{1}{2}v$ in time interval t.

Figure 1. Drawings from Oresme’s “De configurationibus qualitatum.”

Graphing change

It was Nichole Oresme (c. 1360) who invented a new element in these arithmetical descriptions. He introduced the graphic representation. Oresme worked at the University of Paris, and also studied changing qualities. He was not primarily interested in what actually happens, but in how one could generally describe what happens. For instance, he described ways to display the distribution of heat in a beam (see Figure 1): think of a line along the beam and imagine at every point of this line the heat at that position in the beam represented by a line perpendicular to the beam. The length of this second line displays the heat at that position in the beam. These perpendicular lines constitute a geometrically flat shape. This shape signifies the distribution of the heat and its area is a measure for the total heat in the beam. A constant temperature is displayed by a rectangular shape, while a uniform change from low to high is displayed by a triangular shape (or a trapezoid). Oresme reasoned with and compared changes in qualities using such geometrical shapes.

Oresme also applied this graphical technique to motion. His remarkable way of thinking can be seen by the way in which he defined velocity as a quality of objects that can be pictured against time (the dimension over which the velocity of the object varies). Thanks to this choice, the area of the geometrical shapes had many similarities with the current velocity–time graphs. The perpendicular lines represent instantaneous velocities and the area of the shape can be interpreted as total distance travelled. With this graphical reasoning Oresme proved the Merton rule. These discrete graphs simplify, conceptualise and illustrate reasoning about motion.

Some mathematicians argue that this proof of the Merton rule is not valid since instantaneous velocity had to be defined as a differential quotient and only then distance travelled can be deduced by integration. The Dutch historian Dijksterhuis discussed this and defended Oresme by stating:

"It is a situation which occurred regularly in the history of mathematics: mathematical concepts are often — maybe even usually — used intuitively for a long time before they can be described accurately, and fundamental theorems are understood intuitively before they are proven." (Dijksterhuis, 1980, p. 218).

Free fall revisited: Galileo

Until the sixteenth century, it was commonly accepted that the time needed for an object to fall to the ground was inverse proportional to its weight. This was still a legacy of Aristotle's theory. In 1586, Simon Stevin published his Beghinselen der Weeghconst (Principles of weighing). Stevin opposed this theory and described an experiment with two falling lead balls of different weight that touched the ground at exactly the same time. In this experiment he tested Aristotle's assertion and concluded that it was contrary to this experience.

In 1618, Isaac Beeckman proved a new relation between elapsed time and falling distance that is independent of the weight of the object. He approximated a continuous force that pulled the object as if with little tugs. After each time interval *t*, such a tug increased the velocity by a constant amount *g*. This process was visualised by the graph below (Figure 2), in which the distance travelled in a time interval *t* is represented by the area of the corresponding bar.

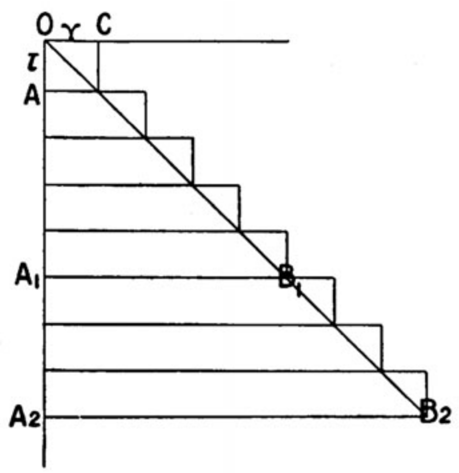

Figure 2. Graph showing Beeckman's reasoning with areas of bars.

When the length of the time interval t approaches zero, the distances travelled in total times $0A_1$ and $0A_2$ are represented by the areas of the triangles $0A_1B_1$ and $0A_2B_2$. These distances are proportional to each other as the squares of the time intervals $0A_1$ and $0A_2$. He still used this reasoning in proportionalities between similar quantities. We compress the relation between time and distance travelled in one equation: $s(t) = c \cdot t^2$.

Beeckman used a discrete approximation of the area. Such approximations were related to Archimedes' methods to determine centers of gravity (c. 200 BC). This is no surprise; since Archimedes' collected works were translated into Latin and published in 1544, Stevin, Kepler and Descartes also used his methods in their publications. Interest in the work of Archimedes was the result of the Renaissance and a rising prominence for mathematical methods.

Galileo (1564–1642) is one of the most famous scientists to work on free fall. In his *Dialogue Concerning Two New Sciences* he wrote about the Aristotelian view on this topic and why this view must be wrong. His argumentation is tied up in dialogues between Simplicio (representing the Aristotelian view on motion) and Salviati along the question what would happen with the falling velocity of two objects of different weight when they are put together.

Galileo used graphs to explain the quadratic relationship between distance travelled and falling time. Galileo tested this relationship with experiments. He knew that sequences of successive odd numbers, starting with 1, add up to a square, and he used ratios of odd numbers between the distances travelled in equal time intervals. This ratio must be 1 : 3 if you divide time into two equal intervals. If you divide the time into four intervals the ratio is 1 : 3 : 5 : 7, etc. With this property he tested the equation that is based on the conjecture that the acceleration of a free falling object is constant. An important step which Galileo made was to reason that the motion of free fall is similar to (in terms of proportions), and can be delayed by, an object rolling down an incline. He probably designed a slide with nails on one side. The distances between the nails were in the same ratio as the successive odd numbers, thus a rolling ball should need the same amount of time to pass each following nail.

Many scientists commented on Galileo's reasoning. For instance, Fermat (1601–1665) believed that an object must have a velocity at the moment of falling, otherwise it would not start moving. This is yet another example which illustrates that their ways of thinking about velocities of falling objects and about instantaneous change were not trivial. It shows that even famous mathematicians during the time of Galileo found the idea that, at the moment of starting to fall, the object could have constant acceleration while its instantaneous velocity should be zero, problematic.

Leibniz and Newton

Two scientists, Leibniz and Newton, were crucial in the development of calculus; they discovered and proved the main theorems of calculus. In the seventeenth century, methods were discovered for calculating maximums and minimums in optimisation problems. These methods mainly concerned polynomials. However, many problems, such as the breaking of light, could not be described with polynomials. The conceptual understanding of the mathematics of instantaneous change developed. How to calculate change became a topic of interest, and Leibniz's and Newton's contributions concerned precisely this issue. Their invention of a literal symbolism was essential for the rapid progress of analytic geometry and calculus in the following centuries. It permitted the concept of change to enter algebraic thought.

The language of Newton was closely related to motions of geometrical entities in a system of coordinates. The y-coordinate denotes the velocity of a changing entity (e.g., an area or a length) and the x-coordinate denotes time. Such a geometrical approach fitted the research tradition in the seventeenth century and might have supported his findings (Thompson, 1994). The embedding of motion and time in geometry is one of the most characteristic features of Newton's dynamical techniques.

Newton used the context of motion to give intuitive insight into the limit process of the proportion between two quantities that tend to zero. He argued that the ultimate proportion of two vanishing quantities should be understood as the velocity of an object at the ultimate instant when it arrives at a certain position. The two quantities are position and time, and he defined the limit or vanishing proportion between change of position and change of time as the instantaneous velocity.

The roots of Leibniz's work were in algebraic patterns in sums and differences and their properties. In 1672, before he formulated the fundamental theorem of calculus, he published on properties of sequences of sums and differences of sums.

Leibniz noticed that with a sequence a_0, a_1, a_2, … and with a sequence of differences $d_1 = a_1 - a_0$, $d_2 = a_2 - a_1$, … $d_n = a_n - a_{n-1}$, he could conclude $d_1 + d_2 + \ldots + d_n = a_n - a_0$. Therefore, the sum of the consecutive differences equals the difference of the first and the last term of the original sequence. According to Edwards (1979), Leibniz refers to this inverse relation between the sequences a_n and d_n in his later work as his inspiration for calculus.

Leibniz introduced a more accessible symbol system for calculus than Newton, a system which we still use today. Leibniz did not write much about the limit concept as a foundation for his symbol system. He illustrated his method in his first article *Nova Methodus* on calculus in 1684 with a graph of an equation with no related context. Leibniz did not define "infinitely small." He interpreted a tangent as a line through two points on a curve that lie at a distance to each other which is smaller than every possible length. Leibniz did not publish this definition in the article, because he thought this to be too revolutionary. He only published the rules to do the calculus and some convincing applications.

L'Hôpital

Marquis de l'Hôpital (1661–1704) was a French mathematician. He is perhaps best known for the rule (which still bears his name) for calculating the limiting value of a fraction whose numerator and denominator both tend to zero. In addition, he is the author of the first textbook on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). The text includes lectures by his teacher, Johann Bernoulli. In 1694 l'Hôpital forged a deal with Bernoulli to give away his discoveries for the book.

L'Hôpital's purpose with his textbook was to convince mathematicians and physicists that differential calculus, which was not yet fully accepted, was a sound and powerful method. The activities in the book show a wide variety in complexity and applications. He translated the problem of the breaking of light into a question about the quickest route of a traveler passing two different landscapes that require different velocities, and showed how this could be solved easily with the new method. Another application concerns the position of a pulley when the weight D is at rest (see Figure 3).

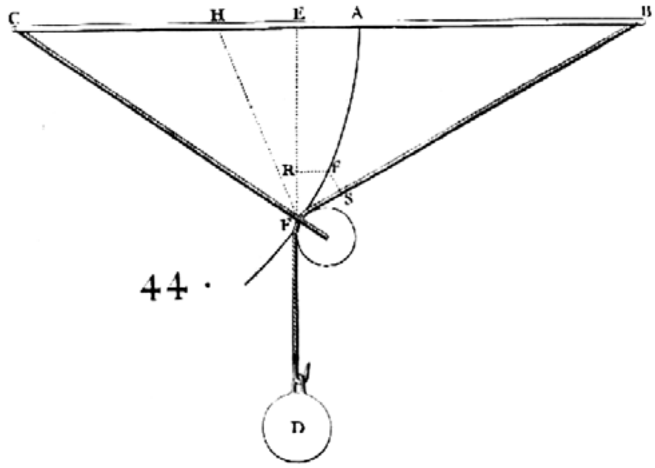

Figure 3. The weight problem in l'Hôpital's Analyse.

We will return to this problem in the next section. Now we would like to finish this historical sketch with a few final remarks. The period before l'Hôpital described a development of calculus that took place over circa two thousand years. It is remarkable that current educational practices hardly reveal anything of this struggle for mathematising change and motion. The methods of Leibniz are taught as an obvious, or natural, way to treat change mathematically.

Looking at history through a didactical lens

In history we do not see the regular calculus textbook approach from limits to differential quotient, from methods for differentiation to methods for integration, and finally to the main theorem of calculus. Instead, the definition of differential quotient as a basic idea underlying calculus is one of the final findings in the history of calculus. This inversion of history for displaying a mathematical topic in a textbook is precisely what Freudenthal calls an "anti-didactic inversion" (Freudenthal, 1991). Such an inversion has its origins in the expert's point of view on elegance and efficiency. The reader can imagine how these inversions might have anti-didactic consequences.

Oresme's intention was to describe and value changing qualities, one of which was velocity, in order to be able to compare them. He used graphs for displaying and reasoning about changing qualities. He did not define velocity as a compound quantity, nor did he use scales along his two-dimensional graphs. Nevertheless he interpreted areas as distances travelled and used geometrical shapes to compare different kinds of change.

The graphical method made it possible to illustrate the middle-speed theorem (Merton's rule) and to investigate the relation between change of velocity and distance travelled. The method was successful thanks to Oresme's choice to draw a graph with a horizontal time axis. Possibly, his choice was influenced by his trying to depict potential distances travelled.

In this history we recognise a dialectic process of the development of meaning, notations and graphical representations; a process from two-dimensional discrete graphs for describing motion to reasoning about slope and area. Discrete graphs might provide students with meaningful graphical tools that afford them both a way to reason with area and slope, and to invent relations between them before the formalisation with limits.

The methods of Leibniz opened up the possibility of symbol manipulation without examining these symbols and understanding their meaning. This symbolic writing seems to replace conceptual thinking by substituting calculation for reasoning, the sign for the thing signified. However, we note that Leibniz's symbol manipulations were built upon extensive experience with numerical patterns in sums and differences. We assume that his experience underpinned a meaningful use of these manipulations.

History suggests — as Dijksterhuis noticed — that an intuitive understanding of reasoning with discrete graphs, with sums and differences and with area and slope precedes the formal methods of differentiation and integration (Dijksterhuis, 1980). A process of reification can be recognised (Sfard, 1991). However, it is not the graph but rather the activity of summing and taking differences that is reified into the mathematical objects of integral and derivative. Discrete graphs support both the modeling activity and the reification.

Suggestions for instructional design

It might be a natural step to use discrete approximations and to take that as a starting point for reasoning about sums and differences. We assume that, in this reasoning, the use and understanding of graphical characteristics might emerge. In the process of trying to get a handle on change, the method of approximating a constantly changing velocity with the help of graphs plays a key role. These ideas can be exploited in an instructional design by starting a learning sequence with patterns in discrete graphs (Doorman, 2005; Gravemeijer & Doorman, 1999).

In the context of motion, students can have the opportunity to contribute to the idea that successive displacements are a measure for change (velocity) and that they add up to the total distance travelled. According to the discrete inspirations of Leibniz, students can study the relationship between sums and differences with graphs.

Let graph S be given (Figure 4) as a graph of distance travelled. Define displacement

$$D(k) = S(k+1) - S(k).$$

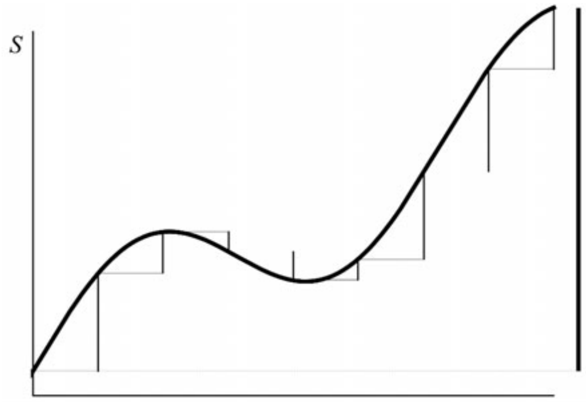

Figure 4. Total distance travelled as a sum of displacements.

From the graph it can be seen that the sum of all displacements

$$D(0) + D(1) + \ldots + D(n-1) = S(n) - S(0).$$

In general the discrete case of the main theorem of calculus is:

$$\Delta S(k) = D(k) \text{ and } \sum D(k) = S(n) - \mathrm{S}(0).$$

Students can use graphic calculators when applying this property to functions. From graphs and tables (or algebra) it is possible to deduce that

$$\Delta 3^k = 3^{k+1} - 3^k = 2 \cdot 3^k$$

and, with the property it can be proved that

$$\sum 3^k = \tfrac{1}{2} \cdot (3^n - 3^0).$$

These activities of taking and relating sums and differences for grasping change prepare students for the relation between integral and derivative.

In addition, the scenery of a lesson might take place in Galileo's time and the search for a relation between time and falling distance (e.g., Polya, 1963). In this context, the problem of how to visualise the motion of an object that moves with varying speed can be posed. While struggling with this problem, the students (or the teacher) may come up with the idea of symbolising instantaneous velocities with bars.

Next, the students are told the story of Galileo, who assumed that the motion of a free-falling object was with a uniformly accelerating velocity. They are asked to graph the discrete approximation of such a motion (see Figure 5). Subsequently, Galileo's problem is posed: What distance is covered by the object? Here, after solving and improving discrete approximations, the students are expected to make the connection between the area of the bar-graph, and the area of the triangle that is created by the continuous graph:

$$s(t) = t \cdot \frac{10t}{2} = 5 \cdot t^2$$

This resulting equation reveals the quadratic relation between time and distance that Galileo used to test his hypotheses empirically.

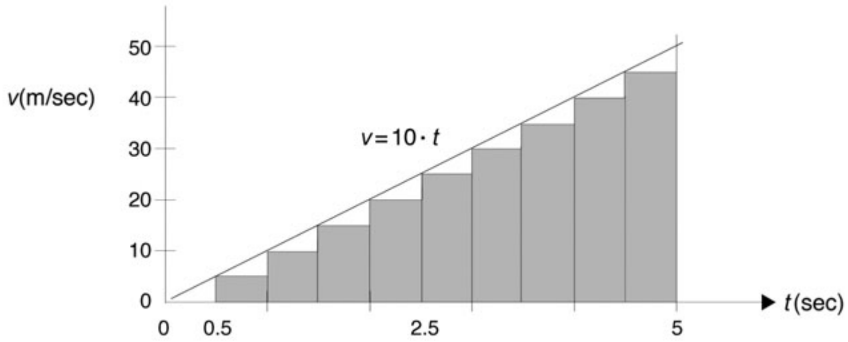

Figure 5. The discrete approximation of free fall.

Our final suggestion is to involve students in the deal between l'Hôpital and Bernoulli and use the "weight problem" from l'Hôpital's textbook (Van Maanen, 1991). This problem can be a starting point for discussing the power of methods in calculus. Equipment needed can be borrowed from physics colleagues. The instrument is displayed in Figure 3. *CB* is a horizontal bar. A rope *CF* of length *a* tied to *C* carries a pulley at *F*. A rope of length *b* attached at one end to *B* passes over the pulley and carries a mass at its free end *D*. One supposes that the pulley and the ropes do not have mass. L'Hôpital took $a = 0.4$ and $b = 1$. The problem is to determine the equilibrium position of the weight *D*. The analytic solution uses the principle that the weight will search for the lowest possible position. The only thing you have to do is to introduce a coordinate system, to choose an independent variable *x* that fixes the position of the pulley, to express the distance of the weight to *CB* in *x*, and to maximise this distance using differentiation.

After modeling the problem situation and determining *x* in such a way that the problem can be translated into an optimisation problem, the solution process is rather straightforward and does not depend on geometrical tricks. This is precisely the strength of the general and algorithmic methods in calculus. What makes this weight-problem interesting is not just the mathematics, but also the story behind it: a means of fighting for the acceptance of calculus. It shows how attention shifts from teaching various geometrical solution procedures to modeling and heuristics for clever ways to choose *x*.

Conclusions

This historical study provides us with indications of conceptual thresholds, how they were overcome, and how calculus evolved in connection with the physics of motion. Comparison of our present-day mathematics with older methods enables us to value our modern notations and to establish more coherence with physics. History shows us that mathematics is more than a fixed collection of truths, facts and algorithms. Reasoning with discrete graphs as a support for learning calculus might prevent the algorithms becoming disconnected from their roots (Doorman & Gravemeijer, in press).

In addition, history appears to be a source for interesting mathematical problems and suggestions to tackle conceptual difficulties. The problem of the pulley in l'Hôpital's book was part of a propaganda machine; it was a weapon against conventional methods. Such problems — and the motivation behind them — provoke discussions about mathematics, its methods and its value for society.

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