A lesson in number pattern

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Following is a guided investigation into the spatial relationships between the centres of the squares in a Fibonacci tiling. It is essentially a lesson in number pattern but includes work with surds, coordinate geometry and some elementary use of complex numbers. The student worksheet is shown in Figure 1.

The investigation could be presented to students in a number of ways according to the teacher's requirements and the needs of the students. It will give students a contextual and hence a practical application of the previously mentioned work

The investigation is punctuated throughout with dividers which indicate possible breaks in it where teachers could assess or monitor student progress.

The investigation could be used simply as class-work or even as an analysis assessment task.

Guided investigation/application task

On the grid paper provided, the position of the x and y axes have been indicated but not drawn in to avoid confusion with the diagram of squares surrounding the origin. The diagram consists of three squares drawn in the following order.

- Square #1. A unit square in the first quadrant.
- Square #2. Another unit square adjacent to and on the left-hand side of the first.
- Square #3. A two unit square adjacent to the others and directly below them.

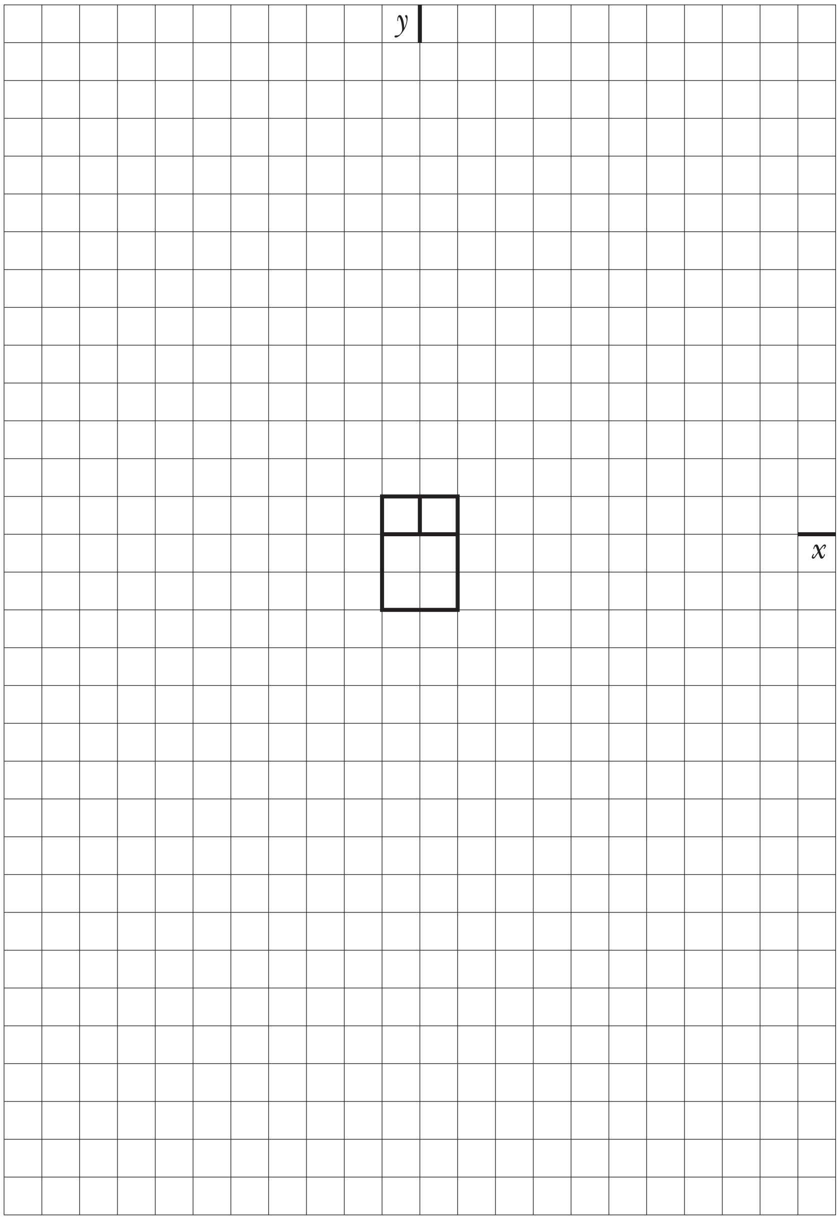

Figure 1

Your tasks

1. Draw square #4, a three unit square, adjacent to and to the right of the existing diagram. Note the pattern: the squares, increasing in size, are added to the diagram in an anticlockwise manner.

2. Draw squares #5, #6, #7 in order.

3. Mark on the diagram the centre point of each square and on a separate table record the coordinates of each centre. Use vulgar fractions rather than decimals. Call the centre point of square #1, C_1 and the centre point of square #2, C_2 and so on.
   You will be investigating the spatial relationships between the centres of the squares.
4. Look for visual patterns in the centre points. Briefly mention those that you see.

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5. Determine the mathematical equation of the line which passes through the odd centres.

6. Determine the mathematical equation of the line which passes through the even centres.

7. Prove that the two lines found in (5) and (6) are perpendicular to each other.

8. Find the coordinates of the intersection point of the lines found in (5) and (6) Call this point B.

9. Calculate exactly (use surds) the distance from B to each of the centre points; i.e., calculate the lengths of $\overline{BC}_1$, $\overline{BC}_2$, … $\overline{BC}_7$.

10. Find the following ratios and write the results separated by commas
$$\frac{\overline{BC}_2}{\overline{BC}_1}, \frac{\overline{BC}_3}{\overline{BC}_2}, \frac{\overline{BC}_4}{\overline{BC}_3}, \ldots \frac{\overline{BC}_7}{\overline{BC}_6}$$

11. Look for and record a pattern in the sequence created at (10).

12. Use the pattern to extend the sequence at (10) with three extra terms; i.e., find
$$\frac{\overline{BC}_8}{\overline{BC}_7}, \frac{\overline{BC}_9}{\overline{BC}_8}, \text{ and } \frac{\overline{BC}_{10}}{\overline{BC}_9}$$

13. Calculate the decimal equivalent of the terms in the sequence created at (10) and (11) correct to 3 significant figures.

14. What do you notice about these terms?

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15. Write down the number correct to three significant figures to which the sequence created in (10) is converging.

The following sequence is called the Fibonacci sequence

F_1,	F_2,	F_3,	F_4,	F_5,	F_6, …
1,	1,	2,	3,	5,	8, …

16. What connection does this sequence have with Figure 1?

17. Write down the next three terms of the Fibonacci sequence; i.e., F_7, F_8, F_9.

18. Calculate and record the results of the following ratios, using decimals

$$\frac{F_2}{F_1}, \frac{F_3}{F_2}, \frac{F_4}{F_3}, \frac{F_5}{F_4}, \frac{F_6}{F_5}, \frac{F_7}{F_6}, \frac{F_8}{F_7}, \frac{F_9}{F_8}$$

19. Write down the number, correct to 3 significant figures to which these ratios appear to be converging (compare with (15)).

The number in (15 and 19) has an exact value of

$$\frac{\sqrt{5}+1}{2}$$

and is approximately equal to 1.61803.

This number is a special constant in mathematics and it is given the symbol ϕ or Phi. Phi occurs regularly in mathematical models of situations occurring in the real world in areas such as biology (growth), art, geometry, music and more.

Imagine the diagram you have with squares and their centres C_1, C_2, etc. to be an Argand diagram or, if you prefer, the Complex Plane with the origin at point B. C_1 would be represented by the complex number

$$\left(\frac{1}{10}+\frac{3}{10}\times i\right) \text{ or } \frac{1}{10}(1+3i)$$

20. Write down the complex numbers which represent the points C_1, C_2, C_3, C_4, C_5.

Consider the following difference equation

$$C_n = \left(F_{n-2} + 3F_{n-1}\right)\frac{(1+3i)}{10} i^{n-1}, \ n \geq 3$$

where F_1, F_2, F_3, … represent the terms of the Fibonacci sequence.

21. Verify that this difference equation does in fact generate the positions of the centre of squares by substituting $n = 3, 4, 5$.

Teaching point

In answer to task (4), some students will mention that the centres appear to be on a spiral. If teachers wish to elaborate on this, the following information leads to an explanatory demonstration in conjunction with an overhead projector and appropriate graphing calculator. Note that the spiral accumulation point is point B on the student worksheet

1. Make a scatter graph of the first six centre of squares as found in (20); use ZOOMSTAT followed by ZOOMSQUARE.
2. Put the graphing calculator in polar coordinate mode and enter the following equation:

$$r = e^{(c_\theta + k)}$$

where $c = \dfrac{2\ln(\phi)}{\pi}$ and $k = \dfrac{1}{2}\left(\dfrac{\phi+1}{10}\right) - \dfrac{2}{\pi}\ln(\phi)\tan^{-1}(3)$

3. Set $\theta_{min} = 0$, $\theta_{max} = 10$ and $\theta_{step} = .05$. This will produce the graph shown in Figure 2.

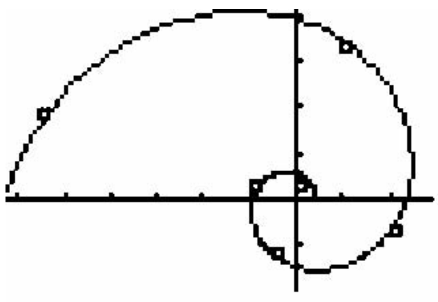

Figure 2