

Trigonometry from a **DIFFERENT ANGLE**

Michael Cavanagh

Macquarie University

<michael.cavanagh@mq.edu.au>

I read with interest the article on teaching trigonometry recently published in *The Australian Mathematics Teacher* (Quinlan, 2004). The article reports on a lesson given by a student-teacher in which the pupils were involved in a practical activity designed to introduce the tangent ratio and demonstrate its utility in some real-life contexts. Quinlan (2004) concludes with some general principles for introducing new mathematical concepts, ideas which he was fortunate enough to have learned when he completed his teacher training in the 1950s. The author also suggests that teachers begin by allowing students to explore concrete examples of a concept before presenting its definition, and that the formal terminology and symbolism associated with the concept should be introduced much later, after students have developed a sound grasp of the basic ideas.

Re-thinking classroom practices

My recollections of the mathematics methodology subjects I undertook in the early 1980s are quite different. I remember being encouraged to adopt a very expository style of teaching in which each new concept is introduced by its formal definition. The teacher should then explain a few carefully chosen examples for students to copy into their books, and then provide plenty of graded practice exercises from the textbook for students to complete. It is what Mitchelmore (2000) calls the ABC approach: where abstract definitions are taught before any concrete examples are considered. So, for many years, my teaching of trigonometry in Year 9 began with exercises in identifying opposite and adjacent sides in right-angled triangles, definitions of the trigonometric ratios and the mnemonic SOHCAHTOA, then lots of work on calculating unknown sides and angles, all devoid of any realistic context. Finally, right at the end of the topic, I gave the class some word problems involving applications like angles of elevation and compass bearings.

It was only when I undertook further study some years later and was exposed to alternative ways of thinking about the nature of mathematics and its pedagogy that I began to reassess my classroom practice. There was

no blinding light or sudden conversion but, over time, I did make some significant changes in my teaching. In my trigonometry lessons this meant not following the textbook so slavishly, changing the order in which students tackled the basic ideas associated with right-angled triangles, and reconsidering the kinds of classroom activities I provided for students. I was also mindful of the *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2002) and the advice on professional practice in Domain 3. In particular, I wanted to use a variety of teaching strategies and try to take account of students' prior mathematical knowledge. The purpose of this article is to outline briefly some of the elements of my new approach and how I developed them.

Introducing the ratios

First, I thought it important for my Year 9 students to understand that “sine”, “cosine” and “tangent” are *ratios* whose value depends on the relative size of the sides in a right triangle. I used a diagram like Figure 1, found in many textbooks, and asked the students to measure BF , CG , DH , and EI , the lengths of the sides opposite the marked acute angle, θ . Then the students measured AF , AG , AH and AI , the lengths of the hypotenuse in each triangle. Finally, I asked the students to divide the values for each of the opposite sides by the hypotenuse in $\triangle ABF$, $\triangle ACG$ and so on, until they obtained approximately the same value in each case, and so I was able to explain that they had found the sine ratio! This was not a very auspicious beginning at all and the students were unconvinced by my explanation but they accepted it and we moved on to repeat the process for the two remaining ratios. In hindsight, this approach was still too abstract and provided no rationale for measuring those particular sides to obtain the three ratios. In fact, I am not even sure that students actually see a series of separate triangles ($\triangle ABF$, $\triangle ACG$, $\triangle ADH$ and $\triangle AEI$) in a diagram like Figure 1 because the shapes are superimposed on each other. I needed to find another way.

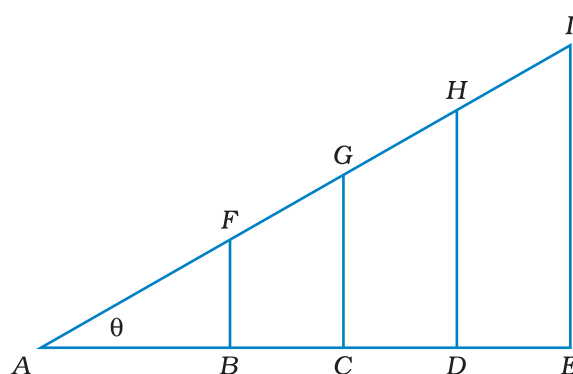


Figure 1. Diagram to show constant values for the sine ratio.

Trigonometry and coordinate geometry

Before teaching trigonometry the next year with my next Year 9 class, I started to think about other ratio contexts familiar to students and began to focus on gradients of straight lines. Prior to learning about trigonometry, students have typically done some basic work on coordinate geometry and are familiar with gradient as the ratio of “rise over run”. They also know that

the gradient of a straight line is constant, so any two points on the line can be used to determine the gradient ratio and the result will always simplify to the same value. This appeared promising, but first the students needed to link the gradient of a line and its angle of inclination. So I prepared a worksheet on 2 mm grid paper showing various straight lines, all leaning to the right, and in the first trigonometry lesson I asked the students to find the gradient of each line and to measure the angle it made with the direction of the positive x -axis as another way to describe the steepness of the line. At this stage, I just wanted the class to notice that the value of the gradient and the size of the angle increased and decreased together and that each measure provided a reasonable way of expressing the slope of the line. Figure 2 shows a diagram that summarises the elements of this approach. Most students measured the angle θ in the position where it is shown in Figure 2, though some chose the corresponding angle formed between the straight line and the x -axis.

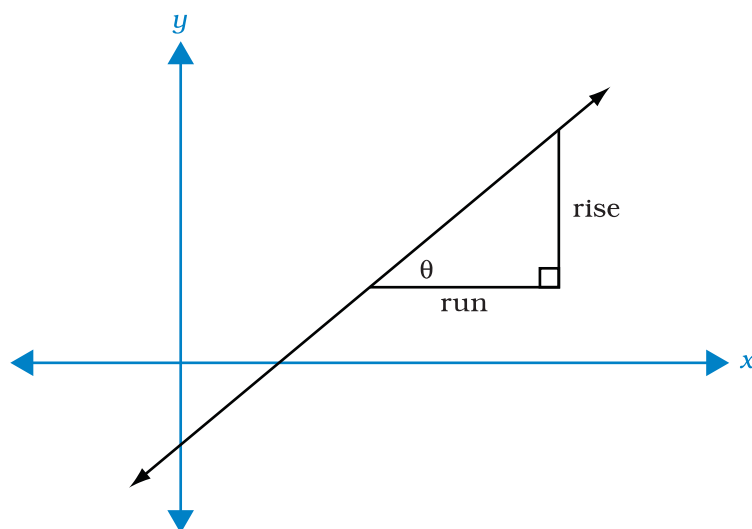


Figure 2. Relating trigonometry and coordinate geometry.

Next lesson I asked students to work in small groups and think about whether they could find the gradient of a line if they knew only its angle of inclination. Students soon recognised that the size of the angle would be sufficient information to draw a line on grid paper and choose a couple of points from which the gradient could be calculated. But could such a line be drawn uniquely? Some students were unsure that any line with the required slope would do so I reminded them about the constant nature of linear gradients and we had a discussion about the equal gradients of parallel lines.

The fact that any number of lines with the same slope could be drawn led nicely to another discussion about how the size of the angle between the line and the horizontal is the same no matter where it is measured and I asked the groups to think about the gradients of various lines inclined at 45° to confirm this. I noticed that the students in one group had started to draw lines without bothering to construct the coordinate axes—they were drawing right-angled triangles! It was provident that the group had this insight because it saved me from having to suggest it and ideas that come from the students themselves are more satisfying and sometimes more influential in shaping the thinking of their peers. So I asked this group to present their findings to the class and we discussed how the right-angled isosceles triangles they drew could be used to represent a straight line inclined at 45° to the positive direction of the x -axis.

The other students were now happy to draw triangles to represent straight lines and gradients as it saved having to rule up axes all the time and so the process of abstracting the underlying mathematical ideas and linking them to trigonometry had begun. We discussed how the gradient of a line could be greater or less than 1 depending on whether the angle of inclination was more or less than 45° . I provided the groups with more grid paper and asked them to investigate gradients of lines inclined at 10° , 20° , 30° and so on up to 80° . It was only after the class were nearly finished this activity that one student commented that we did not need to draw all those triangles because the 10° triangle also included 80° as its complement—something I probably should have foreseen.

Now the students compiled all of their results on the board and decided that taking the average value for each angle would be a good way of dealing with any inaccuracies that might have occurred in the measurements or calculations. The class had thus developed a primitive table of tangent ratios for multiples of 10° and I asked the students to think about why the gradient of a vertical line could not be calculated in order to anticipate a much later discussion about the tangent of 90° being undefined. But, for now, it was time to think about how to use this new found table of values.

The height of the flagpole

In the following lesson, I asked the students to work in small groups to devise a method for finding the height of the flagpole in the school playground and I encouraged them to think about using the table of values from the previous lesson. Some students were unsure about how to measure angles in a practical context like this so I also had to show them how to operate a clinometer. All of the groups eventually concluded that if they could measure a distance from the base of the flagpole to a point where the angle of elevation to the top was close to a multiple of 10° then they could use the values in the table to calculate the height of the pole. It was interesting that some groups recognised that their own height would need to be accounted for (or that they would need to measure the angle while lying down) while other groups were completely unaware of this potential problem.

The students went to the playground and took their measurements. Then we returned to class and the groups performed the calculations required to obtain a value for the height of the flagpole which they wrote up, together with an explanation of the methods they had used and a justification for their result. There was a large spread of values from the sublime to the ridiculous, but I think it was a worthwhile activity to show the students a practical application of the work they had been doing.

I returned the students' work in the following lesson and we talked about the difficulties associated with having only a small number of ratio values to work with—some groups took quite a while to find a place where they were satisfied that the angle of elevation was close to the nearest 10° . If only we had more values to choose from! Now was the time to reveal that all along we had been working on a branch of mathematics known as trigonometry and that the values we had calculated in our table were called the tangent ratios of the angle. Not only that, but if the students looked closely at their calculators they would see a button labeled “tan” and they could use this key to generate tangent ratios of angles more quickly and reliably than by hand. I asked the students to check the accuracy of the

ratios in our table using their calculators and we discussed likely reasons why a couple of our values were slightly astray. Then I showed the students how we might have solved the flagpole problem using a calculator and stressed the kind of setting out that I wanted in their working. Finally, they tried a worksheet containing similar problems where the angle of elevation and horizontal distance were known and the height of various objects needed to be found.

A rationale for learning about sine and cosine ratios

In the following lesson, the students worked on more word problems based on realistic situations. However, this time the height of the object was given and we needed to calculate the distance from the base. Then we considered a circumstance where the height and the distance from the base were known to see if we might be able to find the angle of elevation. This required the students to learn a new sequence of calculator keystrokes and some new ways of setting out their working, but they were quite comfortable with this. Some practice exercises followed and then I proposed a new problem for the students to consider: “A ladder, 3.5 metres long, is leaning against a wall. The ladder makes an angle of 60° with the ground. How can we use trigonometry to find how far the ladder reaches up the wall?” The situation is shown in an abstract diagram in Figure 3 and it is clear that the tangent ratio will not help here, hence the need to introduce the sine and cosine ratios.

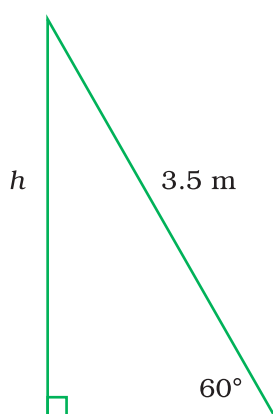


Figure 3. A schematic diagram to represent a ladder leaning against a wall.

Conclusion

The introduction to the topic took slightly longer than I had planned but I felt it was important to proceed slowly and give the students time to think about what they were doing. Linking the tangent ratio to the familiar concept of the gradient of a straight line was successful and provided a useful starting point for teaching about trigonometry. Beginning only with the tangent ratio rather than all three ratios together was particularly beneficial because it avoided the need for the SOHCAHTOA rule until after students had plenty of time to think about the concepts. In the past, I have often felt that students switch off once they have learned the mnemonic and they stop trying to make sense of the work because they have a simple rule they can follow almost without thinking.

I was able to gain back some of the extra time spent at the beginning of the unit when I introduced the other ratios because students did not need as much practice with them. I had already covered angles of elevation in some detail so that work did not need to be done again. I also spent less time dealing with errors and misconceptions from students because they had a solid grounding in the concepts. I have a hunch they might even remember their work in trigonometry over the long term as well.

The journey from my early “ABC” teaching days is ongoing and I continue to look for new ways to help students think more deeply about concepts and see the applications of the mathematics they study. I agree with Quinlan (2004) that such an approach is far more meaningful and productive than starting with a definition.

References

AAMT (2002). *Standards for Excellence in Teaching Mathematics in Australian Schools*. Adelaide: AAMT.

Mitchelmore, M. (2000). Empirical is not mathematical! *Reflections*, 25(2), 13–15.

Quinlan, C. (2004). Sparking interest in trigonometry. *The Australian Mathematics Teacher*, 60(3), 17–20.

From Helen Prochazka's

Scrapbook

Born of man's primitive urge to seek order in his world, mathematics is an ever-evolving language for the study of structure and pattern. Grounded in and renewed by physical reality, mathematics rises through sheer intellectual curiosity to levels of abstraction and generality where unexpected, beautiful, and often extremely useful connections and patterns emerge. Mathematics is the natural home of both abstract thought and the laws of nature. It is at once pure logic and creative art.

Calvin Clawson in his book
Mathematical Mysteries (1996)

[upon losing the use
of his right eye]
“Now I will have less
distraction.”
(Leonhard Euler,
1707–1783)
In H. Eves, 1969,
“In Mathematical
Circles”, Boston:
Prindle, Weber and
Schmidt.

“Do not worry about your difficulties in
Mathematics. I can assure you mine are still
greater.” (Albert Einstein)