DISCOVERY

with Neville de Mestre

Sudoku

Sudoku is a logic puzzle that has appeared in many newspapers in recent years. In its introductory form it consists of a 9×9 grid in which the digits 1 to 9 inclusive are each to be placed nine times in the 81 separate cells of the grid. Each row and each column may not have any digit repeated. If these were the only rules, then the solved puzzle would be called a Latin square. This name was invented by the Swiss mathematician Leonhard Euler (1707–1783) who first created one on a 3×3 grid using the Latin numeral characters I, II and III. Ask your students to create one, and they should produce a result similar to Figure 1.

I	II	III
II	III	I
III	I	II

Figure 1

Discuss this Latin square with your students, noting that other arrangements are possible. However they will be not all that different from the one shown above, since they will all have a common symbol along one of the main diagonals (III in the above example). It is an interesting problem to prove this, and will introduce your students to a well-known mathematical method of proof. Simply try to start with no symbol repeated in either main diagonal.

Now the Roman (or Latin) numerals I, II, III could be replaced by letters A, B, C and the puzzle remains the same. This is the reason that the accompanying explanation of the rules of Sudoku in many newspapers claims that solving Sudoku requires no mathematics. However, this is an incorrect statement, for although no calculations seem to be required, the solution of a Sudoku puzzle may require the use of logical reasoning, deduction, and "reductio ad absurdum" techniques which are all part of the rich tapestry of mathematics.

For your general information, Euler's work on Latin squares forms the basis of a branch of mathematics called combinatorics. This area of mathematics is associated with permutations, combinations and probabilities and proves to be useful in real-world applications of statistical experimental design, digital electronic codes, and tournament arrangements for sporting and recreational activities such as golf and contract bridge.

Back to Sudoku, which has an extra constraint compared with a regular 9×9 Latin square. Each of the nine non-overlapping 3×3 sub-grids along the edges and in the centre of the larger grid must also contain the digits 1 to 9 without repetition.

To enable a Sudoku puzzle to be solved, particular digits or clues are fixed in place in some cells at the start. The number of these varies, but they should be placed so that the solution is unique. At this stage no one has created a solvable 9×9 Sudoku puzzle with fewer than 17 given clues, but this lower limit has not been proved mathematically. The number of clues usually ranges from 24 to 32, with many newspaper Sudokus having 26. The newspapers usually also give a degree of difficulty for their Sudokus ranging from *easy* to *very hard*. The degree of difficulty does not depend exclusively on the number of clues given, but must take into account the number, variety and complexity of the logical steps needed to complete the solution. A typical Sudoku Latin square puzzle is shown in Figure 2 with 26 digits (clues) initially assigned, and usually in an asymmetric pattern.

Before any attempt is made to further analyse the 9×9 Sudoku problem in general, it is suggested that your students should look at a simpler related problem in the sense suggested by Polya (1973).

	5		4			8		
7		4						
	6			2	5	7	1	
9				7				
4								1
				6				2
	7	9	6	1			2	
						1		3
		3			2		9	

Figure 2

1			
		2	
			4
	3		

Figure 3

Place the digits 1 to 4 inclusive in a 4×4 Sudoku Latin grid (see Figure 3), so that each digit occurs exactly four times, but no digit occurs more than once in any row, column or 2×2 corner sub-square.

Your students could now be asked to find all the sixteen possible arrangements with the digits 1, 2, 3, 4 in that order across the top row. Then they should show that the top row can be rearranged in 4! = 24 different ways. Hence the total possible 4×4 Sudoku Latin squares is 16 × 24 = 384. The number for all 9×9 Sudoku Latin squares is much, much larger, more than six billion trillion (see Semeniuk's article in *New Scientist*, 2005).

Why not set your students the task of creating some 4×4 Sudoku puzzles, and you can then copy the better ones onto a sheet for the rest of the class to attempt? Discuss the strengths and weaknesses of each created task. Perhaps your students will conclude that when the clues 1, 2, 3, 4 are assigned one to each 2×2 sub-square in such a way that each column and each row has only one digit then the solution is unique and easy to obtain (see Figure 3).

However when the clues 1 to 4 are arranged all in one column or row or sub-grid then the solution is not unique. It is a useful exercise in developing logic skills for your students to find the minimum number of assigned cells (clues) so that there is only one possible answer and the degree of difficulty is harder than in Figure 3.

Finally, here is a little help in solving Sudoku puzzles like the one shown in Figure 2. Make a larger copy of the puzzle so that there is space in each cell to record a few numbers near the edges of each cell. Count the frequency with which each digit is present in the clues. Start with the most frequently occurring clues and record the possible cells for these digits to occur in the remaining empty cells. Continue for all other digits in descending order of frequency.

If any incomplete cell has only one entry, fill it in. Then remove it from any associated row, column or sub-square.

Next, see if two cells in a row have only two possible digits. Remove these digits from any incomplete cell in the same row. Repeat this operation for columns and then 3×3 sub-squares. Your students can now easily extend this technique to three cells with only three digits. Most Sudoku puzzles can be solved by the above techniques. However if you need more, then go to the web and type Sudoku into a search engine such as Google.

Happy discoveries!

References

Polya, G. (1973). *How to Solve It.* New York: Princeton University Press.

Semeniuk, I. (2005). Stuck on you. *New Scientist, 24*, 31 December, 45–47.