The mathematics of sundials

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As early as 3500 years ago, shadows of sticks were used as a primitive instrument for indicating the passage of time through the day. The stick came to be called a “gnomon” or “one who knows.” Early Babylonian obelisks were designed to determine noon. The development of trigonometry by Greek mathematicians meant that hour lines could be determined arithmetically rather than by geometry, leading to more sophisticated sundials. In the first century CE, the Roman architect and engineer Vitruvius described several types of sundials in his *De Architectura* (Lennox-Boyd, 2006, p. 32), including hemispherical, conical and planar dials. Sundials are often constructed to commemorate special events, for example, the Bicentennial Park sundial in Sydney. The many common designs of sundials, such as vertical, horizontal and analemmatic dials, can all be derived by projections of the basic equatorial dial (Lennox-Boyd, 2006). In this article I show how trigonometry can be used to calculate the positions of the hour lines for vertical and horizontal sundials, with a particular focus on the mathematics underlying a recently-constructed unique horizontal sundial at Piazza Italia in Melbourne. The words “vertical” and “horizontal” are used in their normal sense, that is, in the direction of gravity and at right angles to this.

Equatorial sundials

Let us imagine the Earth as a giant sundial (Figure 1). The Earth’s axis is tilted at an angle of 23.5° to the plane of its orbit around the Sun. As the Earth rotates on its axis, the shadow of a vertical stick at the pole would form a circle on the surface of the Earth parallel to the equator. If the circle is divided into 24 equal hour marks, the position of the shadow around the circle would give the time. Sundials based on this principle are called equatorial sundials.

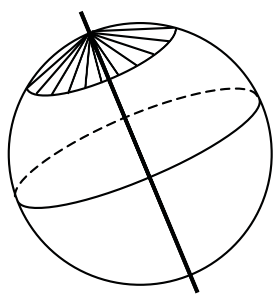

Figure 1. Representing the Earth as a giant sundial.

In any sundial, the part that casts the shadow is called the gnomon. For an equatorial sundial, the gnomon must be parallel to the Earth’s axis. In Figure 2, P is in the southern hemisphere at latitude L. The diagram shows why the angle that the gnomon makes with the horizontal is equal to the latitude. The dial with the hour marks must be perpendicular to the gnomon. In the southern hemisphere the gnomon will point to the south celestial pole (in the northern hemisphere, it points towards the pole star).

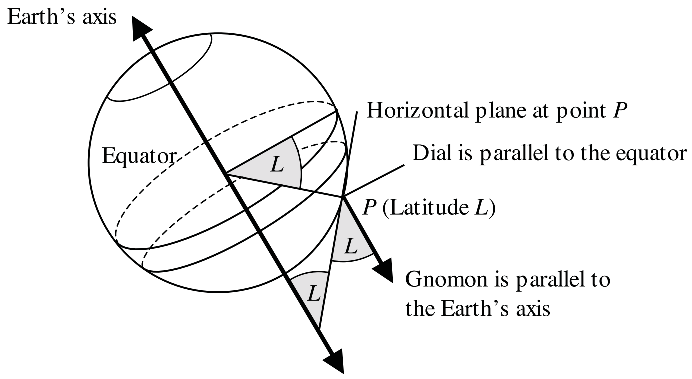

Figure 2. Inclination of the dial and gnomon of an equatorial sundial.

Figure 3 shows an equatorial sundial for the southern hemisphere, with the gnomon making an angle with the horizontal equal to the latitude. The hour lines are equally spaced, with 15° intervals. At midsummer the Sun is directly in line with the gnomon so there is no shadow. Between the spring equinox (22 September) and the autumn equinox (22 March), the shadow falls on the upper face of the dial. Between 22 March and 22 September, when the Sun is lower in the sky, the shadow falls on the under side. The equatorial sundial therefore needs a face on both sides.

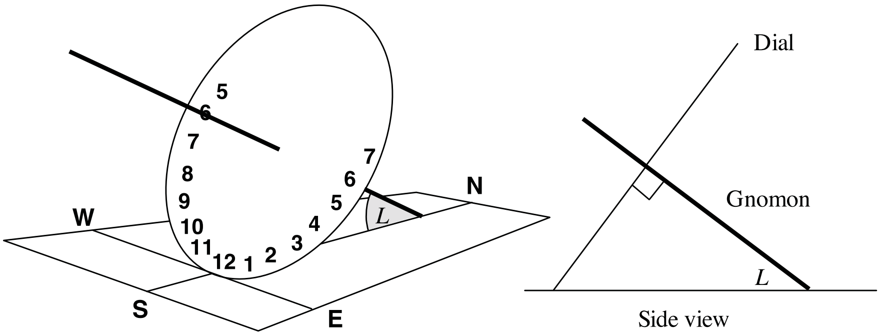

Figure 3. Equatorial sundial for the southern hemisphere.

Vertical sundial

For a vertical sundial, the circular equatorial dial is projected onto a vertical plane as an ellipse. The semi-major axis, a, is the radius of the equatorial dial. If b is the semi-minor axis of the ellipse,

$$\cos L = \frac{a}{b}$$

$$b = \frac{a}{\cos L}$$

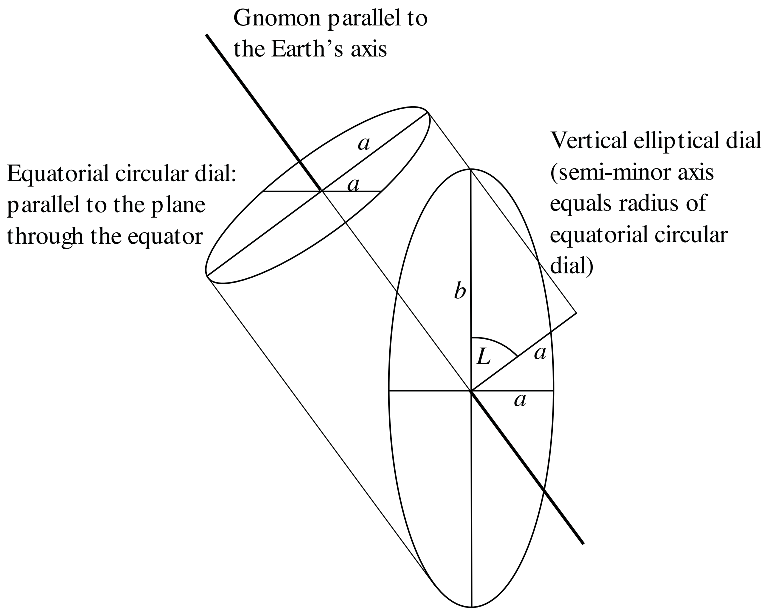

Figure 4. Projection of the equatorial dial to form the ellipse of the vertical dial.

The gnomon of the vertical sundial makes an angle of 90°–L with the vertical (that is, an angle L with the horizontal), as shown in the side view in Figure 5.

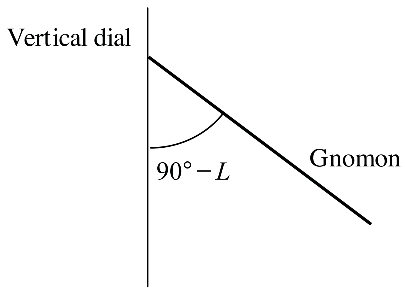

Figure 5. Side view of gnomon of vertical sundial.

Calculating the angle hours for the vertical sundial

In the southern hemisphere, the vertical dial is north-facing. Unlike the equatorial dial, the hour angles are not equally spaced. In Figure 6, angle T is the hour angle measured from the north–south line around the equatorial dial. For example, at 11 am, T = 15°, at 10 am, T = 30°, and so on. Angle H is the projected hour angle on the ellipse of the vertical dial. Using the equation for an ellipse and applying trigonometry in triangles OAC and OBC, we can find H in terms of L and T.

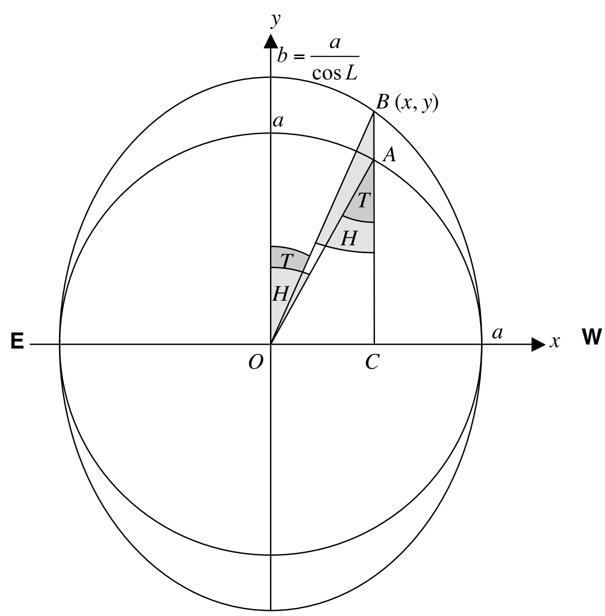

Figure 6. Relationship between the hour angle T of the equatorial dial and the projected hour angle H of the vertical dial.

Equation for the ellipse is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2\cos L}{a^2}=1 \quad \left(\text{substituting } b=\frac{a}{\cos L}\right)$$

$$x^2+y^2\cos L=a^2$$

but:

$$x=a\sin T \quad (\text{from } \Delta OAC)$$

$$y^2=\frac{\left(a^2-a^2\sin^2 T\right)}{\cos^2 L}$$

$$=\frac{a^2\left(1-\sin^2 T\right)}{\cos^2 L}$$

$$=\frac{a^2\cos^2 T}{\cos^2 L}$$

$$y=\frac{a\cos T}{\cos L}$$

Note that $x=a\sin T, y=\dfrac{a\cos T}{\cos L}$ are the parametric equations for the ellipse.

From ΔOBC

$$\tan H=\frac{x}{y}$$

$$=a\sin T\div\frac{a\cos T}{\cos L}$$

$$=\frac{a\sin T\cos L}{a\cos T}$$

$$=\tan T\cos L$$

$$H=\tan^{-1}\left(\tan T\cos L\right)$$

The hour angles, H, for 6 am to 12 noon for a vertical dial in Melbourne (latitude 37.7°) are shown in Table 1. The angles are symmetrical about the noon line for 1 pm to 6 pm.

Table 1

Time	T (°)	H (°)
12 noon	0	0.0
11 am	15	12.0
10 am	30	24.6
9 am	45	38.4
8 am	60	53.9
7 am	75	71.3
6 am	90	90.0

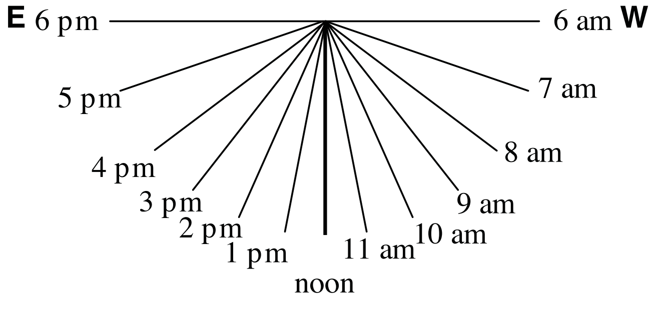

Figure 7. Hour angles for vertical sundial.

Figure 8 shows a vertical sundial on the almost north-facing wall of Box Hill Library in Melbourne's eastern suburbs. The shadow indicates the time is approximately 10:20 am.

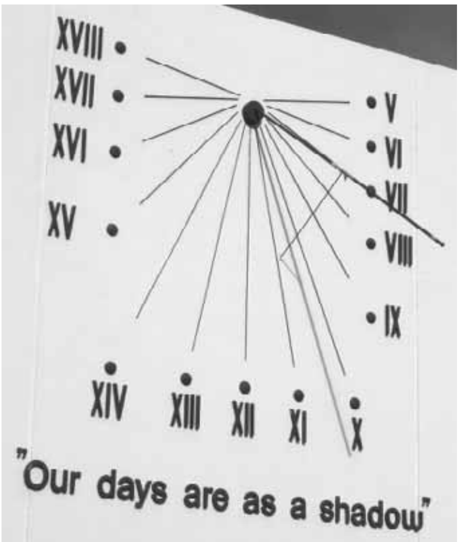

Figure 8. Vertical sundial on Box Hill Library, Victoria.

Horizontal sundials

For a horizontal sundial, the circular equatorial dial is projected onto a horizontal plane as an ellipse (Figure 9). As for the equatorial and vertical sundials, the gnomon makes an angle L with the horizontal. The semi-minor (east–west) axis is a, the radius of the equatorial dial. If b is the semi-major (north–south) axis of the ellipse,

$$\sin L = \frac{a}{b}$$

$$b = \frac{a}{\sin L}$$

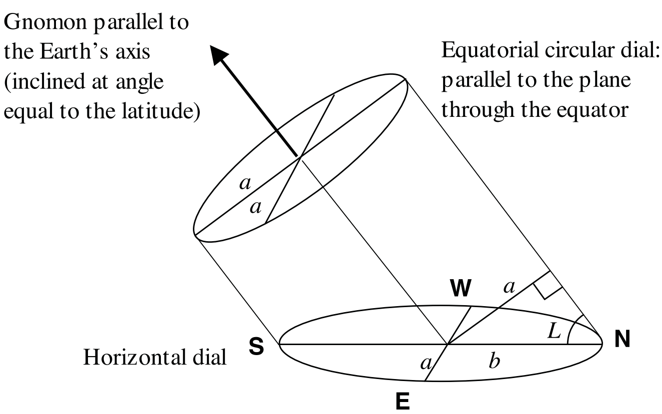

Figure 9. Horizontal projection of the equatorial dial parallel to the Earth's axis.

Calculating the hour angles for the horizontal sundial

In Figure 10, angle T is the hour angle measured from the north–south line around the equatorial dial. Angle H is the projected hour angle on the ellipse of the horizontal dial. Using the same approach as for the vertical sundial, the parametric equations for the ellipse for the horizontal sundial are

$$x = a\sin T,\ y = \frac{a\cos T}{\sin L}$$

The hour angles are given by

$$H = \tan^{-1}\left(\tan T \sin L\right)$$

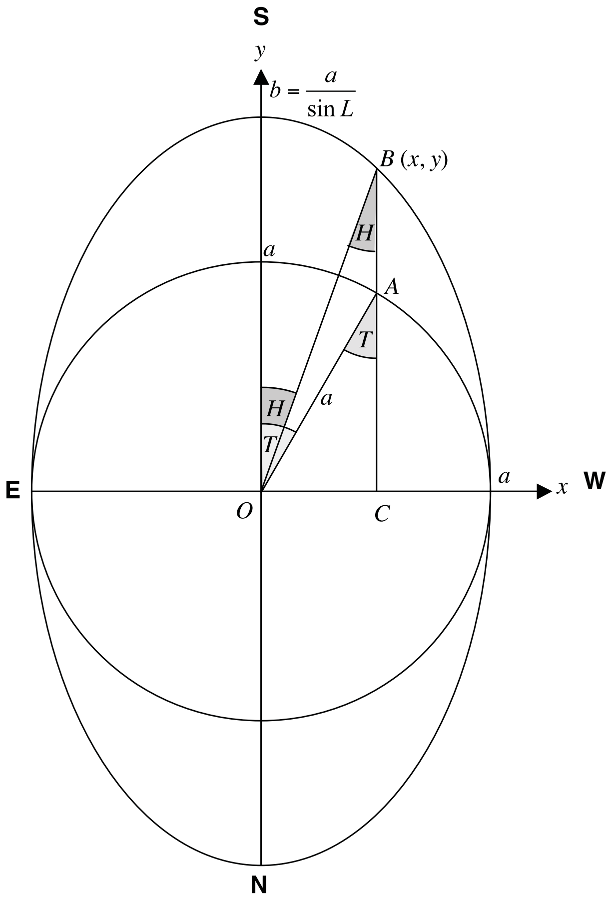

Figure 10. Relationship between the hour angle T of the equatorial dial and the projected hour angle H of the horizontal dial.

The hour angles, H, for a horizontal sundial in Melbourne are shown in Table 2. The latitude 37.7° has been used. For 11 am, for example, $H = \tan^{-1}(\tan 15° \sin 37.7°) \approx 9.3°$. Unlike the vertical sundial, the numbers are anticlockwise for a horizontal sundial in the southern hemisphere. Figure 11 shows the spacing of the hour angles for the horizontal sundial for Melbourne.

Table 2. Hour angles for a horizontal sundial in Melbourne

Time	T (°)	H (°)
6 am	90	90.0
7 am	75	66.3
8 am	60	46.6
9 am	45	31.4
10 am	30	19.4
11 am	15	9.3
12 noon	0	0.0

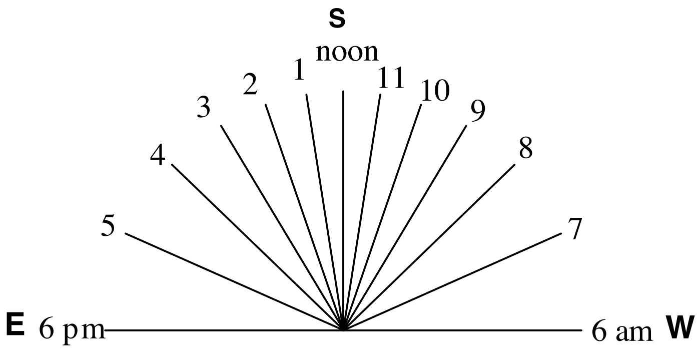

Figure 11. Spacing of hour lines for a horizontal sundial in Melbourne.

Garden sundials are typically horizontal dials. A large-scale horizontal sundial has been constructed in the New South Wales town of Singleton.

Horizontal sundial at Piazza Italia

The horizontal sundial *Solaris* (see Figure 12) at Piazza Italia in the inner Melbourne suburb of Carlton has a unique design. Instead of a gnomon set at the angle of the latitude, a person stands on the north–south line at a calculated position according to their height and acts as a vertical gnomon.

If we consider a right-angled triangle formed by the human gnomon and the distance d m from the centre of the sundial at which the person stands, the hypotenuse of this triangle makes an angle of 37.8028° (the local latitude) with the horizontal (see Figure 13).

Figure 12. Solaris at Piazza Italia, Carlton.

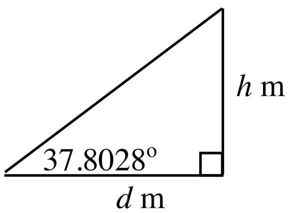

Figure 13. Relationship between height and position.

For example, for a person of height 1.80 m,

$$\tan 37.8028° = \frac{1.80}{d}$$

$$d = \frac{1.80}{\tan 37.8028°}$$

$$d \approx 2.32$$

So a person of height 1.8 m would stand on the north–south line at a position 2.32 m from the centre of the sundial. The positions for heights from 1.00 m to 1.90 m are shown in Figure 14.

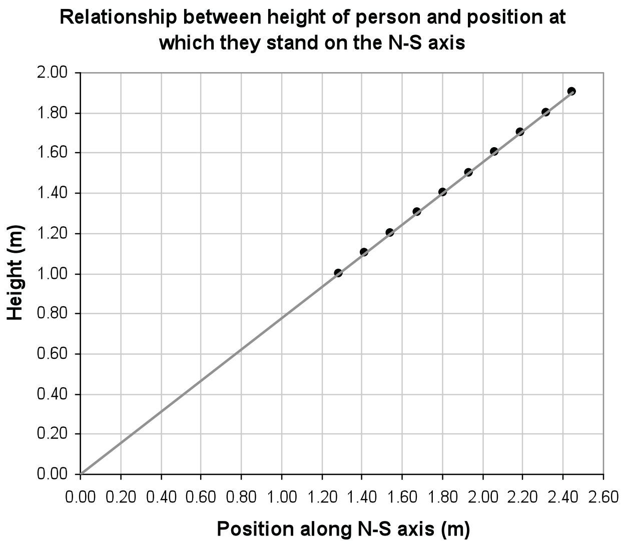

Figure 14. Relationship between height of person and position at which they stand.

Intersection of the shadows with the hour lines

If the vertical (human) gnomon were to be replaced with a right-angled triangle with its hypotenuse making an angle L with the horizontal, then the shadow of the hypotenuse at any given hour would fall along the hour line. The tip of the person's shadow would meet the hour line as shown in Figure 15.

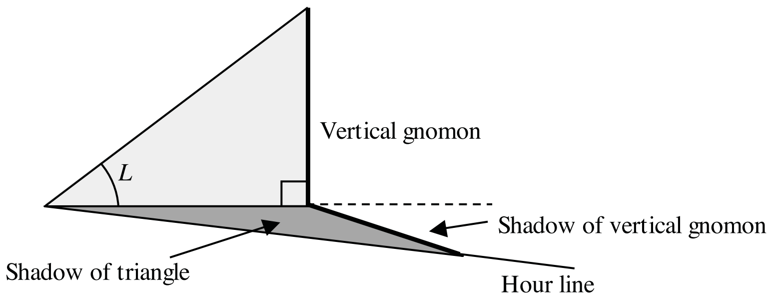

Figure 15. Tip of shadow of human gnomon meets the hour line.

The time at Piazza Italia is therefore indicated by the hour line on which the tip of the person's shadow falls (see Figure 12).

Figure 16 shows the view from above of the hour line, OP, and the shadow, GP, of a person standing on the solar clock at G. These two lines intersect at

P. The coordinates of *P* can be found by solving simultaneously the equations for *GP* and *OP*. The angle *Z* (the Sun's azimuth) that the shadow of a vertical gnomon makes with the north–south line depends on the latitude, the time of day and the declination of the Sun, which changes through the day and through the year. Angle *Z* can be calculated according to the following formula (Budd & Sangwin, 2000), where *T* is the angle hour measured clockwise from the north–south line, *L* is the latitude, and *D* is the declination of the Sun.

$$\cot Z = \sin L \cot T - \frac{\tan D \cos L}{\sin T}$$
$$= \frac{\sin L \cos T - \tan D \cos L}{\sin T}$$

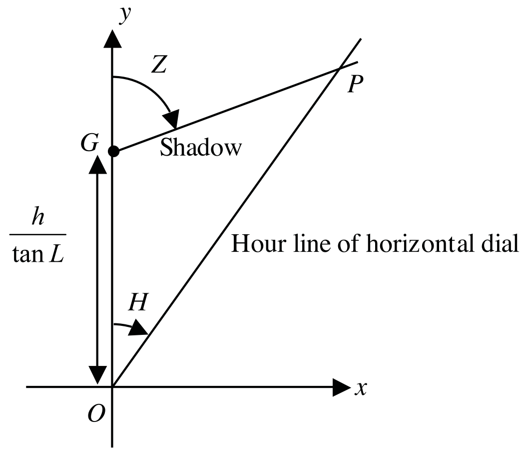

Figure 16. Intersection of person's shadow with the hour line.

Equation of hour line *OP:*

$$y = x \tan(90° - H)$$
$$= \frac{x}{\tan H}$$

But tan $\qquad \tan H = \tan T \sin L$

So $\qquad y = \dfrac{x}{\tan T \sin L} \qquad (1)$

Equation of shadow *GP:*

$$y = \frac{h}{\tan L} + x \cot Z$$

Substituting for cot *Z*,

$$y = \frac{h}{\tan L} + \frac{x(\sin L \cos T - \tan D \cos L)}{\sin T} \qquad (2)$$

From equations (1) and (2),

$$\frac{x}{\tan T \sin L} = \frac{h}{\tan L} + \frac{x(\sin L \cos T - \tan D \cos L)}{\sin T}$$
$$\frac{x \cos T}{\sin T \sin L} = \frac{h \cos L}{\sin L} + \frac{x(\sin L \cos T - \tan D \cos L)}{\sin T}$$

Multiplying both sides by sin T sin L,

$$\begin{aligned} x\cos T &= h\sin T\cos L + x\sin L\left(\sin L\cos T - \tan D\cos L\right) \\ &= h\sin T\cos L + x\left(\sin^2 L\cos T - \tan D\cos L\right) \\ h\sin T\cos L &= x\left(\cos T - \sin^2 L\cos T - \tan D\cos L\right) \\ &= x\left(\cos T\left(1-\sin^2 L\right) - \tan D\sin L\cos L\right) \\ &= x\left(\cos T\cos^2 L - \tan D\sin L\cos L\right) \\ &= x\cos L\left(\cos T\cos L - \tan D\sin L\right) \\ h\sin T &= x\left(\cos T\cos L - \tan D\sin L\right) \\ x &= \frac{h\sin T}{\cos T\cos L - \tan D\sin L} \end{aligned}$$

Substituting to find the y-coordinate of the intersection,

$$\begin{aligned} y &= \frac{x}{\tan T\sin L} \\ &= \frac{h\sin T}{\tan T\sin L\left(\cos T\cos L - \tan D\sin L\right)} \\ &= \frac{h\cos T}{\sin L\left(\cos T\cos L - \tan D\sin L\right)} \end{aligned}$$

Hence the coordinates of the tip of the person's shadow will be:

$$\left(\frac{h\sin T}{\cos T\cos L - \tan D\sin L}, \frac{h\cos T}{\sin L\left(\cos T\cos L - \tan D\sin L\right)}\right)$$

For *Solaris*, $L = 37.8°$. Figure 17 shows a plot of the hour lines and the positions of the shadow tip during the day for the equinoxes and the summer and winter solstices ($D = 0°$, 23.5° and –23.5° respectively) for a person of height 1.80 m standing at G (0, 2.32).

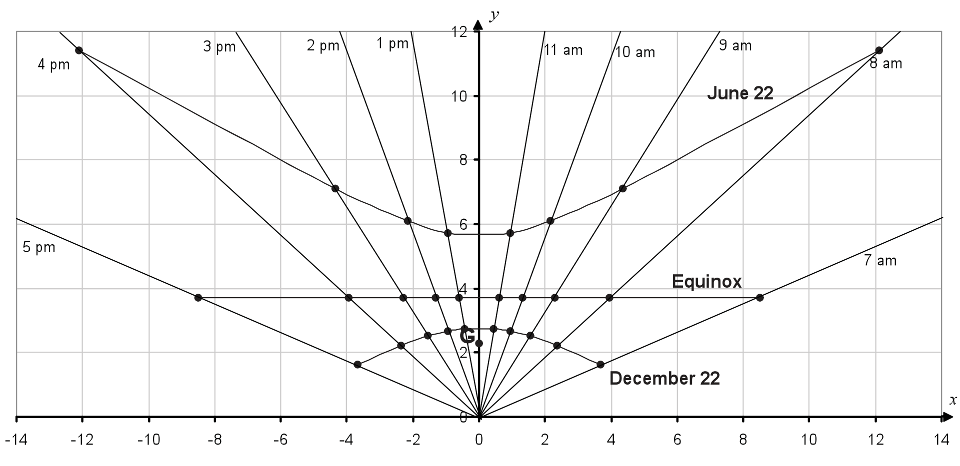

Figure 17. Solaris hour lines and the positions of the shadow tip during the day for the equinoxes and the summer and winter solstices.

The time indicated by *Solaris* is the solar time at Piazza Italia. Melbourne's longitude is approximately 5° west of the Australian Eastern Standard Time (AEST) meridian. If the Earth rotates 15 degrees every hour, 5 degrees corresponds to 20 minutes, that is, local solar time in Melbourne is approximately 20 minutes behind AEST, so 20 minutes must be added to the time indicated by *Solaris*.

Analemmatic sundials

A further type of sundial is the analemmatic dial, which has a horizontal dial and a vertical gnomon. Like the horizontal sundial, the analemmatic sundial is derived from an equatorial sundial by projecting the equatorial ring onto a horizontal plane, but it is an orthogonal projection onto the horizontal plane rather than in the direction of the Earth's axis. Examples of interactive analemmatic sundials are to be found at the Mount Annan site of the Royal Botanic Gardens, Sydney and at Torquay in Victoria.

Corrections to be applied to sundial time

Reference has been made already to the need for correcting sundial time for the local longitude if that differs from the longitude on which local time is based. Sundial time also needs to be adjusted slightly for two further reasons:

- the Earth's path as it revolves around the Sun is an ellipse, rather than a circle and the Sun is not quite at the centre of the ellipse.
- the Earth's axis is tilted at an angle of 23.5° to the plane of the elliptical orbit.

References

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