PROPORTION in middle-school mathematics: *It's everywhere*

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Proportional thinking is the mathematical basis of a wide range of topics in the middle-school mathematics curriculum. While the concept is obvious in the traditionally-named ratio and proportion sections, proportional thinking is also the key to such diverse topics as rate, gradient of a linear function, similarity, trigonometry and percentage, to name but a few. The separation of topics and the lack of focus on the underlying structures of concepts are key problems in the presentation of mathematics to students. In his *Principles for the Design of Teaching*, Alan Bell (1993) discussed the idea of "structure and context". This principle highlights the need to help students to see that the same mathematical structures can occur in different contexts. If the structure is recognised in a new context, then solution methods similar to those used previously can be applied to the new context. In this article we firstly outline ways of working in proportional situations and then demonstrate how similar solution methods for problems based on proportion can be adopted in a range of contexts. We argue that "proportion" is not really a topic in its own right, but rather the concept underlying a wide range of topics.

Proportional thinking

Susan Lamon (2006a, 2006b) has provided a comprehensive exposition on the learning and teaching of fractions, ratios and proportional thinking based on her own research and that of others over many years. The discussion that follows builds on that work.

Consider a simple ratio problem.

> A cordial drink is made up of a mixture of 1 part cordial to 5 parts water. How much water is needed to be mixed with 40 mL of cordial to make a drink of the required strength?

In many parts of the world students have been taught to solve such problems by writing and solving a proportion equation.

$$\frac{1}{5} = \frac{40}{x}$$

Students are often taught to cross-multiply and solve the equation. The idea of proportion is usually not part of their thinking. These problems, sometimes named 'missing value' problems, all involve three known values and an unknown fourth value.

A problem of this type can be thought of as involving two measure fields or variables, in this case millilitres of cordial and millilitres of water. The problem situation may be represented in a proportion table.

M_1 cordial (mL)	**M_2** water (mL)
1	5
40	x

There are two ways of thinking of the proportional relationship between the two variables. Use of the 'within' strategy involves thinking that in the first field the quantity of cordial is multiplied by 40. Therefore, in the second field the quantity of water must also be multiplied by 40 to maintain the relationship. The thinking is multiplicative (× 40) rather than additive (+ 39). We have 40 times the base amount of cordial, so we need 40 times the base amount of water to go with it.

Within

	M_1 cordial (mL)	**M_2** water (mL)	
	1	5	
× 40			× 40
	40	x	

The 'between' strategy involves looking across the two fields and thinking of the multiplicative change that links the quantities in the two fields. Because we always need 5 times as much water as cordial, we multiply the new quantity of cordial by 5 to find the required amount of water.

Between

M_1 cordial (mL)		**M_2** water (mL)
	× 5	
1		5
40		x
	× 5	

Some problems need to be solved using the operation of division, the inverse of multiplication, as in this example.

> A bulk pack of sweets contains 1.5 kg and costs $3.60. If the sweets are repackaged into 250 g packs, what price should each pack be to break even?

The problem has two measure fields: mass and price. We can find the factor in the mass field by dividing 1500 by 250 (within thinking).

	M_1 mass (g)	**M_2** price ($)	
× 6	250	x	× 6
	1500	$3.60	

To solve the problem, the inverse operation has to be used. Students may be familiar with this idea from 'backtracking' number sentences and equations.

	M_1 mass (g)	**M_2** price ($)	
× 6	250	x	÷ 6
	1500	$3.60	

So the cost of 250 g should be $3.60 ÷ 6 = $0.60. The solution of this problem relies on the student's thorough understanding of the operation of multiplication and its inverse, division.

Diverse problems

'Rate' problems appear in many forms in middle-years mathematics, examples being speed, price per kilogram and flow rates. There has been a tendency to approach such problems with a formula and to rely on students successfully manipulating the formula. However, constant rate problems can be thought of as proportions and solved that way, at least initially. Consider the following example.

Mince is priced at $6.99 per kilogram. How much can I buy for $10?

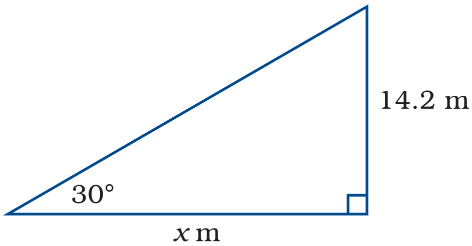
We can think of two fields, mass (kg) and cost ($).

M_1 mass (kg)	**M_2** cost ($)
1	6.99
x	10.00

Either strategy could be used to solve the problem but we would normally think about rate situations using a 'between' strategy as a rate is expressed in a unit combining both fields ($ per kg). In this case, the inverse operation is used to solve the problem.

M_1 mass (kg)	**M_2** cost ($)
× 6.99	
1	6.99
x	10.00
÷ 6.99	

Problems involving the use of trigonometric ratios are closely related to rate problems. Think of finding an unknown side in a right-angled triangle using the tangent ratio.



Saying that tan 30° = 0.57735 means that for every one unit of length along the side adjacent to the 30° angle, there is 0.57735 of a unit of length along the opposite side, a proportional situation. The tan ratio for 30° is 0.57735 : 1 (opposite side : adjacent side). The two measure fields here are the opposite side and the adjacent side and the solution can be found in the same way as in all the previous examples.

$\mathbf{M_1}$ opposite (m)	$\mathbf{M_2}$ adjacent (m)
0.57735	1
14.2	x

Conversion problems, such as those involving currencies, also use proportional thinking. Imagine changing US$100 to AU$ if the current rate is AU$1 = US$0.7943. The difficulty for students is knowing whether to multiply or divide and setting out the problem as a proportion can help. The approach here usually involves 'between' thinking: for every one Australian there are 0.7943 United States dollars. So to come back from US$ to AU$ use the inverse operation.

$\mathbf{M_1}$ AU$	$\mathbf{M_2}$ US$
1	0.7943.99
x	100

× 0.7943

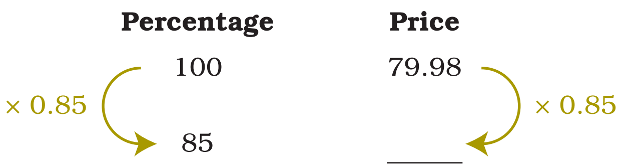
÷ 0.7943

$$100 \div 0.7943 = 125.8970,$$

therefore US$100 = AU$125.90

Note also the use of the word 'rate' in the currency conversion problem. The two variables here are both of the same type (money) and by the usual definitions of rate and ratio this would have been classed as a ratio problem. The use of such labels is discussed in the conclusion of this paper.

Percentage calculations also involve proportional thinking. A percentage decrease problem such as applying a discount of 20% to a marked price of $79.98 to find the sale price can be represented in a table.



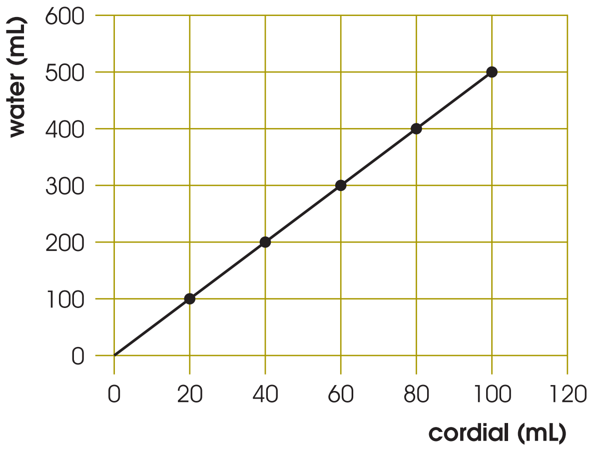
This representation helps learners to think of the association between the original price and 100%. They then think about the factor needed to change 100% to 85%.

More representations

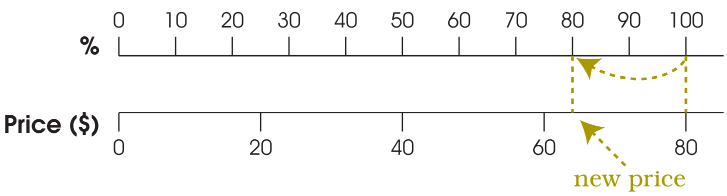
Proportion tables involving the three known pieces of information and the unknown value have been demonstrated. In learning about proportion, students can also represent a situation in a more extensive table for which they calculate a range of values for the variables. For example, for the cordial problem (1 : 5), students could be asked to complete this table.

Cordial (mL)	1	20	40	60		
Water (mL)	5				400	500

The cordial situation can also be represented with a graph. Links can be made between the ideas of proportion and linear function with the observation that the points generated in any proportion situation lie in a straight line. The decision about whether it is legitimate to join the points relates to the idea of continuous variables.



Proportional situations can be represented with parallel number lines. Here is the representation for the problem in which a price is reduced by 20%.



This representation provides students with a visual means of estimating the result of a percentage change and is particularly valuable with a percentage increase.

The proportion equation was mentioned earlier in the article. The indication from research is that the representation of proportion problems with an equation should be delayed because its use tends to stifle proportional thinking. Students going on to more advanced mathematics should learn to solve proportion equations but only after working extensively with proportional thinking.

Conclusion

In this paper we have demonstrated how similar solution methods can be applied to a range of problem types that are all based on proportional thinking. There are many other problems that share this mathematical structure and showing students the structural similarities can help them to develop reliable solution methods. In addition, several representations are available to support learners to solve these problems. Mathematics materials and teaching has traditionally compartmentalised problem types, with different solution methods being used for structurally similar problems. We ask whether it really matters if one type is called a ratio problem and another is called a rate problem, or a proportion problem? What really matters is that students recognise that all these problems involve two related variables and all can be solved in a similar way.

References

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