

David Hough, Ph.D., Editor  
Missouri State University  
Springfield, Missouri

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**Evolution of a Teacher's Problem Solving Instruction:  
A Case Study of Aligning Teaching Practice with Reform in Middle School Mathematics**

Anthony Rickard  
Department of Mathematical Sciences and School of Education  
University of Alaska Fairbanks

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**Abstract**

Problem solving continues to be a major focus of K-12 mathematics education reform. This case study shows how a teacher's middle school mathematics instruction evolved from teaching problem solving as computation exercises to teaching how to select and implement combinations of problem solving strategies. The teacher's thinking about problem solving instruction also evolved to encompass different understandings about what a problem is, what problem solving means, and what teachers' and students' roles are in teaching and learning problem solving. The result of this evolution was that the teacher's instruction became more closely aligned with reform goals for problem solving. This finding suggests that reform efforts should incorporate strategies and support for teachers to revise and refine their own instruction through reflection and dialogue with colleagues as they try out approaches and activities in their classrooms for teaching problem solving.

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**Teaching Mathematics as Problem Solving: More Than 20 Years of Reform**

Problem solving has been a major focus of mathematics education reform for more than 20 years. With the publication of *How to Solve It*, George Polya (1945) argued that problem solving should be a legitimate topic in teaching and learning school mathematics and presented his well-known heuristic—understand the problem, devise a plan, carry out the plan, look back—as a coherent framework for problem solving. However, it was not until the publication of the National Council of Teachers of Mathematics (NCTM) 1980 Yearbook, *Problem Solving in School Mathematics*, that problem solving moved to the forefront of mathematics education reform, with NCTM asserting that problem solving should become "the focus of school mathematics" (NCTM, 1980, p. 1). With the publication and enormous influence of the NCTM *Standards* series, comprised of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991), *Assessment Standards for School Mathematics* (NCTM, 1995), and *Principles and Standards for School Mathematics* (NCTM, 2000), problem solving has become established as an essential reform goal for teaching and learning K-12 mathematics. As NCTM (2000) notes, "Problem solving is central to inquiry and application and should be interwoven throughout the mathematics curriculum to provide a context for learning and applying mathematical ideas" (p. 256).

During the 1980s, however, it became apparent that a weakness in problem solving reform is that problem solving itself is not well defined and may be used to describe or justify a wide array of teaching and learning practices. Branca (1980), for example, argued that problem solving in school mathematics can be viewed as a goal, a process, or as a basic skill, with each interpretation having different implications for teaching and learning problem solving. Another factor contributing to ambiguity about problem solving is that what constitutes a mathematical problem is relative to the solver. For example, finding the area of a rectangle with a length of 12 units and a width of 7 units may be a routine calculation for a student who knows the "length times width" formula for finding the area of a rectangle, but may be a challenging problem for another who does not. In other words, "... a problem for you today may not be one for me today or for you tomorrow" (Kilpatrick, 1985, p. 3). At the end of the 1980s, Stanic and Kilpatrick (1989) observed:

The term *problem solving* has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and of why we should teach mathematics in general and problem solving in particular (p. 1, emphasis in original).

To better support reform efforts, sharper definitions of what problem solving is (or should be), that include what students' problem solving activity in K-12 classrooms should look like, have been developed and disseminated. The National Council of Supervisors of Mathematics (NCSM) offers a representative and generally accepted definition of problem solving:

Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations ... problem solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error (NCSM, 1989, p. 471).

One reason the above definition clarifies what problem solving means is that it helps to distinguish problems from *exercises*. Exercises typically entail practicing a procedure (e.g., an algorithm) or rehearsing specific facts or concepts (e.g., multiplication facts or definitions), to build proficiency and quickly obtain a correct answer. So, for example, finding the answer to a rote multiplication "problem" like  $7.8 \times 2.9$  would be an exercise for a student who has learned the standard multiplication algorithm with decimal numbers. However, using the multiplication algorithm as a tool to explore and express the relationship between the length, width, and area of rectangles could be a challenging problem for the same student (see Rickard, 1996). Advancing reform further, the NCTM *Principles and Standards for School Mathematics* (2000) provides detailed examples and vignettes of teaching and learning problem solving in grades PK-2, 3-5, 6-8, and 9-12 classrooms.

Clarifying what problem solving is and distinguishing between problems and exercises is critical to advance the reform goal of teaching and learning mathematics as problem solving. For example, if a teacher views problems as computation exercises and problem solving as using memorized algorithms to determine the correct answer, then K-12 mathematics reform goals are undermined. In such a classroom, mathematics is typically taught and learned as memorizing definitions and procedures that are disconnected, rarely based on understanding of underlying concepts, and are often not well understood by students or the teacher (Rickard, 1995a). For example, Burns (2000) has documented that middle school students often have thin, incomplete, or erroneous understandings of algorithms (e.g., the algorithm for adding decimal numbers) because they are asked to memorize rules that are frequently not justified (e.g., the decimals must be "lined up" to add decimal numbers correctly) and may not address the concepts that justify the algorithm so that it makes sense (e.g., representing decimals as fractions to see that lining up decimal points ensures adding fractions with common denominators).

Prior research has shown that even teachers who express support for teaching and learning mathematics as problem solving and distinguish, at least to some extent, between problem solving and exercises may still spend a substantial amount of instructional time teaching rote algorithms and having students complete computational exercises (e.g., Rickard, 1998). Moreover, while a growing body of research suggests that students who use mathematics curricula that is aligned with the NCTM *Standards*, including mathematics as problem solving, tend to outperform students using traditional textbooks (e.g., R. Reys, B. Reys, Lapan, Holliday, &

Wasman, 2003; Rivette, Grant, Ludema, & Rickard, 2003), how teachers shape curriculum to meet the needs of their students is informed by their own understanding of mathematics content, problem solving, classroom discourse, and other factors (Lappan & Briars, 1995; Rickard, 1993; Rivette et al., 2003). Given the wide variability of how teachers may understand problems and problem solving, it is important to examine the process of how a teacher might explicitly unpack her or his own understanding and teaching of problem solving. Furthermore, it is critical to investigate how such a teacher might work to align her or his teaching practice with reform goals to inform efforts to support all teachers' implementation of problem solving in K-12 mathematics classrooms.

### **Developing a Case Study: Background and Methodology**

Investigating how a teacher grapples with his own understanding and teaching of problems and problem solving, how his understanding and teaching evolves as measured by reflection, discussion, and observation of his teaching, is the focus of this case study. This line of inquiry, which utilizes case study methods, fits within a substantial body of research on teaching and learning mathematical problem solving, teachers' use of *Standards*-based mathematics curricula, and informs how teachers' understandings about problem solving shape how they teach problem solving and support their students in developing problem solving skills (e.g., Grouws, 1985; Lester, 1994; McCaffrey, Hamilton, Stecher, Klein, Bugliari, & Robyn, 2001; Reid, 2002; Silver, 1985; Theule-Lubienski, 1997; Wilcox, Lanier, Schram, & Lappan, 1992). To investigate these kinds of questions, researchers have used case study methods to examine, for example, how teachers' beliefs shape their instruction as they attempt to implement state-level school mathematics reforms (e.g., Wiemers, 1990; Wilson, 1990, 2003) and how teachers' beliefs interact with innovative mathematics curricula to shape their teaching (e.g., Remillard, 1991; Rickard, 1993, 1995a, 1995b, 1996). Other researchers have compared and analyzed multiple case studies of different mathematics teachers to study how issues connected to problem solving play out in different classrooms (e.g., Putnam, Heaton, Prawat, & Remillard, 1992). Taking a case study approach to investigate a teacher's understandings and teaching of problems and problem solving, therefore, makes sense in light of prior case study research which has explored related issues.

In developing this case, I observed Bob Fern (a pseudonym), a sixth-grade mathematics teacher, over a six-month period. I observed and audio taped four of Bob's lessons each month and also took field notes during his teaching. Later, usually the same day, I would expand my field notes, review the audio tape, and transcribe selected excerpts for analysis. I also audio taped, reviewed, and transcribed portions of informal conversations I periodically had with Bob where we discussed problem solving. I regularly made copies of students' written work—data collected from students are used to further inform my analysis of Bob's teaching and to gauge changes in his understanding and approach to teaching problem solving. For example, the kinds of problems Bob assigned his students over the six-month period (e.g., whether they were exercises or problems) and how he engaged students with problem solving (e.g., questions he asked students, how he responded to students' questions) helped me to make connections between his understanding of problem solving and his classroom instruction.

I drew on all of these data sources (i.e., field notes, transcripts of lessons, conversations, and interviews, samples of students' work) as I developed the final case study. I used the multiple sources of data to triangulate my analysis and develop assertions about how Bob was thinking about problem solving and about changes in his practice in teaching problem solving. For example, when Bob expressed specific ideas about teaching and learning problem solving in interviews or conversations, I looked for evidence in his teaching (e.g., kinds of exercises or problems assigned, how he engaged students with the exercises or problems) and in his students' work (e.g., solutions students developed for exercises or problems, to what extent students revised their solutions, the kind of mathematical reasoning students provided) to understand how and to what extent these ideas were implemented in Bob's practice. I would also probe these ideas and aspects of Bob's practice further in subsequent conversations and interviews. A limitation of this case study is that its findings are not necessarily generalizable to other teachers. However, how a particular teacher's understandings and teaching of problems and problem solving evolve over time can provide insight into issues that may impact the process of how

other teachers' thinking about problems and teaching problem solving is shaped and may change. Such insight can support reform efforts in mathematics education focusing on teaching and learning problem solving.

### A Case Study of Bob Fern

I guess the basic question is, exactly what is problem solving? Is problem solving, by definition, using manipulatives? What exactly are we talking about when we talk about problem solving—  
Comments by Bob Fern, first week of this study.

A teacher with more than 20 years of experience, Bob Fern teaches sixth grade in a large middle school (grades 6-8) situated in an urban area in a midwestern state. Originally built in the 1930s, the school was expanded during the 1960s to accommodate growing numbers of students. Enrollment was about 1,100 during the time I worked with Bob Fern. The student population of Bob's middle school is diverse—approximately 50% of the students are Caucasian, 30% are African American, and 20% come from Asian, Hispanic, and Native American ethnic backgrounds. The predominant socioeconomic range of students' families is low income to blue collar. At the time of this study, about 50% of the student body qualified for the school's free or reduced-cost lunch and breakfast programs. Bob's schedule includes three mathematics classes, one social studies class, and one reading class. Class periods at Bob's school are 50 minutes long. Each of Bob's classes roughly approximates the ethnic and socioeconomic diversity of his school.

#### *Emphasis on Exercises and Computation*

Bob Fern was motivated to participate in this case study because he had not taught mathematics in six years. He told me that he was particularly interested in problem solving and felt that having someone else in his classroom regularly would push him to reexamine and revamp his mathematics teaching. Bob is confident in his own knowledge of mathematics (he completed several mathematics courses in college) but still attended several workshops on teaching middle school mathematics during the previous year to refresh himself for teaching mathematics. During these workshops he encountered different understandings of what problem solving is:

**AR:** So, was it in these workshops that you first began thinking about problem solving?

**Bob:** Well, not really (pause)—I had heard "problem solving" [makes quotes in air with fingers] before and I just assumed that it meant that we should be getting kids to solve problems. But now I'm not so sure—I mean, I've heard people talk about problem solving very differently.

**AR:** You mean, like at the inservices or—

**Bob:** Yeah—other teachers, even the presenters. One guy showed us how to use word problems—you know, just regular story-type problems you solve with a quick computation—and called it problem solving, but another person had us work with manipulatives and build models—she said that typical word problems really weren't "real" [makes quotes in air with fingers] problem solving.

**AR:** Hmm ... Interesting, so different ideas about teaching problem solving? I mean, with different ideas like that what do you think about problem solving in terms of how we should focus on it or teach it?

**Bob:** I guess the focus right now for me is very fuzzy. I have an (pause) innate need to make sure the kids know some of these basic skills—I'm not ready to ditch the computational aspects totally (pause) and yet I need to get away from it [i.e., computation] more than I have—I have to find that happy medium.

Bob's comments show that he is interested in teaching problem solving, believes that computational skills are

important, and that computation might not necessarily be addressed by problem solving. Bob's concern that teaching and learning problem solving may happen at the expense of teaching and learning basic skills is not unusual among teachers (e.g., Burns, 2000; Rickard, 1995a). Bob is struggling with how to teach both problem solving and build students' computational skills as he believes that both are important.

During the first month I observed Bob in his classroom, he taught a lesson about how to find a common denominator of two fractions in order to compare them. This lesson, which began immediately after the bell rang to start the class period, is representative of the lessons I observed Bob teach during the first month:

**Bob:** Okay—suppose we have two fractions [pauses and writes  $\frac{3}{4}$  and  $\frac{4}{5}$  on overhead] and we need to figure out which fraction is the largest. Remember how we read fractions? (pause) we read these fractions 'three-fourths' and 'four-fifths' [points to numerator and denominator of each fraction as he reads it]. Any guesses about which fraction is the largest? (pause) Terrel, what do you think?

**Terrel:** I think that, uh ... (pause) that four-fifths is biggest [Note: All students' names are pseudonyms. In selecting pseudonyms I have tried to portray the diversity of Bob Fern's classroom].

**Bob:** So maybe four-fifths is the larger [underlines  $\frac{4}{5}$  on overhead]. How could we write that using our signs for comparing numbers—remember these [writes  $<$ ,  $>$ , and  $=$  on the overhead]? [most students nod] So which symbol would we use to record Terrel's idea that four-fifths is the larger fraction?

**Lamaya:** Use the 'less than' sign [traces the  $<$  symbol in the air with a finger].

**Bob:** Like this [writes  $\frac{4}{5} < \frac{3}{4}$  on overhead]?

**Lamaya:** Yeah (pause)—I mean no. It should be the other way!

**Bob:** Like this then [smudges out the  $<$  sign and changes it to read  $\frac{4}{5} > \frac{3}{4}$ ] [Lamaya nods] Chang, what do you think—do you agree?

**Chang:** Well, yeah (pause). That says that (pause) four-fifths is greater than three-fourths.

**Bob:** Alright. So now we've got Terrel's guess recorded. But how do we know if Terrel's guess is right? (pause) I mean, how do we find out if four-fifths really is bigger than three-fourths? [long pause with no responses from class] Well, what we do is convert each fraction so that we have a common denominator [writes common denominator on overhead]. Has anyone heard of this before [again, no responses]? Okay, here's what we do. First, you multiply the denominators of your two fractions together [circles the 5 in  $\frac{4}{5}$  and the 4 in  $\frac{3}{4}$  to indicate the denominators] and then you multiply the denominators together [writes  $5 \times 4$  on the overhead] and what do you get?

**Class:** Twenty [about half the class responds]!

**Bob:** Good. Now multiplying the denominators together gives us our common denominator, so our common denominator is going to be 20 [writes  $\frac{4}{5} ? \frac{3}{4}$  on the overhead and below writes  $\frac{\quad}{20}$ ]. Now, since we multiplied the five by four to get 20, we need to multiply the numerator, which in this case is four, by four also and we get [pause while class choruses the product] 16—good. Because we multiplied four by five to get 20 we need to multiply three by five also to get [pause while class choruses product] 15—good! So we can write four-fifths and three-fourths with a common denominator of 20 [fills in numerators so the number sentence reads  $\frac{16}{20} \quad \frac{15}{20}$ ]. Now, is greater than the right symbol to put in here (pause)?

**Lorrie:** Yep! Cause 16 is bigger than 15 [pauses while Bob puts in > sign so that the number sentence reads  $16/20 > 15/20$ ]. Yeah that's right!

**Bob:** Good! You convert to the common denominator so you can just look at the numerators—you can compare two fractions by just looking at the numerators as long as the denominators are the same—so Terrel's guess was right [Terrel smiles and waves at class mates]!

Bob went on to do another example of how to convert to a common denominator to compare two fractions. He then passed out a ditto with 40 problems where students were to compare two fractions (i.e., problems like  $3/8 \_ 2/3$ ) and fill-in the correct <, >, or = symbol. There were five additional problems on the ditto where three or four fractions needed to be arranged in sequence of smallest to largest.

Students worked for the rest of the class period on their ditto as Bob circulated around the room answering questions. As he interacted with his students, Bob continued to focus on whether or not they were correctly executing the procedure he had demonstrated on the board. For example, Bob remarked to one student, "Be careful to multiply each fraction, top and bottom, by the denominator of the *other* fraction." He also announced to the class, "Do your multiplication carefully—if you're having trouble remembering your multiplication facts than that means you should practice them at home." Notably, Bob did not address comparison of fractions conceptually (e.g., using Cuisenaire rods to construct and compare  $4/5$  and  $3/4$ ) nor did he show a case where the denominators of the compared fractions are not relatively prime (e.g., where the denominators have a factor other than 1 in common, like  $1/6$  and  $3/12$ ). Without such conceptual justification for the procedure, Bob's students must rely on memorization, which further emphasizes the rote computational nature of the comparing fractions lesson.

After the bell rang, I asked Bob how he thought the lesson had gone and to what extent he felt the lesson focused on problem solving. He replied:

Yes, I would say that they were doing problem solving in the lesson, because (pause)—I mean, if you have a *problem* and you're trying to *solve* it, then, I guess, you're doing *problem solving*. The kids were learning to solve the problems, to compare the fractions with different denominators by finding a common denominator. Most of the class was able to do that successfully, at least by the end of class, so, I guess, the lesson went pretty good.

Bob believes that he is teaching his students about problem solving in the comparing fractions lesson. But, from a reform perspective, the lesson focused on using computational exercises (not problems) to practice a procedure to get the correct answer. In the comparing fractions lesson, what reformers call exercises Bob is thinking of as problems.

### ***Emphasis on Problem Solving Strategies***

Over the next two months, Bob's students began to use a classroom set of calculators that were purchased by the school. Bob explained to me that he was using calculators because, "... my focus is problem solving and not computation." Bob claimed calculators would allow his students

... to spend more time solving problems and less time just doing tedious computational stuff. (pause) I wish they could do calculations faster and more accurately with paper and pencil than they can, but, I guess I'm thinking now that, well, being able to get the right answer to a division problem with a calculator is just as good as being able to find it with a paper and pencil.

By the beginning of the third month of this study, Bob had begun to routinely distribute the calculators to his students at the beginning of each class. Bob instructed his students to, "Use your calculator when it will save you time—sometimes it takes more time to use the calculator, like if you need to find  $3 \times 5$ —you shouldn't need a calculator for that." This shift in Bob's practice is important because it marks when he explicitly decreased his emphasis on computation and increased his emphasis on problem solving. For Bob, the calcula-

tor is a tool that allowed him to balance his dilemma of wanting to teach problem solving yet not neglecting the development of his students' computational skills. In essence, using calculators, Bob decided, would allow his students to perform accurate computations while freeing up more time for problem solving.

What motivated Bob to use calculators in his class and ratchet up his emphasis on problem solving is that he had become increasingly concerned that the students seemed to have had so little prior experience with problem solving strategies that it was difficult for them to solve problems other than routine computation exercises (e.g.,  $34.2 \times 67$ ) or word problems requiring only one (or at most two) arithmetic operations (e.g., "If my groceries cost \$26.71 and I give the clerk a \$20 bill and a \$10 bill, how much change should I get?"). In effect, Bob believed that his students needed to know more mathematics than only being able to get correct answers to computation exercises as in the comparing fractions lesson. Bob decided to address this by teaching his students specific problem solving strategies (e.g., guess and check, draw a diagram, work backwards). Bob expressed that by modeling specific strategies for his students and then providing them problems where they would need to select and apply one or more problem solving strategies they would build their skills as problem solvers and move beyond routine word problems.

A representative example of how Bob taught problem solving is the coin problem lesson in which he introduced the "guess and check" problem solving strategy, the first strategy his students learned. Bob began the class by writing this problem from the Lane County Mathematics Project (1983) on the overhead:

*If you have 9 coins that are together worth 58 cents, what are the kinds of coins you can have?*

**Bob:** Okay (pause)—everyone read this problem carefully. [Students become quiet and read the problem] Does anybody think that they have an answer to this problem? (pause) [a student raises his hand] Carlos—what do you think?

**Carlos:** I think that you could have two quarters and eight pennies.

**Bob:** Okay, why do you think two quarters and eight pennies is an answer? [Writes "2 quarters" and "8 pennies" on the overhead.]

**Carlos:** Well (pause)—if you got two quarters and eight pennies than you have 58 cents!

**Bob:** So two quarters and eight pennies is 58 cents—does everyone agree with that (pause) [some students nod in agreement] Okay, some people don't agree? (pause) Cindy, don't you agree with Carlos?

**Cindy:** Yeah, two quarters and eight pennies is 58 cents, but you've got to have nine coins and in Carlos's answer he's got 10 coins!

**Bob:** Hmmm ... Does everyone see what Cindy is talking about here? Carlos's answer does give us the right amount of money, but his answer doesn't have the right number of coins. Carlos, what if you knew that one of the coins was a 50-cent piece—then what coins could you have?

**Carlos:** Oh! (pause) You'd have one 50-cent piece and eight pennies! [lots of students nod and Bob writes "1 50-cent and 8 pennies" on the overhead.]

**Bob:** Does everyone agree with that? [almost all students nod] Okay, Bianca can you explain why you agree with Carlos's new answer?

**Bianca:** I guess—cause you have nine coins—and you've got fifty-eight cents, because you have 50 cents and then 8 cents more for the pennies.

**Bob:** Very good! Now what Bianca just did for us is check the answer to see if it meets the conditions of the problem. You have to have nine coins and fifty-eight cents—a correct answer has to meet both of those conditions. Carlos's first answer only met one condition but the second solution met both. [Writes "Guess and Check" on overhead below problem] What we've just done here is use a problem solving strategy called guess and check—you guess a solution to the problem—like Carlos did with his first answer—and then check it against the conditions. Now, if your answer doesn't check like Carlos's first answer didn't then you go back and make a new guess and go back again and check it. You keep guessing until you have an answer that checks. Any questions? (pause) Okay—now I want you to work in your small groups [students' desks are arranged in clusters of four that are their small groups] and (pause) find two solutions [holds up two fingers] to this problem *without* using a 50-cent piece.

As his students began working, Bob circulated around the room monitoring each group's work. Bob reminded each group to make sure that their solution met all three conditions—nine coins, a total of 58 cents, no 50-cent piece. After about 15 minutes, each group had found two answers and Bob chose a group to explain their solutions. After the group's explanations, Bob pointed out that, "This is an example of how a problem can have more than one answer." The class spent the rest of the period working on a handout of four problems similar to the coin problem. Bob instructed the class to solve all four problems using the guess and check strategy. During the next week, when Bob taught the problem solving strategy "make a systematic list", I asked him to what extent he was concerned about his students' computational skills and he replied, "Well, I am concerned a little, but I think that there's enough computation in the problems we're doing with the strategies that they won't get too rusty on their computation facts." Bob said that a problem the class spent most of the period on during the second week (on a day I was not present)—"How many rectangles are there that have an area of 72 square units?"—was an example of a problem that has, in his words, "lots of multiplication facts in it."

For the next six weeks, Bob and his class worked on six problem solving strategies in this order: guess and check, make a systematic list, make a model, solve the problem in multisteps, draw a diagram, and look for patterns (see Lane County Mathematics Project, 1983). Bob taught his class one strategy per week and on no occasion during that time did students work on paper-and-pencil computation exercises. During this period, Bob's emphasis shifted away from teaching students computational procedures with exercises, as in the comparing fractions lesson, to learning, applying, and practicing specific problem solving strategies as in the coin problem. In a way, Bob was still teaching his students procedures—i.e., specific problem solving strategies (procedures) instead of specific computational procedures. However, Bob's instruction with problem solving strategies focused substantially on making sure his students understood and could make sense of the strategies as reflected in the discourse of his classroom. Moreover, the strategies were applied in meaningful ways where students needed to explain why using the strategy and the answer it produced made sense. None of this occurred in Bob's teaching of computational procedures with exercises as in the comparing fractions lesson. For all of these reasons, Bob's teaching of problem solving strategies indicated that his practice was now more closely aligned with problem solving reforms.

### ***Emphasis on Reasoning and Problem Solving***

After he had covered the six problem solving strategies, Bob continued teaching problem solving but without prescribing specific strategies to use to solve problems. Bob informed me that he wanted his students to be able to "devise a plan" (Polya, 1945) by selecting an appropriate problem solving strategy or a combination of strategies. Bob increasingly encouraged his students to reason carefully as they tried to decide how to solve a problem and then to assess whether they were making progress toward a solution. For example, Bob would often ask the whole class or individual students questions like, "What conditions would a solution to this problem have to satisfy?", "Why are you using 'make a systematic list' [or other strategy] as your approach?", and "How is your strategy working?" By encouraging his students to reason, Bob was teaching what Schoenfeld (1985) calls *heuristics* and *control*—i.e., he was encouraging his students to think carefully about the strategy they would use to approach the problem (heuristics) and then to monitor their own progress to see if the approach was moving them toward a solution or whether they should go back and try another approach (control). Bob felt that using combinations of problem solving strategies was a "logical next step" for his students in learning about problem solving.

The dartboard problem is a typical example of how Bob encouraged his students to reason as part of problem solving. The lesson was taught during the last month of my observations in Bob's class. At the beginning of the period, Bob put this problem on the overhead:

*Carmen needs 12 points to win a dart match. Carmen gets to throw 4 darts and assume that all of Carmen's darts always hit the target. When a dart hits the target, it scores 5 points, 3 points, 2 points, or 1 point. What are all the possible ways that Carmen's darts can score on the target and win the match?*

**Bob:** This is the situation. [draws a dartboard on the overhead composed of four concentric circular regions labeled 1, 2, 3, 5 to illustrate the dartboard] The first thing I want you to think about is what a solution would look like—and explain why it's a solution. (pause) Becky, what do you think are some different solutions for how Carmen could win this dart match?

**Becky:** Well, all of her darts could hit the three.

**Bob:** [makes four dots in the 3 region] Okay—and why do you think this would be a solution?

**Becky:** Because then she'd have her 12 points. [nods from some students]

**Bob:** Sounds good—are there any other solutions that you can think of? (pause) Well, is there another way to get 12 points?

**Marcus:** Yeah! Like, she (pause) or he could throw two 5s and two 1s—that would be four darts and it would make 12 points too!

**Bob:** Very good—Marcus just explained why two 5s and two 1s would be a solution too—does everyone agree with that? [nods from class, then Becky raises her hand] Yes, Becky?

**Becky:** Well—I mean (pause) there are lots of solutions because Carmen could also get 13 points, or 14 points, or 15 points, or whatever!

**Bob:** Good point! Does everyone see what Becky is saying? (pause) Becky and Marcus showed us that there are at least two ways for Carmen to score 12 points to win the match, but there are also different scores that Carmen could get bigger than 12, like this [puts three dots in a different color in the 3 region and one in the 5 region]—how many points is this?

**Class:** Fourteen!

**Bob:** Yes—so as you work on this problem in your small groups, you need to each come up with your own solutions to the problem but you can compare with others in your group. You need to *find all the different scores* Carmen can get to win the dart match and *all the ways Carmen can get each of those scores*. Go ahead and get to work, I'll be around to see how you're doing.

Bob began circulating around the room as students got out pieces of paper and began trying to solve the dartboard problem. As I watched and listened to students, I noticed that most groups soon found all the scores Carmen could get to win the match: 12, 13, 14, 15, 16, 17, 18, and 20. Bob had Terrel explain to the class how his group came up with these scores:

Carmen needs 12, so 12 would be okay. But, you can always get more points. (pause) And the best you could do would be 20 from hitting the bullseye [i.e., the 5 region] all four times—but (pause) you can't get a 19 because the next score from 20 (pause) I mean down from 20, would be three 5s and a 3 (pause) and that's 18.

Terrel continued to explain examples of how Carmen could throw darts to get each of the other winning scores.

As the class continued to work on the problem, most students combined the "make a systematic list" strategy with the "solve the problem in multisteps" strategy (see Lane County Mathematics Project, 1983). Most students first looked at each individual winning score Carmen could throw and then used the systematic list strategy to record all the possible dart throws for getting that score. For example, students in one group were trying to find all the ways Carmen could throw a score of 12 if exactly one bullseye (i.e., one dart landing in the 5-point region) was scored, then two bullseyes, etc. As he was circulating around the room, I asked Bob how he felt the class was going. He replied, "I think this is going very well—we had a nice discussion and they're really getting into this." I asked Bob near the end of the period if he thought all of the students would be able to come up with all of the possible winning throws and he said, "I'm not so concerned that everyone gets every possible solution—my goal in this lesson is for all of these kids to reason about solving the problem and know whether they're getting closer to solving it." Throughout this lesson, Bob's students were reasoning with problem solving strategies as they used the strategies to formulate, investigate, and refine conjectures about the problem and potential solutions. Bob supported his students' reasoning with problem solving strategies by asking questions that helped his students refine their thinking and justify their ideas.

### **Evolution in Teaching Problem Solving**

Polya (1968) argued that problems could be classified into four major types:

- One rule under your nose
- Application with some choice
- Choice of a combination
- Approaching research level (p. 139).

Polya maintained that the potential effectiveness of problems for helping students learn about problem solving increased going from "one rule under your nose" problems to "approaching research level" problems (Kilpatrick, 1985). In Bob's lesson on comparing fractions, all the "problems" were "one rule under your nose" exercises—he showed his students a procedure and instructed them to execute the procedure to get an answer for every exercise. However, when Bob began teaching different problem solving strategies, he would model a strategy and then expect his students to apply it and monitor their own progress as they used the strategy to solve problems. For example, Bob stressed the importance of students checking their guess when using the guess-and-check strategy to monitor their progress in solving the coin problem by deciding if the proposed solution met the conditions of the problem. As students learned more strategies they were poised to do "application with some choice" problems because more heuristics were available to them. Finally, the dartboard problem is an example of a "choice of a combination" problem because students combined the "make a systematic list" and "solve the problem in multisteps" strategies to generate solutions. Polya's classification framework for problem types shows how Bob engaged his students with problems that increased in potential to help them build problem solving skills. In terms of mathematics education reform, over the six-month period of this study, as demonstrated by the three representative lessons, Bob selected increasingly more "worthwhile mathematical tasks" for teaching problem solving (see NCTM, 1991).

The table below summarizes how Bob's understanding of teaching and learning problem solving changed over the course of the study, as demonstrated by the three sample lessons. The table also shows how each of Bob's lessons fit into Polya's (1968) framework, as argued in the above analysis. Finally, the table draws on the *Principles and Standards for School Mathematics* (NCTM, 2000) to identify how each of Bob's lessons address, or do not address, reform recommendations for teaching and learning problem solving in middle school mathematics:

<b>Sample Lesson</b>	<b>Comparing Fractions</b>	<b>Coin Problem</b>	<b>Dartboard Problem</b>
<b>Bob's Understanding of Problem Solving</b>	Problem solving is about getting answers to computational exercises to develop "basic skills."	Problem solving is about learning strategies and then choosing and applying appropriate strategies to solve problems.	Problem solving is about reasoning through a problem, developing and articulating a solution strategy, and then reflecting on solutions.
<b>Polya (1968) Problem Type</b>	One rule under your nose.	Application with some choice.	Choice of a combination.
<b>Problem Solving Reform Recommendations (NCTM, 2000)</b>	Lesson reflects practice of a rote procedure, superficial problem solving; does not reflect NCTM (2000) reform recommendations for problem solving.	Lesson reflects NCTM (2000) reform recommendation to "Apply and adapt a variety of appropriate strategies to solve problems" (p. 256).	Lesson reflects NCTM (2000) reform recommendations to apply strategies to solve problems and to "monitor and reflect on the process of mathematical problem solving" (p. 256).

### **Aligning Teaching Practice with Reform and Implications for Professional Development**

Teachers are the key to improving mathematics education. ... Regardless of the curriculum or the assessment process in a school district, the person in charge of adapting materials for a particular classroom and student is the teacher. It is through teachers' efforts that students have opportunities to learn mathematics. ... [T]eachers need access to high-quality materials, the support of parents, and ongoing, focused professional development. —Glenda Lappan, President's Message, October 1998 *NCTM News Bulletin*

Teachers will be the final arbiters of the progress or stagnation of mathematics education reform, including teaching and learning mathematics as problem solving. The case of Bob Fern demonstrates how an experienced teacher can more closely align his teaching practice with reform goals for problem solving. However, such alignment should not necessarily be anticipated for all or even most teachers in addressing problem solving or other reforms in their classrooms. For example, researchers have shown that teachers can resist mathematics education reform, have insufficient resources, content knowledge, or confidence to implement reforms, or develop inappropriate conceptions that may limit the impact of curriculum materials or policies intended to serve as vehicles for reform (c.f., Rickard, 1995b, 1998; Putnam et al., 1992; Remillard, 1991; Wiemers, 1990). Unpacking Bob Fern's alignment process can provide insights for how other teachers might work successfully toward aligning their practices with problem solving reforms.

Multiple factors fueled the evolution of Bob Fern's teaching of mathematics as problem solving: Bob had access to curriculum materials that supported teaching and learning mathematics as problem solving; his district provided him with inservice opportunities to learn more about problem solving; and, perhaps most importantly, Bob is confident in his own mathematics background and he wanted to learn more about problem solving to balance development of his students' computational and problem solving skills. In addition, as demonstrated in his comments and in conversations, Bob engaged in ongoing reflection and dialogue with me about

his practice and how he was addressing computational skills and problem solving with his students. These aspects of Bob's alignment process are reflected in other research findings that show reflection and dialogue with colleagues about mathematics education reforms help motivate and sustain professional development (e.g., Rivette et al., 2003). In my work with Bob, reflection and dialogue—i.e., carefully and routinely thinking about his mathematics teaching and sharing his thoughts and ideas with me, respectively—helped to motivate and sustain the evolution of his teaching to increasingly reflect the reform goal of teaching and learning mathematics as problem solving.

Employing reflection and dialogue as tools for professional development is not new (e.g., Brookfield, 1995; Schon, 1990). However, engaging in reflection and dialogue over the six months of this case study is different than most professional development experiences in school mathematics. The substantial majority of professional development in K-12 mathematics is short term in nature, usually taking the form of discrete workshops or inservices lasting anywhere from a couple of hours to an entire day (e.g., such as Bob experienced). Some professional development experiences for teachers are longer term, typically one- or two-day weekend courses or one to three week summer institutes (e.g., see Rivette et al., 2003). Access to such experiences is often limited, however, and when available tends to be costly.

The sustained reflection and dialogue Bob engaged in was a form of long-term professional development that, together with enabling conditions such as access to supporting curriculum materials, inservices, being confident in his mathematics knowledge, and openness to revising his practice, resulted in more closely aligning his practice with reform goals. Moreover, while these enabling conditions are important, it is not certain that Bob would have been as successful in aligning his practice with problem solving reform goals without sustained reflection and dialogue. For example, Bob remarked to me on multiple occasions that he pushed himself to rethink his teaching frequently and that he found the conversations that he and I had to be extremely helpful. Additionally, some teachers' who do have access to high-quality curriculum materials to support teaching and learning mathematics as problem solving and/or have a strong mathematics background, but who do not engage in both sustained reflection and dialogue with one or more colleagues, do not address teaching and learning mathematics as problem solving in their teaching consistently or at all (e.g., Burns, 2000; Rickard, 1995a, 1995b, 1996, 1998).

The case of Bob Fern suggests that ongoing reflection and dialogue can play powerful roles in ongoing professional development for teachers to support them in revising and refining their teaching to more closely align their practice with problem solving reform goals. However, this finding should be extended by investigating how reflection and dialogue with colleagues can be refined to address the needs of teachers specifically seeking to align their teaching with problem solving or other mathematics education reform goals. For example, what are key issues that reflection and dialogue about teaching and learning mathematics should address (e.g., what kind of curriculum materials are needed to support problem solving, subject matter knowledge of mathematics)? Who should be working with teachers to support on-going reflection and dialogue (e.g., mentor teachers, mathematics or teacher educators, school administrators)? How and to what extent does teaching and learning mathematics as problem solving address high-stakes testing and federal requirements that schools demonstrate "adequate yearly progress" (e.g., No Child Left Behind Act)? For Bob Fern, reflection and dialogue centered on the roles and balance of computational skills and problem solving in his teaching and students' learning, but these may not be the most critical issues for all teachers—what would an assessment process look like that would determine different teachers' specific needs to align their practice with problem solving and then gauge their progress?

The case of Bob Fern shows that utilizing reflection and dialogue to support sustained professional development holds potential to help attain ambitious reform goals for school mathematics. The role of reflection and dialogue in aligning teaching and learning mathematics with reform goals for problem solving hold particular promise because they necessarily focus on teachers' individual practices. And it is the individual practices of teachers that will ultimately determine how deeply problem solving and other mathematics education reforms take root in classrooms and in students' learning.

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