Exploring the possibilities of using

# tic-tac-toe to think and communicate about mathematics

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Doing mathematics, and thinking about how you are doing it at the same time, are not the easiest things to do. It is even more difficult if students are not aware that they should be attempting both processes at the same time. They are likely to concentrate on the immediate task of "doing" the mathematics, rather than trying to access the deeper process. Yet it is this deeper process that is really at the heart of mathematics. In turn, accessing this deeper process requires in part some command of the appropriate rational/logical language so communication with yourself and others can proceed effectively and efficiently. This article discusses the possibilities of using students' explorations of the traditional strategy game "tic-tac-toe," and some extensions, to set up situations for students to discuss and examine this process.

A possible way of examining this deeper process of "reflecting on how you are doing mathematics" is to create an analogous context. Of course, asking students to change contexts adds another potentially confusing element. Hence the teacher will need to have clear links that allow the students to see that what they are doing in one context applies to the other analogous context. Hence the student will be solving a strategy game, but also will be thinking about *how* they are solving the strategy game. By analogy, students will appreciate that the solving of mathematical problems will parallel their attempts at solving the strategy problem. Crucially, also by analogy, their discussion of how their solution process progresses for the strategy game will unveil some of the deeper processes at the heart of mathematics.

In a strategy game context, many students will have the notion of an overall general strategy, and some particular variations if special circumstances arise. Hence one starts with "common knowledge" that the students already have, a good place to begin.

A creative decision for the teacher is choosing the particular game or games to analyse. This is the same type of process for the teacher, and just as important, as when they choose a particular and graduated set of mathematical problems for students to attempt. Such mathematical problems should enable the accessing of the deeper strategies, as well as the specific content. Likewise, the game analysis should allow students to latch on to appropriate game strategies so these can be discussed easily. Some games that may be of use for this are those that have easy variations for which strategies can be identified with little difficulty. The games can be gradually modified so more complicated versions emerge, which in turn means that the beginning strategies need to be reanalysed and changed as well, and so the process progresses.

Just playing the games is not sufficient, although necessary, just like doing the mathematics is not sufficient. The process of playing the game needs to be thought through in a collegial manner by the players. Certainly, they need to be prepared to debrief their playing of a game at its conclusion. More importantly, they need to discuss the possible options in trial games when they are exploring and learning how to play. This means a certain cooperative attitude needs to be built up, as well as a command of the appropriate language with which the players can communicate effectively and efficiently about the playing of the game.

## The simple version

One such game that seems useful to explore is "tic-tac-toe," also called "noughts and crosses." There is the common version of this game that most students know by the time they are in their fourth year of school. It is played on a simple two-dimensional playing mat in the form of a 3 by 3 grid (Figure 1) which divides the playing mat into nine cells. Two students take turns placing an "x" or an "o" in any open cell. The winning player is the first to have three crosses or three noughts in a row, column, or diagonal. It is not all that difficult for such students to discover that there are in fact definite ways that this game can be played that are advantageous.

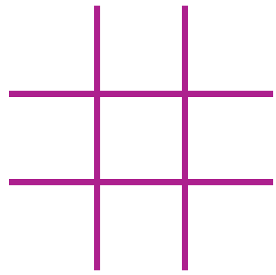

Figure 1. Playing mat for tic-tac-toe.

What teachers sometimes overlook when students play this game is to encourage students to analyse their playing strategies, and further, to verbalise these strategies to themselves and to others. Hence it becomes clear to the group why some strategies for playing are advantageous. For example students soon learn that playing in the centre square can be a decided advantage. They may come to this conclusion by observing who wins after playing many games. But unless they are encouraged to think through why this is an advantage, they may never understand it has something to do with how it restricts many possible plays of their opponent, and paves the way for them to play in optimal cells.

Often students wish to concentrate on obvious matters, such as the position of playing the next piece, which of course is also still crucial, if you are to win. Part of the strategy of getting students to articulate their reasoning should also include aspects that students might overlook. For example, whether it is an advantage to go first or not, and why or why not. After all, going first is part of the decision process, even if the players agree to randomise this choice in some way by, say, tossing a coin. Other important factors of the analysis to be drawn out in discussion include the need to recognise the symmetry of the playing mat. This means that many plays that might at first look different really are the same, if the playing mat is rotated by a quarter or a half turn. This notion of equivalence of relationships is crucial in understanding mathematics.

It also can be useful to have the students think carefully not only about the playing strategies, but also about the underlying rules of the game. For example knowing how to start, and knowing when the game is finished, are obvious but crucial factors, as they are in solving mathematical problems. In doing mathematics being able to start a problem by recognising that a particular set of conditions has to be met is important. Perhaps more important is knowing when you have finished a problem. Many teachers can recall students who have arrived at an appropriate solution, but do not recognise this and have continued to work, often with growing frustration. This can be seen for example with quite simple algebra problems when students do not recognise equivalent expressions.

Another crucial rule in strategy games that has a parallel in mathematics is knowing what sequential moves are allowed. In tic-tac-toe this is quite simple in that each player takes alternate turns. The other simple rule here is that the marker or counter needs to be placed on the mat. One analogous mathematical process to this is knowing the next step in a numerical algorithm, which may be for example moving from the tens to the hundreds column.

To emphasise these rules and the bearing they have on the game, it can be instructive to allow students to change them as they wish, although the players should mutually agree on such changes before a game starts. For example, instead of taking turn and turn about, students might choose to take two turns at once and then swap. This normally does not eventuate in a very interesting game. Another possible scenario to explore, which does add interest, is not to start with the normal playing mat. The players agree to define the playing mat as a 3 × 3 grid, but let the grid emerge as a result of the first few plays. Figure 2 shows how the first four counters could be played, but the playing mat has not yet been fully defined. The three columns of the grid are now known, but whether the third row is above or below the crosses and the noughts that have been played, is still to be determined. On the fifth play, although not obliged to do so, the first player using "x" could define the final format, and probably should do so, otherwise they will, in all likelihood, lose the game. I leave it to you to determine the better position to place the next "x."

Figure 2. The undefined playing mat.

This notion of altering the rules and seeing whether an interesting game does eventuate has a parallel in mathematics. At a fundamental level the procedures, algorithms, etc. that students often rote learn are actually accepted conventions within the discipline of mathematics, because they are useful to mathematics. This is quite a different approach to that which sees the "laws" or practices of mathematics as if they were cut into stone and are immutable. Mathematical conventions are chosen, because "it makes sense in some way to do so," by the practitioners within a global framework of ideas that define what we mean by mathematics in our cultural context.

During the playing of the game, students will hopefully be talking, rather than silently trying to just win. It is in this discussion between players, which is the externalisation of their joint thinking, that they articulate to each other possible moves and the veracity of them. In doing so, they finally come to grasp some of the underlying principles of the game. Attention needs to be paid to this talk. Of course, plenty of the language that is used in this context will refer to spatial ideas, as well as talk that indicates that the students are thinking visually. This is as it should be. However the students will, or should, be making frequent use of logical connective words such as "but," "if," "however," and "not," and combinations such as "if…

then…” This type of vocabulary is at the heart of efficient logical, rational mathematical discussion (Zepp, 1989). It is language that does not get much, or in some places no, emphasis in the mathematics curriculum (Clarkson, 2003). But teachers so often readily assume that students can use this language. Strategy games like tic-tac-toe, played in a collegial manner, should be wonderful opportunities to encourage students to use this type of language, and to ensure they are doing so accurately.

## An extended version in 3-D

Another way in which students can examine some of the underlying issues of mathematics is to play more complicated versions of a strategy game they know well. In the extended versions, the rules, plays, and general procedures need to grow out of the simpler version. However there needs to be sufficient changes so that students are challenged to re-examine what they have been doing in that simpler version. Hence in constructing more complex games, there is the opportunity for creative teachers and students to examine more closely underlying principles, and use more intensively the rational/logical language of mathematics. In mathematics this happens when students move from just working with positive whole numbers to integer and rational numbers. In doing so some laws and procedures do not alter. But some do and others are added. It is the same when students move from the realm of arithmetic to the beginning of algebra.

For the game of tic-tac-toe, another version can be played with two sets of cubes. Two-centimetre wooden cubes work well with one set painted. The playing frame this time is not a 2-D playing mat, but a 3-D imaginary cube with dimensions 3 by 3 by 3. Thus there is a possibility of playing 27 cubes. The easiest version is to actually start with a 2-D playing mat, which clearly defines where the base of the playing frame is (Figure 1). Each player takes turns to place a cube where they wish (see Figure 3). The final object is placing three cubes in a straight line. The lines can be vertical, horizontal or on any of the diagonals.

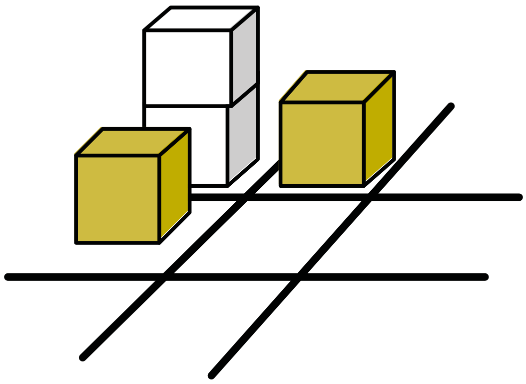
Figure 3. A 3-D game of tic-tac-toe after two plays by each player have been completed.

Not all positions are available for use immediately. In particular, players cannot suspend a playing cube in mid air. Such a position can only become available if the underlying cube has already been played, which in turn would allow a player to place another cube on top of it: hence the name for this version, “On-top-tic-tac-toe.” This notion embedded in the rules of the game, of only allowing certain plays if other conditions have been met, is analogous to a number of situations that occur regularly in mathematics. Such comparisons can be instructive for students.

Interestingly for the “On-top” version of the game, there does not seem to be a winning strategy. However there are best moves that can enhance the possibility of victory. One of these is to try and play one of your cubes in the centre of the playing frame. Again the notion of a “best” play, without any assurance that this will inevitably lead to success, is mirrored in solving good mathematical problems as opposed to practice examples. It is best for students to reflect on the notion that in solving mathematical problems there are not always foolproof procedures that will lead them always to the correct answer.

As in the 2-D version, the notion of students altering the rules can lead to interesting challenges, enhancing the notion that some rules only lead to trivial results, while others can set up intriguing situations. Following the above suggestion in the 2-D game of discarding the playing mat at the beginning, and defining the playing frame as play proceeds, leads to very challenging games since from the beginning of play so many more possibilities for play are available. Another interesting version is to change the end rule. This time the first player to have three cubes in a straight line loses the game. A simple parallel in mathematics of changing the "playing rules" is that of changing the base of counting numbers from 10 but keeping all other "rules" in place. Some older teachers will recall stories when multi-base arithmetic was in vogue, and how some students were taught each "new" base system, as if it were quite different to all the others. Little effort was given to explore the underlying generalities across bases. In such a discussion it can become quite clear that such a change to a binary base was crucial for computer technology. However the use of many other bases, although quite logically possible, has not proved to be useful or beneficial in any clear manner for most people.

## A harder 3-D version

To change the game to a yet more complicated situation, but one that links directly to the easier version, "Double-on-top-tic-tac-toe" can be introduced. In this version, the same scenario as "On-top" is used, but the playing pieces are changed. In this version, two cubes, one of each colour, are glued together to result in a playing piece which when played, places a cube of your own colour into the playing frame, but at the same time a cube of your opponent is also being played (see Figure 4).

Again adaptations of the different variations of play used in the earlier 2-D and 3-D versions of the game may be tried out in this new context. This is something that is often a useful way to begin a new mathematical problem. If the problem looks reasonably similar to one that has been completed before, then try to use the same procedure that was successful in the past. If it does not work, then start trying a variation of that approach. Finally, if that fails to produce progress, start looking for a completely new approach. It would be useful to underline this point of similarity in the different contexts, both for game playing and mathematics.

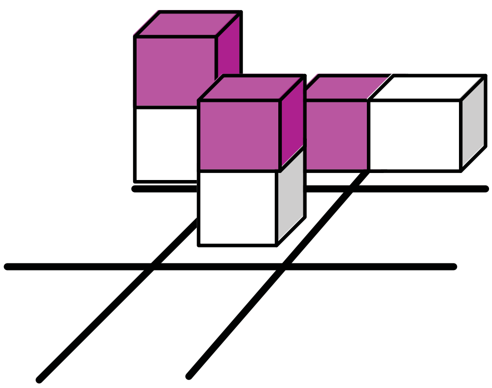

Figure 4. The state of play after the third play in "Double-on-top-tic-tac-toe."

Interestingly in this new version not all things follow smoothly from the earlier 3-D situation. Most players believe that winning the centre position of the playing frame is no advantage in this new game. In fact it can be a positive disadvantage. It also seems that the more enjoyable variation of this playing situation is to have as the goal, to force your opponent to have "three in a row."

## Summary

Playing games in class can bring fun into learning. It can also give a break from normal routines, which teachers and students do need from time to time. The game situation is often so much less threatening that it has immediate appeal to students — but that of itself can lead to confusion on the students' part. They sometimes can think that playing games is only that, playing games. However, playing strategy games is not just a break from "real work." Justifying playing strategy games because they just might "help students learn how to think" is also not good enough. These situations can be used by creative teachers as wonderful opportunities for students to think deeply about mathematics and how it works. If the students are going to experience productive "cross-over" thinking between the analogous contexts of strategy games and mathematics, they need to be asked to think explicitly about the similarities and differences. Such thinking is rarely self-generated by students. In this way teachers will certainly be addressing those elements of the curriculum that at times are seen as crucial but difficult to deliver. For example, in the Victorian curriculum, the section "Working Mathematically" would probably be in this category, and for Queensland curriculum addressing the headings of "Knowledgeable person with deep understanding," "Complex thinker" and "Effective communicator"are not always seen as easily done. There are such sections in all the other state and territory mathematics curricula.

The use of strategy games in a teacher's repertoire should occur as a natural part of the style of teaching mathematics; just like problem-solving should be a characteristic of teaching, not a specific topic area. Hence the teacher should have other ways, as well as this one, of exploring how to think about what mathematics is and its deeper processes. Explicit discussion with students about these notions should be encouraged from time to time. This will also mean encouraging students to use metacognitive strategies to explore their own ideas on the nature of mathematics.

In this article the game of tic-tac-toe has been used as the example. There are other games that can be used this way, including such games as Hex (Clarkson, 1984). However the crucial factor in these situations is for teachers to know that they are creating a context in which students can think more deeply about the nature of mathematics. This will only happen if the students use the appropriate rational/logical language that is at the heart of so much mathematical communication and of course, if they enjoy the experience of playing the game.

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