Fractions as division

The forgotten notion?

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explores a construct of fractions that is very useful but often neglected.

About fifteen years ago, I discovered an interesting activity in some materials that Malcolm Swan from the Shell Centre (University of Nottingham, UK) had developed for the English National Curriculum Council in 1991. The activity, one which has been used by several presenters in professional development workshops in Australia in recent years, involves sharing chocolate in a problem solving context. Although I have seen it used in a variety of ways, I will describe one way in which I use it with teachers and middle school students:

The chocolate block task

I place three small chairs out the front of the classroom as shown in Figure 1. I explain to the group that I am placing one block of chocolate on the first chair, two blocks on the second, and three on the third. I deliberately use chocolate that is not already subdivided into separate pieces, as this would "blur" the concepts which I hope will emerge.

I ask ten volunteers to leave the room, spreading the chairs out to give plenty of room. I then invite the ten to return, one at a time, and choose a chair at which to stand, knowing that when everyone has entered the room and made a decision they get to share the chocolate at their chair. I explain to them that the assumption is that "more chocolate is better," an assumption that most teachers and almost all

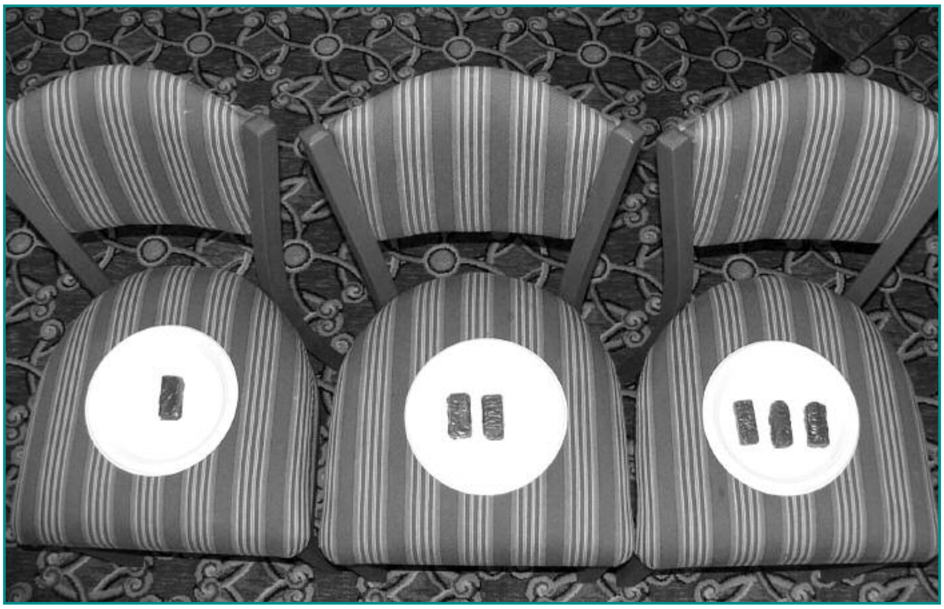

Figure 1

selves. It is possible that the last person is faced with three equivalent alternatives if there are, respectively, 5 people standing with the 3 blocks, 3 people standing with the 2 blocks, and 1 person standing with the 1 block. In each case, the person would get half of a block of chocolate wherever they choose to go.

The calculation involved in sharing the one block is, of course, relatively straightforward, as is the case where the two blocks are shared between two or four people. However, there is usually at least one situation for which most people are not able to give an immediate answer about how much each person would get. This might be, for example, two blocks shared between three people (see Figure 2) or three blocks shared between five people.

My experience is that most middle school teachers and almost all middle school students use the same strategy to share two blocks between three people. They solve the problem by “mentally breaking”

middle school students accept readily!

Interestingly, the first couple of people to enter often choose the chairs with either one or two blocks of chocolate. Possibly, they think that there is a trick involved and this is some kind of reverse psychology; or is it that they think, “With fractions, the bigger it is, the smaller it is”? (see Roche, 2005)

When we are down to the last two (Belinda and Sandy, say), I ask them each in turn to pause before entering, and ask the rest of the class, who have been observing, to decide where they think Belinda should go and why. I invite individuals to explain their reasoning, and then ask Belinda to move to where she wishes. I then pose the same question in relation to Sandy’s decision, and after a similar discussion, I ask Sandy to take her place.

I then invite the individuals or group at each chair or table to discuss how much chocolate they would finally get, and how they know. The remainder of the class is also asked to discuss how much chocolate participants at each chair would receive.

Of course, there are many different ways in which the ten people might distribute them-

Figure 2

each block into three pieces, yielding the correct answer of two-thirds of a block each. Few will explain that two things shared between three must be two-thirds of a block each, without calculating.

I then ask the total group, “How many people knew before they came to class or our session today, without calculating, that two blocks shared between three people must mean two thirds each?” Invariably, less than one-tenth of the group claim to know. This leads me to one of my main intentions in posing the task.

I believe that the notion of “fraction as division” or “fraction as quotient” is not a common construct in most people’s minds. If we understand, for example, that one meaning of 2/3 is “2 divided by 3,” then strategies in the above activity become obvious quite quickly, assuming that there are not complicated comparisons such as comparing two blocks shared between three and three blocks between five (i.e., comparing 2/3 and 3/5).

Ironically, in teaching students to convert 3/7 to a decimal, we encourage them to use a calculator to divide the 3 by the 7, thus invoking the construct of fractions as division, without possibly thinking about why. Similarly, 17/5 can be represented as the mixed number 3 2/5, and students tend to “convert” this by dividing 17 by 5 — the same principle again.

As an interesting postscript to this activity, I would like to share a powerful visual image that was suggested by a teacher in Queensland, who had just participated in this activity with me. After my debrief, he asked the indulgence of the volunteers for a moment. He asked each group to lift up their chairs with the chocolate on them. For the chair with three blocks of chocolate and five people, he showed us in a powerful visual form: the three blocks of chocolate on top; the chair which formed the vinculum of the fraction (the line dividing the numerator and the denominator); and the five people underneath. He had created a stunning and hopefully memorable image of three blocks shared between five giving three over five or 3/5. It was one of those “magic” moments, and the other teachers present and I were all most impressed.

Key constructs of fractions

In what follows, the focus will be on fractions specifically, although the research is often framed in terms of rational numbers. [Rational number is the name given to any number that can be expressed as the ratio of two whole numbers. So, for example, 0.6, 4/7 and 158% are all rational numbers, while the square root of 7 is not, as it can not be expressed in this way.]

Fractions are difficult to teach and to learn. There is a substantial research literature on key concepts of fractions and how these develop, and yet this research has had little impact on state and national curriculum documents and even less impact on classroom practice. Much of the research on fractions is contained within broader research programs in relation to the rational number system.

It is not that research has not focused on the notion of fractions as division, equal sharing and partitioning (see, e.g., Empson, 2001; Gould, 2005; Siemon, 2003). The issue is the lack of impact, to this point, of research on practice.

In recent years, the power of the one-to-one assessment interview as a professional learning tool for teachers has been increasingly recognised in Australia and New Zealand (Bobis, Clarke, Clarke, Gould, Thomas, Wright, Young-Loveridge & Gould, 2005; Clarke, Roche & Mitchell, 2005; McDonough, Clarke & Clarke, 2002).

As part of a research project, the team at Australian

Catholic University (Melbourne) has been developing a range of task-based, one-to-one interview assessment tasks in fractions and decimals. These tasks focus on the "big ideas" which underpin these topics and common misconceptions, (e.g., Clarke, Roche, Mitchell & Sukenik, 2006; Mitchell & Clarke, 2004; Mitchell, 2005, Roche, 2005, 2006; Roche & Clarke, 2004). In this research, we have been careful to try to address important interpretations or constructs of fractions.

Thomas Kieren (1976) identified seven different interpretations (or constructs) of rational numbers. Different scholars have summarised or re-framed these over the years (e.g., Post, Cramer, Behr, Lesh & Harel, 1993). For the purposes of this article, I want to focus on fractions specifically, and consider five different interpretations of them (see Figure 3).

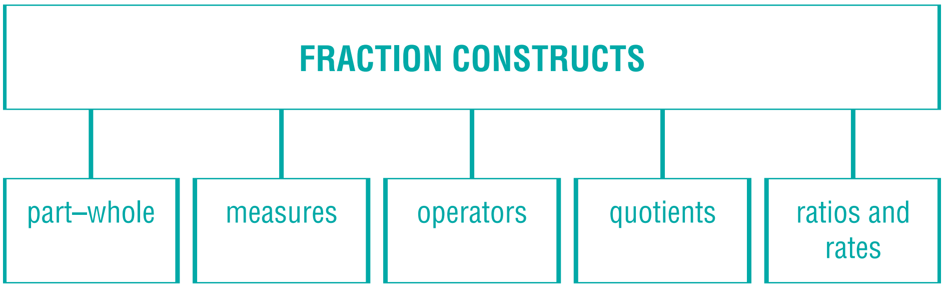

Figure 3

In introducing students to fractions, I believe that the part-whole comparison dominates the way in which fractions are presently taught and therefore learned. In focusing on part-whole, many teachers provide students with prepared wholes (sometimes discrete models such as counters, sometimes continuous models such as commercial "pies" or rectangles), and then ask them to identify and name particular parts of the whole. There is usually less emphasis within this part-whole focus on flexible movement between whole and part, and between one part and other part (e.g., if the brown is 4/3, what rod is one? If the purple rod is 1/2 which rod represents 3/4?)

In the same way that quotients (or fraction as division) may not be given sufficient attention, the notions of measure (which can be thought of as "fraction as a number that can be placed in its appropriate position on the number line with whole numbers, decimals, etc.") and operator which enlarges or reduces the size of something (e.g., determining 3/4 of 28 metres, where the fraction is operating on the 28) are often absent.

Kilpatrick, Swafford and Findell (2001) argue that from a student point of view, rational numbers (and therefore fractions) are not single entities but have "multiple personalities… the task for students is to recognise these distinctions, and at the same time to construct relations among them that generate a coherent concept" (p. 233). They note the relative complexity compared with whole numbers, and that students need to understand that the numerator and denominator are related through multiplication and division, not addition.

Students also need to come to grips with a variety of discrete and continuous models, and some are more helpful than others. Within the part-whole interpretation, circular models are extremely common, when our research and that of others (see, e.g., Moss & Case, 1999; Witherspoon, 2002) have found that this is a potentially unhelpful model, given the difficulty with which students "cut up" the circle, and interpret the result. Take a moment to consider how you would use circles to convince a middle school student of the relative size of 2/3 and 3/5, or fold a circle into thirds. A nice classroom activity which gets at the notion of fraction as operator is the "Estimation of Fractions" activity from the Mathematics

Curriculum and Teaching Program materials, which involves initially finding 2/5 of the way across the blackboard (Lovitt & Clarke, 1988).

Taking the activity a little further

In debriefing the chocolate game, I also encourage students to think about their answers to the following questions:

- (directed to the individuals at the chairs) If, at the end, you had the choice to move to a different chair, would you do so?
- (directed to the students watching the task) Where would you choose to stand in the queue? Is it best to go first or last?
- What strategies would you use if you were in the line?

I then either act out or discuss the following two situations:

1. The case where the chocolate is out of the packet and clearly already subdivided. In this discrete case, the students are now using the fraction as an operator notion, as they calculate, say, 1/3 of 24 blocks or 1/5 of 20 blocks.
2. The case where there are more blocks of chocolate at a chair than people. In this case, a context is available to discuss improper fractions, where the mixed number equivalent is either obvious or can be easily determined. For example, five blocks shared between three is 5/3, but by breaking each block into three, we can see that each person would get five thirds, which could form one and two-thirds blocks.

No doubt, the reader may have thought of other directions in which this activity might proceed, but even to this point, I would claim that this one lesson has enabled the emergence of a range of important ideas, and broadened hopefully the notion of what fractions are all about for many students.

Another problem worth posing to middle school students

A task that forms part of the interview we are developing, originally taken from Lamon (1999), which usually brings out a range of interesting responses, is the following:

We explain that three pizzas are shared evenly between seven girls, while one pizza is shared evenly between three boys. The question is posed: "Who gets more pizza: a boy or a girl?"

Students who understand the notion of fraction as division in a way that they can use it, will quickly conclude that each girl gets 3/7 of a pizza and each boy gets 1/3 of a pizza, and then think about which of these two fractions is larger. If they "benchmark" to one-half, they will usually conclude that 3/7 is larger because it is almost one-half. Not surprisingly, in light of the earlier discussion, few students or teachers use this strategy, unless they have recently played the chocolate game!

However, other more interesting methods often arise. For example, we are always pleased when a student will give one of the following two responses:

- "Well, three boys share one pizza, so I'll give the first three girls the first pizza, and the next three girls the next pizza, leaving a whole pizza for the seventh girl, which must mean [on average] that the girls get more."
- "Seeing as the boys get one-third each, I imagine dividing the girls' pizzas into thirds, which would give a total of nine thirds. As there are seven girls, there will be two thirds left over, so [on average] the girls get more."

Taking the activity a little further, students can be challenged to follow up their answer to "who gets more?" with a quantification of how much a girl and boy get, respectively. Further, how much more does one get than the other?

In an easier variation of this task, we asked 323 Grade 6 students to indicate how much each person would get if three pizzas were shared between five people (Clarke, Roche, Mitchell & Sukenik, 2006). Only 30% of students at the end of the Grade 6 year could give a correct answer to this question. 12% did so mentally, while 18% used a drawing to reach their answer.

The reader is invited to try any of the problems given in this article with individuals, small groups or the whole class, and explore their potential for assessing and developing student understanding, leading to a broader, more connected and applicable notion of fractions. As Kilpatrick, Swafford, and Findell, (2001) note, "sharing can play the role for rational numbers that counting does for whole numbers" (p. 232).

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