GOOGOLS and Infinity

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intrigues us all with his tales of very large numbers.

An interesting task for young children — and other people — is:

> Write the largest number that you know.

Incidentally, the obverse of this task — write the smallest number that you know — is also very interesting. This is likely to result in either a declaration about some number being as close as possible to zero, or else being about the obvious negative-number equivalent of the largest-number answer. Many young children even have some intuitive sense of the concept of 'infinity' as the biggest possible number and paradoxical additions such as 'infinity + 1 = infinity' and 'infinity + infinity = infinity'. Other children may have heard of a 'googol', and believe that it is the largest number possible.

In practice, how this word might be spelled by a child (who has only heard the word), or by an adult (who may have read about it and confused it with the modern Internet search-engine), and what it really means, or who invented it, is unlikely to be known. Currently the 'googol' is also likely to be conceptually mixed up with the deliberate word play inherent in the fanciful name for the popular Internet-search service Google.com.

Credit where credit is due: the 'googol' was popularised by Edward Kasner and James Newman in *Mathematics and the Imagination* (1940, pp. 30–35).

When discussing the idea of large numbers with kindergarten (that is, first-year primary school children), and considering the number of raindrops that would fall on New York in a rainy day, or the number of grains of sand on Coney Island, Kasner and Newman were told of a large number that could be written as a 1, followed by one hundred zeros (pp. 30–31). Later they add that the person who called this kindergarten child's number a 'googol' was Kasner's nine-year old nephew (p. 33). In a witty footnote Kasner and Newman also state that the name is *not* related to the Russian novelist Nicolai Gogol.

Perhaps surprisingly, the idea that there is a funny name for the (arguably?) largest imaginable number seems to satisfy most people. They hear about this bizarrely named monstrous number, and forget the possibility of there being larger numbers.

Obviously there are numbers larger than a mere googol. In fact this same famous Kasner nephew offered the 'googolplex', which is defined to be a 1 followed by a googol of zeros (p. 33). However, more recently, most adults and children who are aware of googol-like numbers, are likely to remember only the googol. As a number, it, and the googolplex, seem to be drifting into the status of urban legend, not often known, and rarely understood, except as a mysterious number-word.

Interestingly, Kasner and Newman point out that a googolplex is much larger than a googol, and larger, also, than a googol times a googol. They note that there are 100 zeros after a googol, so there must be 200 zeros after the 1 in a googol squared, or a googol multiplied by a googol. (You may like to think about, or investigate, the way multiples of 10 multiply together, and the rule for combining the strings of zeros, and also the reasons for the rule.) Importantly, Kasner and Newman stress that googols and googolplexes are *finite* numbers.

Is a googolplex larger or smaller than a googol raised to the power of a googol? (i.e. googolplex < or > $\text{googol}^{\text{googol}}$?). Consider the patterns we get with zeros when we raise numbers to powers of themselves: 10^{10}, 100^{100}, 1000^{1000}, and so on. If we could count aloud for long enough, starting at zero, we would eventually count past the first million, and then past the million-million, and sooner or later (later, rather than sooner) count past the billion-billion, and, so, eventually, we would reach a googol (the first googol). Then as we continued counting we would eventually reach, and subsequently, count past a googolplex.

Finite numbers, by definition, are those we can reach, at some time, when we count — even if it takes an amazingly long time to do the counting. (If we allow one second to count each number — which is at least a uniform rate of counting, even if it is humanly impossible — we can calculate, in years, how long it would take us to count to a googol. You might like to practise by calculating how long it would take to count to a billion, counting one number each second.)

> To count is to talk the language of [finite] number. To count to a googol or to count to ten is part of the same process… The essential thing to realize is that the googol and ten are kin, like the giant stars and the electron. (Kasner & Newman, 1940, p. 36).

If we have a collection of everyday objects, such as all my toy cars, or all the trees in a hectare-sized pine plantation, or all the molecules of H_2O in a five litre bucket, or all the atomic particles in the universe, we can count the items of the collection, one by one. When we have finished counting, the last number we said is defined to be the *cardinality* (or numerical-

size) of the collection we just counted. This is easy, if sometimes time-consuming, or necessarily hypothetical, with finite collections.

Infinite numbers and infinite collections are different. The first infinite number we can conceive is the number that specifies *all* possible (finite) counting numbers. Imagine that we count forever:

0, 1, 2, 3, 4, 5…

Think about this process, and try to grasp the possibility of thinking of the entire collection of all these finite numbers as one single gathered-together-whole (and infinitely large) 'thing' — *the set of all whole numbers*.

Clearly the collection of counting or whole numbers is infinite.

How many finite numbers are in the collection? The answer is, an 'infinitely countable' quantity of them. This means that our attempt to count them can be described, but continues forever. Think of the numerical-size of the infinite collection of them as a 'number'. Then consider that *that* many, the number or numerical-size (cardinality) that you are thinking about, is the first infinite number. This is sometimes written as a figure-eight, lying on its side: ∞.

> There is no point where the very big starts to merge with the infinite. You may write a (finite) number as big as you please; it will be no nearer the infinite than the number 1 or the number 7. Make sure you keep this distinction very clear and you will have mastered many of the subtleties of the transfinite (Kasner & Newman, 1940, p. 42).

If anyone reading this objects to my discussion, and says that this first infinite number is more of a vague idea, rather than a specific number, remember that any finite number we care to consider is equally no more than an idea!

The first non-zero whole number, whether represented as a spoken word, or recorded as 'one' or as a (Hindu-Arabic) numeral '1', is an abstract concept. It is a 'brain-meaning' that occurs in a human brain. It is the abstract conception of 'one-ness', the cardinality of any set that contains one item. This idea of 'one' may be modelled (represented) by, but is not literally, any single object or event, such as an apple, or a clap of thunder.

A great deal can be, and has been, said about infinity, and infinite numbers. Only two important and interesting points will be made here. First, the familiar, but conceptually daunting collection of all the positive whole numbers (the counting numbers, 1, 2, 3…) contains the equally familiar set of the even numbers (2, 4, 6, 8…). How do we know this?

It is easy to grasp that the even numbers are *half* of the collection of *all* the positive whole numbers; but does this mean there are half as many even numbers as there are whole numbers?

At first the suggestion that half of something is as big as the whole thing seems counter-intuitive. How can half of some collection have as many bits in it as the whole collection itself?

We can use our ordinary counting numbers to count the even numbers:

- 1st even is 2,
- 2nd even is 4,
- 3rd even is 6,
- 4th even is 8, …and so on.

For each even number there is a counting number that describes the even number's position in the ordered sequence of evens. Equally, every counting number is matched alongside an even number. This is a simple algebraic pattern: the n-th counting number corresponds in our process of counting to the even number $2n$.

We know, because we said so, that there are infinitely many positive whole (counting) numbers, so this argument shows there are infinitely many even numbers. That is, a part (a 'half') of the entire collection of positive whole numbers is as numerically large (as countably large) as the entire set itself!

Obviously this does not happen with finite numbers, or collections. A half of a finite collection has an ordinary half as many members in the collection as the entire collection does. Infinite sets are different — very different!

Suppose we start with our infinite set of counting numbers, starting with 1:

1, 2, 3, 4…

Now consider the same set, expanded by adding two more numbers at the beginning:

-1, 0, 1, 2, 3, 4…

Can you show how we can restart our counting of this expanded set, counting -1 as the first number we count, then 0 as the second, and so on?

Next consider doubling our set of whole numbers, but placing whole-plus-a-half between each successive pair of wholes:

0, 0.5, 1.0, 1.5, 2.0, 2.5…

Can you demonstrate a way of counting this set that contains *twice* as many numbers?

Hence we can show that:

$\infty + 2 = \infty$

$\infty + \infty = \infty$

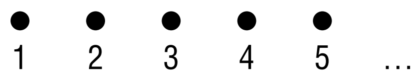
$\infty \div 2 = \infty$

Second, without demonstrating the brilliant proof (which was created by Georg Cantor in the late nineteenth century), it can be stated that:

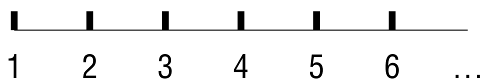
- we can count the positive whole numbers;
- they are *countably infinite*;
- the 'numerical' size of the infiniteness is ∞; *but*
- the *real* numbers cannot be counted!

These so-called 'real numbers' can be thought of as all the possible decimal numbers we can imagine, including some of the mysterious ones such as $\sqrt{2}$ and π, where the decimal parts can be proved to continue, without pattern or repetition, forever — these are *infinite* decimals!

Another way of describing the real numbers (or at least the positive real numbers — an equivalent argument can be used to discuss the negative reals as well), and to contrast them with the whole numbers we use for counting, is to use a row of numbered dots to represent the whole numbers:



Contrasted with this we can use a complete number-line (geometrically this is usually called a 'ray', because it has a specific starting point, but continues as an unbroken line in an infinite direction) to represent the set of positive real numbers. We place dots or upstrokes along the line to represent each of the whole numbers. Each of these whole numbers, as well as each point on the line between successive pairs of whole numbers (that is, we are now including all the joined together geometric points between each pair of upstrokes or dots), represent all the reals that lie between each pair of whole numbers.



This forces us, once we grasp the meaning of Cantor's argument, to accept a second infinite number. This second, larger infinite number is so much larger than our first countably infinite infinity of whole numbers, ∞, that it is an *uncountably infinite* number!

There are also other infinite numbers that are larger still, beyond these first two infinite numbers: they are called the *transfinite* numbers. There is more to bigness in numbers than mere (massively finite) googols!

References and further reading

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APMC

Solutions to famous women in mathematics and technology puzzles

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