From HERE to THERE

The path to computational fluency with multi-digit multiplication

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highlights the critical links between number sense and a child's fluency with mental and written computation when learning how to perform multi-digit multiplication.

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Drawing upon research, theory, classroom and personal experiences, this paper focuses on the development of primary-aged children's computational fluency with multi-digit multiplication. Getting children from "here" (current strategy use) to "there" (a more efficient strategy) is often not a straight-forward path. The critical links between number sense and a child's ability to perform mental and written computation with ease are examined.

Many readers will know the story of the famous mathematician Johann Carl Friedrich Gauss (1777–1855). As a young boy he was prone to daydream in class. One day his teacher decided to punish him for not paying attention. He was asked to add all the numbers from 1 to 100. Much to the annoyance of the teacher, young Carl was able to derive the correct answer in seconds. Fortunately for Carl, he knew a short-cut. He realised that adding pairs of numbers (e.g., 1 + 100, 2 + 99, etc.) all equalled the same number: 101. He figured that there were 50 such pairs, so calculated the total to equal 50 × 101 or 5050.

Recently I related this story to a group of primary school teachers. One teacher immediately asked, "But who taught him that?" This question sparked a discussion about the critical relationship between a person's understanding of mathematics and their computational fluency. The teachers agreed that Carl's in-depth understanding of mathematics enabled him to see patterns and relationships that made the computation more manageable, but that his knowledge of basic facts and the fluency with which he could compute were equally important. The teachers concluded that understanding without fluency can inhibit the problem solving process.

This paper focuses on the development of primary-aged children's computational fluency. It emphasises the critical links between number sense and a child's ability to perform mental and written computation. The case of multi-digit multiplication is used to illustrate these important links.

Computational fluency: Number sense and the standard algorithm

The idea of teaching mathematics for understanding and for meaningful learning to occur has been advocated for over half a century (Brownell, 1935). However, it was not until the 1980s that the term "number sense" was first used to refer to those who had a deep understanding of numbers. The focus on number sense is manifested in the recent and ongoing emphasis in international curriculum and policy documents on mental computation (e.g., Australian Education Council, 1991; National Council of Teachers of Mathematics, 2000). Research has shown that those who are good at mental computation possess a well-developed sense of number (McIntosh & Dole, 2000).

The increased emphasis on mental computation and number sense has seen a corresponding de-emphasis in curricula on standard algorithms. An algorithm is a specified multi-step procedure that produces an answer for any given set of problems and is characterised by long-term practice. While still recognised as important, some Australian state syllabus documents have delayed the introduction of standard algorithms for around two years to allow a focus on mental strategies for as long as possible (e.g., Board of Studies, New South Wales [BOSNSW], 2002). The worry with an early emphasis on standard algorithms is that students will shift their focus to executing convenient procedures rather than on understanding the mathematics.

A concern is that educators will view the development of number sense and fluency in written and mental computation as separate bodies of knowledge requiring separate instruction. In fact, computational fluency, whether employing mental or written methods, and number sense are intertwined and should be developed together. The aim of the following sections is to examine how children develop proficiency in their computational methods while instruction remains focused on learning with understanding.

Understanding the development of children's strategies

While a number of research-based "frameworks" provide excellent descriptions of learning pathways by which children's computational strategies develop, they fail to tell us about how children progress to use a more efficient strategy in preference to another less efficient one. It is imperative that teachers understand how children make this shift.

As children become more competent mathematicians, they develop a variety of thinking strategies for solving mathematical problems. Generally, children initially apply basic counting strategies to help them solve simple numerical problems before moving onto using more complex non-counting strategies. While the strategies that develop usually become more sophisticated as children learn more efficient ways of doing mathematics, it is now well acknowledged that at any one time, a child will use a multiplicity of strategies and that often these strategies will not be the most efficient ones a child is capable of performing. Such inefficient strategies persist because while they may be slow, they eventually yield the correct answer (Gould, 2000). When a child is placed under some form of cognitive demand, such as an imposed time limit, mental fatigue or even boredom, they will often revert to a less sophisticated strategy that they know well and can perform with minimal effort. A nine year-old explained this to me once while I questioned her about her strategies for addition:

> I know when I just have to add a small number — say five or less — then its fast for me to count by ones. But if it's 20 or 30 to add, then I will stop and think of a better way that does not use just ones because I know it will take me too long to count that many. Sometimes I just want to count by ones because it's too hard to think of another way.

I learnt from this little girl that children modify their strategy use according to at least two things: the demands of the mathematical problem and the limitations of their knowledge. Another influence on children's choice of strategy that I have observed during my time in schools is that of textbooks or even teachers themselves. In the attempt to introduce students to a variety of mental and written methods, instructional material may overemphasise or specify the use of a particular strategy or scaffold (e.g., the empty number line) when students are already working beyond what is specified (see Bobis & Bobis, 2005). The challenge for teachers is to encourage the development of, and consistent use of, more efficient and appropriate strategies for solving mathematical problems without it being "too hard" for children. To do this, it is imperative that teachers not only understand what these strategies are, but how a more efficient strategy becomes a student's preferred strategy even when placed in a stressful situation. The diagrammatic representation of Siegler's (2000) overlapping wave theory has helped further my own understanding of how this can be achieved (see Figure 1). I have shared Siegler's theory with practicing and prospective teachers and found it beneficial in explaining how a more efficient strategy can become a child's preferred strategy.

Siegler's (2000) overlapping wave theory is based on three assumptions: (1) children typically use a multiplicity of strategies to solve a single problem; (2) less and more efficient strategies may coexist over prolonged periods of time and not just for short periods of transition; and (3) the relative reliance on existing and more efficient strategies can be changed given appropriate experiences. The first two assumptions are represented diagrammatically in Figure 1. The third assumption is addressed later in this paper.

It can be seen from Figure 1 that at any one point in time, a student may use a range of strategies. However, the relative frequency with which particular

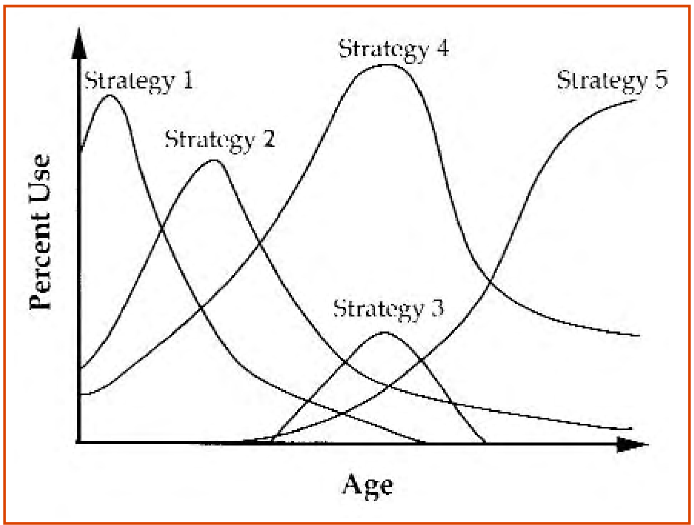

Figure 1. Diagram representing Siegler's overlapping waves model.

strategies are used over time may vary continuously, with new strategies becoming more prevalent and some more inefficient strategies stopping. By following the path of a single strategy, it can illustrate how some strategies will often be used for a prolonged period of time even after more efficient strategies have been introduced. This can be exemplified by a student who uses counting-on by ones to solve simple addition problems such as 7 + 2 as a 5 year-old, and who continues to use the same strategy to solve 47 + 12 as a 10 year-old. Siegler suggests that as a child progressively learns more efficient strategies they pass through four dimensions or components of change. These components range from the initial use of the strategy, which in some cases may at first be used at an unconscious level, to a stable, precise and efficient use of the strategy. The four dimensions along which learning occurs include:

1. the acquisition or introduction of a more advanced strategy;
2. an increased reliance or frequency of use of the new strategy within the set of the child's existing strategies;
3. an increased appropriate choice of the strategy; and
4. an improved execution of the more advanced strategy that can lead to it becoming increasingly automated.

While this model for strategy development is based on the assumption that children learn by doing, it is important to emphasise that simply drilling the strategies is not enough. Understanding is also crucial. We know that the greater the degree of understanding, the less practice that is required to obtain fluency and to sustain the change in strategy use. Additionally, each new strategy competes for a long time with more familiar strategies, so it may not be used consistently as their preferred strategy for some time and there may be occasions when a child seems to regress in their strategy use. In other words, getting children to move from their current array of preferred strategies (the "here" strategies) to a more efficient strategy (the "there" strategies) is not a straightforward process.

The case of Crystal and multi-digit multiplication

I first met Crystal when her Year 6 teacher asked me to assist with the development of an intervention program for a small group of students in her class. These students were experiencing difficulty with the algorithm for multi-digit multiplication and the teacher was unsure what remediation was needed. This section details the journey to computational fluency of one child from that group.

Frameworks describing developmental pathways of children's thinking strategies for addition, subtraction and single digit multiplication are now quite common (see, Bobis, Clarke, Clarke, Thomas, Young-Loveridge, Wright & Gould, 2005) and some are actually embedded into curricula (e.g., BOSNSW, 2002; Van den Heuvel-Panhuizen, 2001). However, much less is known about multi-digit multiplication. Fuson (2003) reports preliminary research that reveals children use a progression of strategies from (a) direct modelling with concrete materials or semi-abstract drawings, to (b) methods involving repeated addition, such as doubling, to (c) partitioning methods. Partitioning strategies normally include the partitioning of one number or both numbers into tens and ones or partitioning by a number other than 10.

The standard algorithm for multi-digit multiplication most commonly used in NSW primary schools requires a number of steps involving multiplication and addition. It also relies on the answers at each step being properly aligned according to their correct place value. Such alignments can be accomplished without any understanding of a number's true value. In Crystal's case, errors in her multi-digit multiplication were the result of a range of factors. The single-digit multiplication work samples in Figure 2 indicate that Crystal could efficiently solve single-digit computations when multiplying by numbers less than 7. However, she did not know all her multiplication facts from 7 onwards, thus hindering her computational fluency. This was later confirmed in an interview with Crystal. She had memorised most facts to 6×10, but seemed unaware of the commutative property of multiplication. Hence, she was unable to see that 6×8 was the same as 8×6. In addition, the work samples indicate that Crystal was not only making procedural errors when carrying, but that she had little understanding of place value when multiplying by tens. This is a very common error in students' execution of the algorithm for multi-digit multiplication and is generally a result of learning the procedure by rote. To overcome these procedural and conceptual errors, Crystal needed to understand the distributive property of multiplication.

A program of work starting with Crystal's understanding of the commutative property of multiplication was implemented by the classroom teacher. It was decided to strengthen Crystal's knowledge base of single-digit multiplication before moving to the more difficult multi-digit multiplication computations. While this initial instruction spanned a few weeks, it is the understanding of the mathematics underlying multi-digit multiplication that is my focus here. It was during our search for a strategy to help Crystal understand the underlying mathematics that the classroom teacher and I learnt most about Crystal's mathematical abilities and about teaching multi-digit multiplication via a number sense approach.

We soon learnt that if Crystal was going to develop an understanding of the distributive property of multiplication, it needed to be presented in a visual form. Early attempts to explain this property through purely abstract means (e.g., 14 × 5 = 10 × 5 + 4 × 5) had little success. Visual representations of double-digit numbers became very cumbersome and messy for

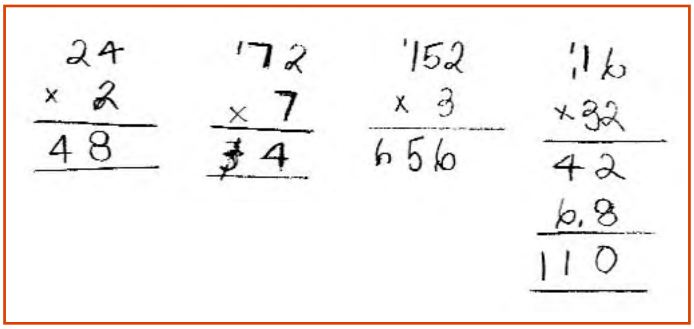

Figure 2. Examples of Crystal's single digit and multi-digit multiplication.

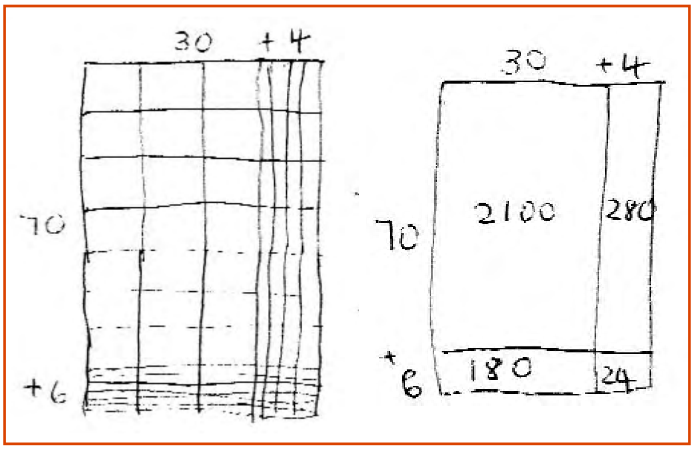

Figure 4. Array structures used to model all combinations in multi-digit multiplication.

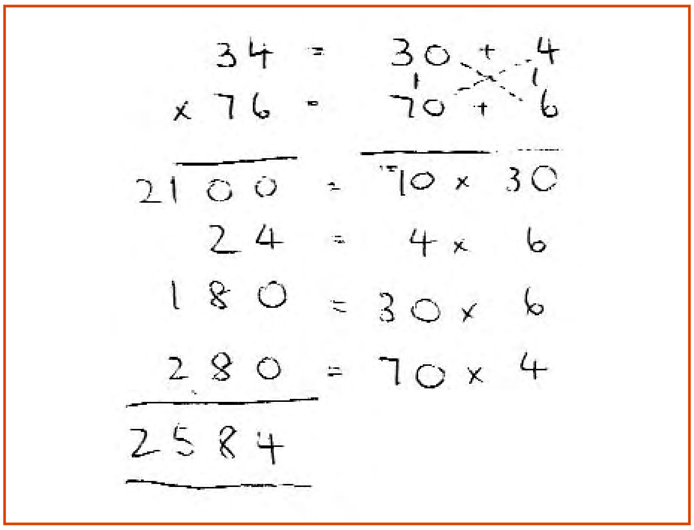

Figure 5. The distributive property is emphasised to assist understanding of the algorithm.

Crystal, thus making the learning and teaching tedious. It was at this point that we encountered a method involving partitioning of numbers according to their place value and a convenient visual model (Fuson, 2003). We started by introducing Crystal to array's incorporating tens and ones (see Figure 3 for an expanded and abbreviated model of an array). The visual representation supported Crystal's understanding of multiplying all the combinations in two double-digit numbers.

The array models scaffolded the introduction of mental strategies involving partitioning, and at the same time provided a convenient representation of the distributive property of multiplication. Within two weeks of instruction, the visual representation of the array was unnecessary and Crystal was able to record her thinking numerically (see Figure 4). As she gained more confidence with this process, Crystal eventually took short-cuts and discarded recordings to the right of the algorithm.

While this sequence of instruction was first introduced to cater for the needs of Crystal and a few other students in the class, the teacher decided to integrate the array model into her regular classroom teaching of multi-digit multiplication. After witnessing the benefits of this process of instruction the teacher interviewed more students from her class to determine their level of understanding of multi-digit multiplication. She was alarmed to find many other students implementing the standard algorithm correctly, but without understanding the underlying mathematics.

Conclusion

High levels of efficiency in computation remain a goal of our mathematics curricula; the process by which it is achieved needs to take account of how students develop a sense of number. The path to computational fluency is not a straight-forward one for most students. However, it is clear that the promotion of number sense is critical to a basic understanding of mathematics and to a child's ability to compute easily.

References

Australian Education Council (1991). A *National Statement on Mathematics for Australian Schools*. Melbourne: Curriculum Corporation.

Australian Association of Mathematics Teachers (1997). *Numeracy = Everyone's Business: The Report of the Numeracy Education Strategy Development Conference*. Adelaide: AAMT.

Board of Studies, New South Wales (2002). *K–6 Mathematics Syllabus*. Sydney: BOSNSW.

Bobis, J. & Bobis, E. (2005). The empty number line: Making children's thinking visible. In M. Coupland, J. Anderson & T. Spencer (Eds), *Making Mathematics Vital* (pp. 66–72). Sydney: AAMT.

Brownell, W. (1935). Psychological consideration in the learning and the teaching of arithmetic. In *The Tenth Yearbook of the National Council of Teachers of Mathematics* (pp. 1–31). New York: Teachers College, Columbia University.

Fuson, K. (2003). Toward computational fluency in multi-digit multiplication and division. *Teaching Children Mathematics, 9* (6), 300–305.

Gould, P. (2000). Count Me In Too: Creating a choir in the swamp. In Australian Council for Educational Research, *Improving Numeracy Learning: What Does the Research Tell Us?* (Proceedings of the ACER research conference 2000, pp. 23–26). Camberwell: ACER.

McIntosh, A. & Dole, S. (2000). Number sense and mental computation: Implications for numeracy. In Australian Council for Educational Research, *Improving Numeracy Learning: What Does the Research Tell Us* (Proceedings of the ACER research conference 2000, pp. 34–37). Camberwell: ACER.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM.

Siegler, R. (2000). The rebirth of children's learning, *Child Development, 71* (1), 26–35.

Van den Heuvel-Panhuizen, M. (2001). *Children Learn Mathematics: A Learning-Teaching Trajectory with Intermediate Attainment Targets for Calculation with Whole Numbers in Primary School*. Utrecht: Freudenthal Institute, Utrecht University/SLO.

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