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What is Mathematical Modelling? Exploring Prospective Teachers' Use of Experiments to Connect Mathematics to the Study of Motion

David J. Carrejo
University of Texas at El Paso

Jill Marshall
University of Texas at Austin

This paper focuses on the construction, development, and use of mathematical models by prospective science and mathematics teachers enrolled in a university physics course. By studying their involvement in an inquiry-based, experimental approach to learning kinematics, we address a fundamental question about the meaning and role of abstraction in modelling when such approaches involve students encountering and resolving experimental error. We use a "tensions" framework to explore the capability of learners to make necessary connections between abstract mathematical models and physical phenomena.

In recent years the use of models in teaching and learning science has been given serious consideration by science education researchers (Halloun, 1996; Hestenes, 1992, 1993; Wells, Hestenes, & Swackhamer, 1995). Likewise, research on the role of models and modelling in mathematics education has also surfaced (Confrey & Doerr, 1994; Doerr & English, 2003; Doerr & Tripp, 1999; Lesh & Doerr, 2003). Support for educational research involving modelling promises to continue (Blum, Galbraith, Henn, & Niss, 2007) and will likely answer many important questions related to student learning of mathematics and science through inquiry. For example, some current research has focused on modelling that supports student learning of statistics in highly contextual and meaningful ways (Lehrer & Schauble, 2002). Most recently, international research communities presented studies of modelling approaches in mathematics classrooms on a global scale and emphasised their impact on learning mathematics (Blum, Galbraith, Henn, & Niss, 2007; Matos, Blum, Houston, & Carreira, 2001).

This paper focuses on what we call "mathematical models"; their construction, development, and use in the classroom through an inquiry-based approach to teaching and learning kinematics. We rely on the view that a scientific model becomes a mathematical model if the model describes or represents a real-world situation with a mathematical construct (or constructs) involving mathematical concepts and tools (Pollak, 2003). A mathematical model is resident in certain domains of mathematics (such as algebra, geometry, and statistics) because of their algorithms and formulae; however, the mathematics involved in the model must be made reasonable in two ways, not only in its mathematical "correctness" with regard to the domain in which it is resident, but also in the real-world situation which it represents (Pollak, 2003). The transfer from scientific to mathematical model also involves identifying and using mathematical constructs such as space and measure as well as other constructs that bring insight to solving a problem or understanding a situation (Lehrer &

Schauble, 2000). We claim that learning with mathematical models not only has practical applications, but also has philosophical and historical relevance in the construction of mathematical and scientific knowledge (Dear, 1995; Sepkoski, 2005).

When reading the literature on modelling, one may ask if mathematical modelling should be considered a proficiency or a competency in learning mathematics. These are two seemingly distinct views of the subject, each with its own related set of research questions (International Commission on Mathematics Instruction (ICMI), 2003) and research paradigms. Consequently, these paradigms lead to different suggestions for instructional methods and the reconciliation of those methods with established instructional goals. We argue that reconciling those methods becomes more complex when one considers the national calls to integrate mathematics and science at the classroom level through authentic activities so that the learning of one subject domain can enhance the learning of the other (National Research Council, 1996, 2000). In actual mathematical and scientific practice, the development and acceptance of mathematical models is complex; yet, national standards call for students to connect mathematics and science to real world phenomena and learn both subjects through authentic activities.

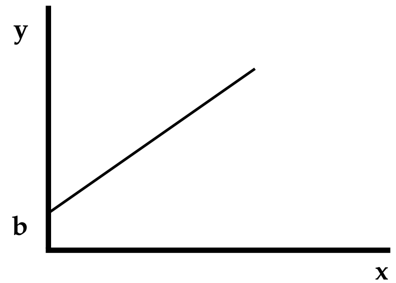
Such an integrated approach reveals the complexities of mathematical modelling based on one key (and related) question posed about mathematical modelling "What is the meaning and role of abstraction, formalization and generalization in applications and modelling?" (ICMI, 2003, p. 11) This question addresses epistemological considerations of why more traditional mathematics and science typically value abstract "truths" over the relationship between mathematics and real phenomena. It may also be interpreted as a need to examine not only cognitive processes and student thinking but also social practice in the classroom. In both cases, the role of abstraction plays a fundamental role and is the focus of investigation.

Does formal, abstract mathematics play a large role in learning with mathematical models? If so, one key question concerns the ways in which students make connections between a formal mathematical model and the phenomenon that they are studying. This issue has been addressed in prior writings. For example, within the body of statistics learning literature, delMas (2004) points out that:

> In the practice of statistics, model abstraction always begins with a context. When this practice is taught in the statistics classroom, the student is dependent on the characteristics of the context to guide model selection and development. In some respects, this may be a more difficult task than the purely mental activity required in mathematical reasoning. During model selection and construction, the student faces some of the same cognitive demands that are required by abstract reasoning while having to check the model's validity against the context. (p. 91)

Another example highlights the importance of learners being required to "fit" their observations to an abstract model in mathematics and physics. Giere (1999) claims that a "technically correct" equation for linear motion can be written – one that involves margin of error (Figure 1). However, he claims "this is not necessarily the best way of interpreting the actual use of abstract models in the sciences" (p. 50).

$$\mathbf{y = a\,x + b}$$



$$\mathbf{d(t) = v\,t + d_0}$$

$$\mathbf{d \pm \delta = (v \pm \delta_v v)\ (\mathit{t} \pm \delta_t t) + (d_0 \pm \delta_d d_0)}$$

Figure 1. Giere's equation for linear motion (p. 49).

The use of symbolic language can disassociate the model from observed phenomena since symbolic language bears its own structure and requires its own rules of use. A similarity between the model and the world must be drawn, but the abstract nature of the model must remain intact. In Giere's view (and perhaps in the view of other scientists), "Mathematical modelling is a matter of constructing an idealized, abstract model which may then be compared for its degree of similarity with a real system" (p. 50). The crux of Giere's claims can be analysed in the context of how that abstraction takes place, especially in light of pre-conceptions, prior knowledge, and experience, which students will not easily dismiss. Furthermore, the realm of physics acknowledges error more readily than mathematics, yet the presentation and use of abstract models in the physics curriculum are common and expected. One may even propose that learning formal, decontextualised mathematical structures is the ultimate goal of mathematical modelling in science. The aforementioned conflicts can make it difficult for classroom interactions to satisfy the goals of the various sciences (including mathematics), which may be in conflict with each other. Another conflict may exist between instructional goals of the education system and goals for robust learning. For example, developing appreciation for experimental error and uncertainty may be a goal in physics and statistics, but not be an important goal in algebra. The theory of reification put forward by Sfard and Linchevski (1994) contends that objectification of symbols is the necessary process for learning algebra; however, others contend that reification is only one view of mathematical reasoning and development. It tends to ignore the historical development of mathematical knowledge and stands in opposition to relevant issues of school reform in mathematics including learning through inquiry (Confrey & Costa, 1996).

In order to help students obtain learning goals, teachers must be able to create and strengthen links between more formal, abstract mathematical concepts and real-world situations where mathematics plays a more applied role. As teachers immersed in a modelling environment move within the realms of personal experience, mathematics, and science (e.g., physics), emerging tensions in student learning (and their own) could become apparent to them. If teachers are to move effectively between these realms, they must make choices on how to relieve resulting tensions within themselves and their students; such choices have a profound impact on the use of modelling approaches in the classroom. For example, teachers who disregard variation in real-time motion data may not be aware that they are failing to meet one of the goals in physics – to account for experimental error. Likewise, teachers who do not advance their students' conceptual development of purely linear (i.e., error-free) models for position and time in force-free situations may lose some critical mathematics and physics understanding for their students. Therefore, teachers immersed in a modelling environment require support and professional development in both content and pedagogical content knowledge (Lehrer & Schauble, 2000; Petrosino, 2003). Otherwise, many teachers may resolve the issue by circumventing the tensions through direct instruction methods that do not facilitate conceptual understanding or abstraction. In some cases, teachers may possibly abandon an inquiry-based approach altogether.

Theoretical Perspective on Learning Motion Through Modelling

The critical theme of comprehending and resolving experimental error as it relates to making a connection between an abstract model and learner experience with physical phenomena is highlighted in this paper. Error can be discussed in abstract terms (e.g., the symbolic combined with reliance on formal mathematical systems or structures) or in terms consistent with physical experience (observations and experiments combined with data interpretation). One hypothetical example that highlights these issues involves a simple experiment where students examine a bowling ball rolling down a hall. The students are given the task of predicting how long it would take the ball to travel 10m if it were not blocked by the end of the hall. They decide to mark the times as the ball passes given locations. One student releases the ball and yells, "Go!" At this signal all the others, spaced at equal distances along the path they expect the ball to take, start their stop-watches. Each one stops her or his watch as the ball passes. Sample data for this hypothetical experiment are shown in Table 1.

First, students create both the table and the related graph of position versus time (see Figure 2).

Table 1
Sample Data from a Hypothetical Experiment Investigating Motion

Time (s)	Position (m)
0	0.0
1	0.5
2	1.1
3	1.6
4	2.0
5	2.3
6	2.5

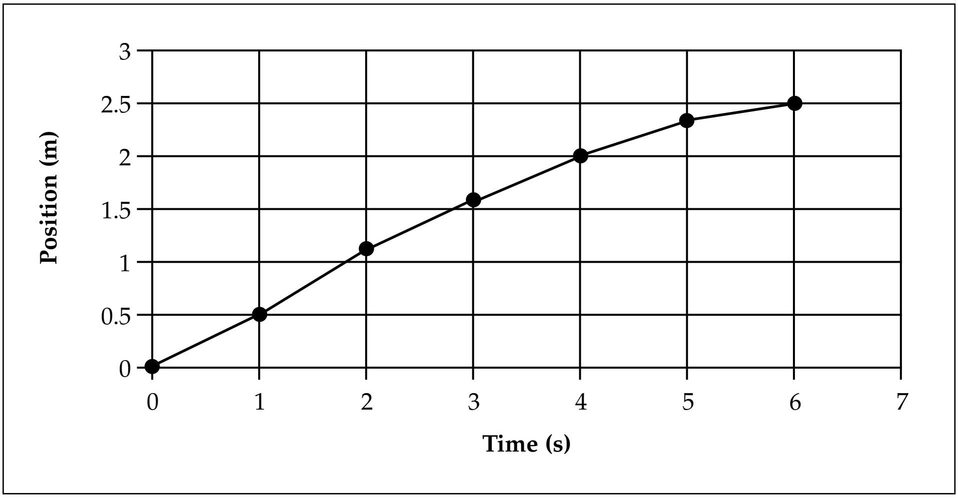


Figure 2. Plot of sample data from hypothetical experiment.

Based on prior knowledge, the students believe that the speed of the ball "should" be constant in this situation. From science classes they have learned that the laws of physics treat real situations as ideal and neglect friction. That leads them to believe that the ball should cover equal distances in equal times. In order to predict the time for the ball to travel 10 metres, they realise that they need to calculate a representative rate of distance per time (or time per distance). They argue that this rate can be found by taking the total distance travelled and dividing it by the time it took to travel that distance.

Other students, however, remember their science laboratory reports and believe that they need to take possible sources of error into account. They argue that the last timer might not have been accurate, and that a better way to find the representative rate would be to calculate it for each consecutive second, and take an average. Yet another group of students recognises that the rate is systematically decreasing. From the graph they can see that the data are not really linear at all, but they still believe that position versus time "should" be a straight line. These students argue that the graph can be replaced by a series of lines from the origin (Figure 3) and that the slope of the middle line will be the characteristic rate.

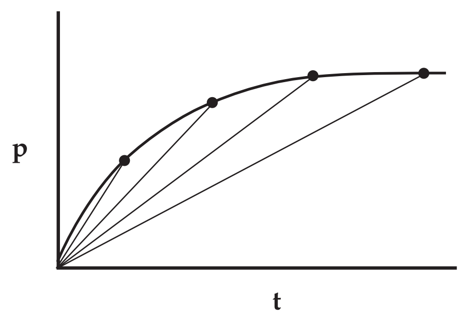


Figure 3. Establishing a characteristic rate for the hypothetical experiment.

They want to extend this line out to 10 metres and predict the time. One of the students in the first group argues that this will not work, though, because the ball's speed will continue to get slower as it rolls further down the hall; it has never yet been as slow as it will be at the 10 metre mark.

The students are at a loss as to how to proceed. Those who believe that sources of error such as friction and human reaction time should be ignored still want to average the rates of change and create one model line. Others suspect that these data really do not fit a linear model, but do not know how to show that the fit is not "good enough," that is, that they need a different model. The data appear to be quadratic (some of them have even made a quadratic fit using their calculators) but the students do not know how to prove this is the right way to go and cannot see why friction should cause the relationship between distance and time to be quadratic in the first place.

Based on student discussions and students' engagement with the task, a teacher might respond to these conflicts by emphasising any one of the following different perspectives on the modelling of physical events:

- No motion in nature truly exhibits constant velocity and it is impossible to measure any physical quantity with infinite precision. A model should reflect these variations and limitations to the greatest extent possible, but all models are limited in their capability to truly describe and predict.

- A mathematical model should never reflect error. Its mathematical structure and nature allows it to describe and predict motion in a way that is generalisable to many situations. Thus, experimental error should be ignored.
- A mathematical model would not reflect error had the students conducted a "perfect" experiment explaining how motion should behave under "ideal" circumstances. Students' personal experience with the experiment limits "true" understanding. Thus, constructing the most ideal situations is the focus of learning.

This hypothetical modelling episode illustrates opportunities for students to consider accuracy and what measurements are "good enough" to use in order to answer a prediction question or generate an abstract mathematical model. In this example, student experience with the phenomenon, along with their prior, formal knowledge of both mathematics and physics, could lead to deeper investigation of tensions among all three areas.

Inherent Tensions in Learning With and Through Models

The tension between scientists' personal experience in conducting motion experiments and mathematical modelling of motion such as free-fall has also been in evidence historically. For example, Dear (1995) outlines a criticism of Galileo's rule of free fall presented by Honoré Fabri, theologian and philosopher. Fabri claimed that Galileo's rule of odd numbers treats physics as mathematics, which Fabri believed was not possible. Dear, explaining Fabri's contention, writes: "The essential problem with Galileo's odd-number rule was that it could not be based on experience, or 'experiences,' because sensory data could never provide sufficient precision to guarantee it" (p. 141). Tensions between a learner's personal experience and the branches of mathematics and physics cannot easily be dismissed especially in the context of constructing mathematical models. For example, personal experience can influence perceptions of what is "concrete" or "real" and what is "abstract." Historically, this perception was a key consideration in the development of critical areas of modern mathematics and was based on nominalism and several views of constructivism. For example, Sepkoski (2005) writes:

Newton's mathematical methodology, particularly in the *Principia*, has been much discussed by historians. I. B. Cohen has described what he calls the "Newtonian style," which involves "the possibility of working out the mathematical consequences of assumptions that are related to possible physical conditions, without having to discuss the physical reality of those conditions at the earliest stages" [1980, p. 30]. This "style" relied heavily on modelling nature mathematically, but the final relationship of those models to physical reality remained a sticky issue for Newton (p. 19). Sepkoski also writes that Sir Isaac Newton "wanted a genuine correspondence between mathematical models and nature" (p. 19).

Based on concerns about learning and teaching, it is important to highlight conflicts that may exist for a learner immersed in the process of constructing a mathematical model. Epistemological tensions or cognitive conflicts may emerge among a learner's real-world experience in contextual inquiry, learning standard concepts in mathematics, and learning standard concepts in a realm such as physics (Figure 4). All three will play a role in the mathematical modelling process since students will not only encounter instruction in both content domains but will also have perceptions, based on prior experience or from the modelling process itself that may not necessarily resemble standard concepts taught in either mathematics or physics.

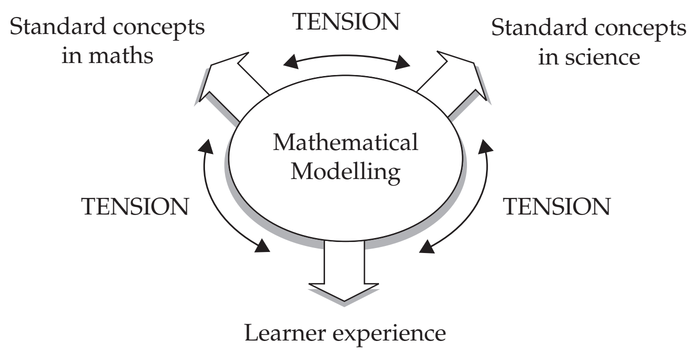


Figure 4. Tensions during the mathematical modelling process.

Similar tensions are identified and discussed by Woolnough (2000) who states, "We would contend that most students, even those who perform well in math and physics, fail to make substantial links between these contexts, largely because of conflicts between the different belief systems" (p. 265). To view and analyse mathematical themes through a constructivist "lens" and with regard to learner goals, we rely on the tensions model in Figure 4. For example, in Figure 5, we view the results of student interaction and learning in the hypothetical experiment presented previously.

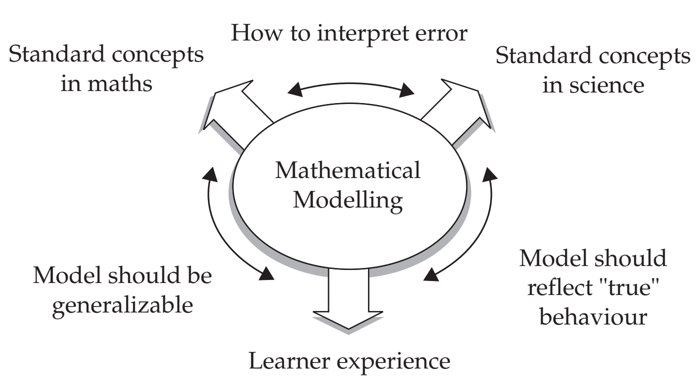


Figure 5. Summary of tensions from hypothetical experiment.

Pollak (2003) states that the crucial problem related to the use of models and modelling in formal classrooms is how to connect mathematics to the "rest of the world." He claims: "What is usually missing is the understanding of the original situation, the process of deciding what to keep and what to throw away, and the verification that the results make sense in the real world" (p. 650).

The researchers' goal, by examining learning through mathematical modelling, is to determine characteristics of epistemological tensions that arise when teachers are immersed in a modelling process to describe and predict a physical phenomenon. Our chief concern is seeing how learners immersed in a modelling environment can make connections between symbolic statements and their apparent physical referent and, particularly, what role experimental error plays in strengthening or preventing that connection. Identification and analysis of this theme could help further understanding about the process of abstraction in modelling: How can a learner make a connection between experience and accepted, abstract models given considerations based on learner experience?

Kinematics as a Learning Context

We examine kinematics (the study of motion) as a content domain. Kinematics is considered a rich topic for investigation as a context for modelling primarily for three reasons:

1. Kinematics provides a very natural context in which to place teachers and students in a familiar activity. A substantial amount of research has been conducted involving motion and the mathematics of change (Stroup, 2002) and further research is needed.
2. Kinematics is a fundamental area of study that links mathematics and physics. Modelling experiments in this domain can foster the development of mathematical concepts such as function while at the same time build understanding of critical ideas such as velocity and acceleration.
3. Kinematics emphasises an important aspect of modelling and creating models – the ability of such a model to describe observed behavior and predict future behaviour.

From a mathematical standpoint, functional reasoning (or cognitive reasoning involving a function concept), may involve a complementarity between representations. Otte (1994) claims: "A mathematical concept, such as the concept of function, does not exist independently of the totality of its possible representations, but it is not to be confused with any such representation, either" (p. 55). Furthermore, a robust understanding of function, presumably, involves a grasp of three distinct representations (equation, graph, data table) and the connections between them (Kaput, 1998). Kinematics, through reliance on a function concept to model motion, provides an opportunity to examine the possible tensions present when learners rely on function representations and attempt to make connections between them during the modelling process. Furthermore, kinematics emphasises an important aspect of modelling and creating models-the ability of such a model to describe observed behaviour and predict future behaviour.

Critical Concepts in Kinematics

Our initial research question (Carrejo, 2004) concerned the depths of understanding that in-service physical science teachers have of two fundamental equations related to kinematics and how that understanding evolves during modelling activities. More specifically, the researchers wished to probe participant understanding of the formulas describing: (a) uniform motion (constant velocity or zero acceleration) and (b) uniformly accelerating motion (constantly changing velocity or constant acceleration). The first formula can be discussed and represented (in a mathematical sense) as a linear relationship between two variables, namely, position (p) and time (t). The latter formula can be represented as a quadratic relationship between the same two variables. Given an understanding (in a physical sense) of position, time, velocity, and acceleration, participants' mathematical background knowledge would allow them to see how these pertinent concepts could be related via the formulas involving standard mathematical symbols.

a) $p(t) = \bar{v}t + p_0$

b) $p(t) = \frac{1}{2}at^2 + v_0t + p_0$

where $\bar{v}$ is the average velocity, v_0 is the velocity at $t = 0$ and p_0 is the position at $t = 0$

These formulas are part of the standard physics curriculum. In many cases, the formulas are written without the function notation, that is using p rather than p(t). Furthermore, these mathematical equations (or functions) are typically introduced through direct instruction, with derivations requiring algebraic manipulation. This is especially true for equation (b) where, arguably, learners may not have an intuitive understanding of certain features of the equation such as $\frac{1}{2}$ and t^2. Learner understanding usually rests on more procedure-driven exercises with the equations. A-priori knowledge of linear and quadratic equations, as well as average velocity and instantaneous velocity (key calculus concepts) and/or geometric structures, are often used to justify the equations in formal ways, yet the relevance of the equations to learner experience could often be overlooked.

Equation (a) may be familiar to students from prior instruction, although it is likely to have been in the form of distance = rate x time, which does not consider the possibility of a non-zero initial position. Most students will have used the concept of average velocity (albeit perhaps unconsciously) to calculate how long it would take to travel a given distance by car given an average speed, or what the average speed was given a distance and a time of travel. There is a subtle difference between this familiar average velocity construct and the one in equation (a) in that the distance is replaced by a difference in positions. In moving to equation (b), even more complex mathematical manipulations obscure the connection to the average velocity. Most students fail to see how the second equation relates to that fundamental, and generally familiar if not intuitive, construct.

Method

Setting

The study took place in the context of a 14-week semester course in physics designed for preservice teachers. The physics course was offered during the fall term at a university in Texas. The course was taught by the university professor using the circuit and optics units from *Physics by Inquiry*TM (McDermott, 1996) as well as a kinematics unit, developed independently and introduced separately from the *Physics by Inquiry* approach to the same topic.

Participants enrolled in the course studied kinematics for a five-week period scheduled near the end of the course. Class meetings were two days per week, and participants met for a minimum of three hours each week within the two-day period. The course was designed to serve as a relevant domain (or content) course for university students seeking careers in mathematics and science education. The fundamental goals of the course as outlined for participants in the course syllabus included the following:

- Developing a deeper conceptual understanding of targeted physical science concepts and creating a coherent conceptual model of the concepts,
- Experiencing physics content through a process of guided inquiry and developing an understanding of how the process of inquiry interacts with student learning,
- Developing an understanding of what is meant by pedagogical content knowledge, and
- Becoming familiar with potential difficulties experienced by students in learning particular topics in physical science, and the effectiveness of various modes of teaching and learning to overcome such difficulties.

Participants

Fifteen prospective teachers, five graduate (masters students) and ten undergraduate (standard four-year college students seeking a bachelor degree), enrolled in the physics course. Majors (disciplines) varied within the group as shown in Table 2.

All mathematics and science education majors were graduate students. One of the science education graduate students held a masters degree in physics. Of the remaining ten undergraduate students, one was a senior (fourth year), five were juniors (third year), and four were sophomores (second year). Six of the fifteen students enrolled held teacher certification.

Design

Physics by Inquiry provides modules in kinematics that involve some laboratory experiments with activities such as rolling a ball on a track and observing a fan belt attached to two pulleys. However, these experiments do not involve collection and analysis of real-time data as an integral part of the construction of the mathematical models for motion. Quantitative descriptions of position and time are only briefly discussed while more emphasis is placed on qualitative graphing (position-time and velocity-time graphs). Furthermore, the experiments can more aptly be described as demonstrations that are followed immediately by introduction of formal (symbolic) mathematics, including precise definitions (e.g., instantaneous velocity) and procedures (e.g., finding the area under a graph). In these modules, the learner is given more guidance through the experiments, which require direct instruction from the facilitator or teacher. There is less emphasis on an inquiry process that might allow a learner to formulate his or her own mathematical models of the physical phenomena.

Table 2
Subject Majors of Participants in the Study

Major/Discipline	Number of Participants
Mathematics Education	1
Science Education	4
Biology	1
Biology (education concentration)	1
Chemistry (education concentration)	1
Mathematics	1
Government	1
Elementary Education	5

In a pilot study (Carrejo, 2004) conducted the previous summer with in-service teachers, the university professor and a Master Teacher developed a kinematics unit based on activities that the researcher had used in college physics courses for preservice teachers and that the Master Teacher had used for Advanced Placement Physics classes. This unit presented a classroom environment for studying mathematical modelling from a constructivist point of view. Building on the assumption that the teachers' prior knowledge might include only a procedural understanding of the equations, the primary goal of the implementation was to facilitate a more conceptual understanding of both equations (a) and (b).

The kinematics unit facilitated a classroom environment for studying mathematical modelling through the tensions lens. The primary goal of the implementation focused on the construction of feasible mathematical models regardless of their resemblance (exact or not) to the standard equations. The university professor and the researcher came to the conclusion that learners from the pilot study possessed sound mathematical constructs and beliefs about mathematical models, but that an evident conflict existed between teachers' prior knowledge of the standard linear and quadratic models and their constructed mathematical models. The professor and the researcher felt confident to use the same approach to modelling in the present study in order to explore the conflict further. Participants were allowed to examine, modify, and re-examine their own models rather than rely on direct instruction techniques to learn the standard models and develop some, if not necessarily full, conceptual understanding. By having participants focus on their own models, we hoped to identify possible sources of conflict between formal models and a real-world phenomenon.

Procedure

The key activities and their objectives resembled those of the pilot study and are shown in Table 3. The preservice teachers worked in groups of two to four members for all activities. Group presentations were required and became the focus of whole class discussions. The course took place in a classroom laboratory equipped with various tools and instruments at the preservice teachers' disposal. Special requests for equipment were considered. At the end of activities three and five, participants worked through problem sets consisting of more "standard" problems based on their experiments. Supplementary problems involved the presentation of similar phenomenon (contexts) and discrete, tabular data, which participants were asked to analyse. Participant's engagement with these problems (both in small and whole group meetings) provided further opportunities for the researcher to analyse and understand their thinking about mathematical modelling and the type of phenomenon they studied in their experiments.

Data Collection

Data collection typically involved whole class and group observations. All sessions were videotaped extensively. Qualitative notes, including researcher reflections, were compiled from this analysis. Classroom artifacts, including representations from individual groups, as well as representations created from whole class discussions were kept and analysed. Participants were interviewed individually upon their completion of the unit. The interview relied on an instrument developed by the researcher. The main goal of the instrument was to probe participants' perceptions of the modelling process they encountered in the course as well as their mathematical conceptions of working with a data set. All questions on the instrument were within the context of modelling motion. All interviews were recorded using a hand-held tape recorder and were transcribed. Artifacts created during the interviewing process were also included in the data analysis.

Data Analysis

Originally conceived by Glaser and Strauss (1967) for social research, a grounded theory approach to qualitative research is similar to other types of qualitative research in that a general area of interest is determined, followed by the formation of a question that is both credible and relevant to the researcher. Mann

(1993) states that grounded theory is a research approach whose goal "is to transform the experiences of one setting into a model that accurately reflects that setting" and yet "be general enough to apply to a range of situations in the context" (p. 134) (the context in this case being a classroom setting). Furthermore, a grounded theory approach does not require a significant change in the setting to "trigger a study" (p. 134). Data can be collected from the normal flow of activity in the classroom while still leaving room for the possibility of making slight changes in the direction of the research based on classroom outcomes.

Table 3
Key Activities for Kinematics Unit

Activity	Objectives
Describing motion	Invent and describe a motion for an object. Create a description of the motion that is detailed enough so that another group could reproduce the motion exactly. Identify the important elements of a complete description.
Measuring constant velocity	Each group must suggest a procedure for describing and predicting what they consider to be constant motion.
Developing a mathematical model for constant velocity	The instructor facilitates class discussion on the proposed ideas and the class determines a common (standard) procedure.
Acceleration with a spark timer	Use a spark timer, a ramp, and a cart to gather position-time data for accelerated motion. Each group must suggest a procedure for describing and predicting what they consider to be "changing" motion, i.e., motion in which the velocity was not constant.
Developing a mathematical model for accelerated motion	The instructor facilitates class discussion on the proposed ideas and the class determines a common (standard) procedure.

The researchers utilised a grounded theory approach similar to that described by Cobb, Stephan, McClain, and Gravemeijer (2001) in their analysis of transcripts from classroom mathematical practices. The first phase of analysis involved examining the video and transcripts chronologically to identify episodes. An episode was characterised as a segment in which a mathematical theme (or perhaps themes) is (are) the focus of activity and discourse (p. 128). Observations and conjectures were developed about reasoning and the context in which the reasoning takes place. As described by Cobb et al. (2001), "the result of this first phase of the analysis is a chain of conjectures, refutations, and revisions that is grounded in the details of the specific episodes" (p. 128).

In grounded theory, three types of coding are typically involved in data analysis:

- Open coding (creating categories for data),
- Axial coding (using open codes and researchers' catalogue of data to determine characteristics or dimensions of categories and create a core category or categories),
- Selective coding (data collection and analysis focuses on the core category and supporting categories).

Representative episodes for the study are presented in the Results section. Through open and axial coding, patterns in thinking as well as emerging mathematical constructs were identified throughout the implementation of the kinematics unit. Selective coding of the results will benefit the researchers for further study. Given the creation of core categories from this study, we will attempt to identify these categories with other learners in different environments who are involved in the same implementation of the unit. For this study, the key episodes need not be interpreted as isolated incidents to support certain claims; rather, they highlight the emergent patterns and constructs that are reflected throughout the data and reflect thinking throughout the modelling process.

The approach to coding data in these studies fits well with constructivist views on learning whereby learners rely on prior knowledge or what preconceptions they may have regarding certain phenomena. The focus is on the learners' construction of ideas, rather than their attainment of established ways of thinking. Learners in general, as with the participants in this study, have time and space to make sense of their experiences. In this sense, the "core" of a grounded theory will remain the same across classroom settings while approaches to data collection and interpretation will reasonably change not only to reflect the setting but also be useful enough to apply to other classroom settings.

One other important aspect of grounded theory that guided the analysis is theoretical sensitivity (Glaser, 1978). Taber (2000) defines theoretical sensitivity as commencing research "with an open mind, so that observations are coloured as little as possible by expectations based on existing theories" (p. 470). For these studies, a constructivist lens supports the necessary theoretical sensitivity. The researchers understand that student construction of knowledge involves more than direct instruction and memorisation of facts (Von Glasersfeld, 2001). Furthermore, an understanding of a "voice and perspective" paradigm (Confrey, 1998) plays a crucial role in interpreting and understanding students' scientific and mathematical views. Voice refers to a student's articulation of a model that may be operating in his/her mind. An observer recognising and acknowledging this articulation makes an interpretation based on his/her own perspective. Interactions with students in this way allow an observer to "rethink" mathematical content and place value on the realisation that the observer (teacher or researcher) is also a learner. By utilising this paradigm, a researcher becomes more "theoretically sensitive" to the study being undertaken without bias and without neglecting emerging categories or themes that are creating a story.

Results

In the pilot study, after using open and axial coding schemes based on the grounded theory approach, the researchers then developed the tensions framework shown in Figure 4. The framework became a "lens" or a means of looking at the main themes or topics discussed by participants and the learning processes involved. We were able to ask: "What tensions exist and why?" In the next phase of the investigation, the present study, these characteristics (codes) became the focus of selective coding and became highly relevant for the researcher examining possible tensions that learners confront when relating physical experience to a mathematical model. Although code names remained the same from study to study, two code descriptions altered slightly to reveal more dimensions of a particular code based on the qualitative data.

- Position – Considering the "location" of an object in motion with more consideration given to thoughts about the initial position of the object.
- Scale – Considering the size of an interval on a coordinate axis. Involved in such considerations are "finer scales" and interpolating data.
- Average – Considering some sort of statistical or numerical average of data points when calculating velocity over an interval.

In summary, the researchers hoped to utilise these codes to develop a grounded theory or be able to identify a core theme that emerged from the analysis. The researcher concentrated on these codes when the unit was implemented again, over a more prolonged period of time and bearing some modifications based on the first implementation.

Upon choosing a motion to create, participants were to perform their experiment, justify that the motion created was constant, and predict where the object would be one, five, and ten seconds after the observed motion stopped (assuming that the motion would continue indefinitely). Table 4 summarises the motions created by the participants.

Table 4
Motions Performed and Considered Constant by Participants

Motion Experiment	Number of Participants
Describing/predicting the motion of a metronome	2
Rolling a wooden ball in a round lid	1
Walking at a steady pace	3
Rolling a bowling ball down the hallway	4
Rolling a small wooden ball down the hallway	2
Moving a book in front of a motion detector	2
Describing/predicting a fixed pendulum swing	1

Students performed their experiments in separate groups, collected data on their experiment and analysed them. They were required to present the motion experiment, the experiment data, and the procedure for answering the prediction question to the class during the following meeting period. Table 5 highlights major areas of concern for participants as they presented and discussed their motions.

Table 5
Student Concerns about Motions Performed and Considered Constant

Motion Experiment	Concern(s)
Describing/predicting the motion of a metronome	Direction reversals, perceived by some to include a slight pause in the motion, do not allow for motion to be described as constant.
	What's happening in-between swings (or in-between the time interval of interest) is not constant.
Rolling a wooden ball in a round lid	An average time may or may not be good enough to use to describe and predict a constant motion or find a general procedure for finding an absolute position.
Walking at a steady pace	Best to ignore motion variation between time intervals (e.g., swinging of arms, "jerky" motion).
Rolling a bowling ball	Friction, affecting the ball's position over time, is a physical consideration that may or may not be resolved by calculating average velocity.
	If actual calculations are not matching theoretical values, then the motion is not constant.
	Calculating a velocity over longer distance and longer time interval makes more sense.
Rolling a small wooden ball	Larger time intervals are better for describing and predicting motion because physical instances over larger time periods make more sense.
Moving a book in front of a motion detector	What's happening in-between time intervals may or may not be constant.
	An infinite number of time intervals may be used to better describe a constant motion.
Describing/predicting a fixed pendulum swing	Despite variation, time values are close enough to each other to pick one of the values that represents the "correct" time.

Following participant presentations, the university professor wanted the class to reach a consensus about how to determine whether or not a motion is constant. The class agreed that for describing and predicting constant motion, using the equation $d = rt$ seemed a feasible approach. Some participants argued that r must be the same for any (arbitrarily small) time interval, but others argued that only the total distance and the time of travel were pertinent, despite variations within the interval. Calculating the rate (the value of r) remained an open issue. Experience with variation in data influenced participants' thinking about the best rate to use when describing and predicting motion. More specifically, conflicting beliefs about using an average rate, a rate based on average measurements, or a "good enough" rate, influenced their construction of a mathematical model. Their beliefs were further tested when presented with more formal physics questions regarding constant velocity and involving data tables. Relying on $d = rt$ as their agreed upon mathematical model, participants, working in groups, approached each problem and presented their results to the class. Presentation and discussion of three critical episodes exemplify the influence the core categories of "scale", "averaging data", and "(initial) position" had on participants' mathematical modelling of uniform motion. Underlying the influence of the core categories were participants' perceptions of what is "good enough" to use for a rate when constructing a mathematical model to describe motion – their direct encounter with the theme of "error" and how to resolve it.

Representative Episodes

Three representative episodes are presented along with interpretation and discussion of each episode. One purpose in presenting these data is to provide a strong indication of the scope of analysis. Furthermore, the episodes along with supporting discussion highlight the open categories assigned to the full set of data and the open categories' relationship to each other as indicated by axial coding and determined core categories.

Episode 1. The first episode involves a simple experiment where prospective teachers examined the motion of a bowling ball rolling down a hallway. As the ball rolled, group members tracked its position over time using metre sticks and stopwatches. For each given moment in time, the members associate a measured position from an accepted starting point. Stephen explained the set up of the chair and the ramp and how the bowling ball is 30 centimetres up the ramp as follows:

> Stephen: What we did is when we first started, our start point was where the ball first initially touched the ground and we would start the timers. And we had metre sticks lined up all along the hall there. The timer would start [at the bottom of the ramp] and then at an arbitrary point, say 200 centimetres, we would say "stop" and the timer would be stopped. We repeated this numerous times. We took each distance...so, say we did 200 centimetres...divided by each time. So distance divided by time would give us a rate that the ball was rolling, was moving along. What we found was...if we did this at 200 centimetres...it didn't apply...we would get a different rate out here at 400 centimetres.

John: We were getting ... so if the average speed is the slope of this line, we were getting the average speed slowing down as it went farther. And our first concern was that what we were looking at was rolling friction – that the further the bowling ball went the more it would slow down.

John drew a graph on the board to answer this question (Figure 7). He made the marks arbitrary (not related to the actual data) but he was merely trying to prove his point. Stephen continued the presentation.

Stephen: We went to 800 centimetres and we got a rate. Then we compared it to 900 centimetres and we got a rate and those rates were equal. So in actuality...If the ball was rolling at a constant rate that we initially measured this is what the graph should look like. [Stephen draws a straight line graph (almost $y = x$) and uses the same axis labels as John.] That's without friction, without anything. We would always get the same rate. And what we were finding was instead of getting a linear graph like that we were getting something that was leveling off [He draws a curve like John's]. And so we thought maybe friction was really coming into play down here, slowing down the ball, slowing down its rate of speed.

The group then explained how they conducted more trials at 600 centimetres away and at 1000 centimetres. At the end of "numerous trials", they felt they were able to get the same rate.

Jimmy: Of course there was some tape standing right here. We stuck some tape up so we could tell when the ball...we wouldn't measure where the edge of the ball passed the point but where the bowling ball was actually touching the ground, so there's both of us [he and John] standing there and we kind of had to guess when it crossed, but we were getting pretty close.

Professor: What do you mean by pretty close?

Jimmy: Well, within 5/100ths or so?

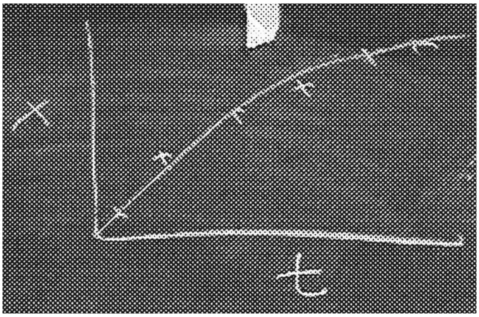


Figure 7. John's graph.

John: Yeah, 5/100ths.

Jimmy: And that's as fast as you could like start and stop a watch on your own.

Professor: So you figure if the numbers are the same to within the amount of time it takes you to start and stop the watch then you're going to say that that's ... you're getting a constant rate?

Jimmy: Well, if somebody else tries to duplicate this, it depends on when they stop it, too.

In actuality, the group determined two different rates, but ones they felt were "pretty close": 135.3 cm/s and 135.0 cm/s. The professor asked the class what they thought about these results.

Dave: Especially when you're doing it by hand with the time and the fact that you all had said, yes, there's friction that is gonna slow this ball down.

Stephen: I think what we did to get away from the friction was use a heavier ball. You see it come into play a lot less from what we observed to the 10 metre mark. It came into play a lot less. If we had let the ball go 20 metres we would have seen probably a lot more friction come into play. But using something that's very dense and very heavy we didn't notice it as much.

Jimmy: Oh, we used the 135.0.

Professor: Why did you use that one instead of the 135.3.

Jimmy: 'Cause when it travels this far it seems a little more accurate.

Dave: Is there really any difference in accuracy?

Jimmy: I think the longer away, the more accurate it would be.

Dave: But he [Stephen] even said that if you had gone out to 20 metres that would have been a lot less accurate. So, why is the longer one more accurate in this case?

Stephen: I think if you try and differentiate between the 135.3 and the 135.0 with our model, you really can't differentiate between the two because we have sources of error in there – the stopping. And the thing is that they were starting at the starting point; both of the timers were starting at the starting point. I was telling them when to stop so I had inaccuracy also come into play when I had to judge when the centre of the ball would pass the stop point. Then they would have to re-click. The two numbers are never going to be exactly the same.

The professor asked Veronica to write their procedure for finding the position of the ball t seconds after their observed motion stopped. The class suggested that Veronica should use 135.15.

Professor: Would that be better and why would it be better?

Lee: It's an average.

Dave: It's just an average of the two numbers you use.

Stephen: But those numbers themselves are averages.

Lee: Well, average some averages and it'll be even closer.

John: I think it would make things worse, actually. I don't think we should use that.

Veronica. It applies [to] something that wasn't really smooth [the roll?].

Stephen: If we could generalise it, it would be better.

Elizabeth: Okay, use the 135.3.

John: No, 135.

Veronica wrote t = 135 x d. It is interesting to note the non-issue of the starting point here. The formal equation could indicate a starting point of 0, although the reference they had was two metres from the end of the ramp. Over time, a consensus for a final answer is difficult to achieve. The participants claimed the motion was not linear, but wanted to come to a consensus on how they would justify such a claim, attempting to make a "fit" with a standard linear model, since they recognised that experimental error is involved.

Episode 2. The second episode involves two consecutive presentations from prospective teachers (in groups of two) after working on a problem given to them on a worksheet:

> Some students are studying the motion of a bowling ball rolling down a lane at the bowling alley. A student with a stopwatch is positioned at the start of the lane, and every two metres after that. Each student stops her watch as the ball passes her. They want to predict how long it will take the ball to reach the pins, 1 metre beyond the last student. Explain how you would help them figure this out, first in words, and then with an equation. Explain why the equation is the right equation to use.

Student 1	0.27 s
Student 2	0.75 s
Student 3	1.25 s
Student 4	1.77 s
Student 5	2.25 s
Student 6	2.74 s
Student 7	3.25 s
Student 8	3.76 s
Student 9	4.24 s
Student 10	4.75 s

Students encountering this problem held discussions related to average, scale, and starting position of the object – originally open codes in the data set that were determined to be dimensions of the larger, core theme of "error". The first group presented its ideas on how to solve the problem. They tried two approaches because they did not know which would be more accurate. The first was to take the time difference between each student, sum the differences and divide by 9 to get an average time between each student. Both group members agreed on an average time value of .4977 seconds when using this approach. The associated rate, thus, became 4.0184 metres per second. Using this rate, both members obtained a value of 5.22 seconds as the final answer to the question. After checking their procedure using their calculated rate to obtain other known values in the table, they encountered what they called "discrepancies:"

> Lee: So there's kind of...we were like, 'Maybe, this isn't the right way to go.' So the other way we tried was taking student 10's measurement of 4.75 seconds and subtracting that from student one's and finding...that gave us 4.48. So, then we divided that by 20 and we came up with a rate of 4.46 [meaning 4.46 = 20 m/4.48]. Then using that rate, our time we came up with 4.708. That doesn't make sense because student 10 is set at 4.75 seconds.
>
> Linda: But, using the average, like, there was too far of a distance. It was like over a 1 second...no...it was like a 1 second distance, wasn't it?
>
> Professor: So, what do you mean?
>
> Lee: Like coming up with this 5.22, it's saying that it took whatever the difference between 5.22 and 4.75 seconds for it to go one metre. Which doesn't make sense to us because on the other ones, the differences, it took 2 metres in a half second. It went two metres in a half second. We're kind of lost.
>
> Professor: So, you're still not happy with your....
>
> Lee: Not happy with either way we went because we found discrepancies.

Following Lee and Linda's presentation, the next group of participants encountered a similar situation calculating average, yet they were explicit in connecting the calculation of average to initial position. They compared their procedure to the previous group's procedure by stating there was a discrepancy in choice of starting point. Stephen's group decided that student one is "at point 0."

> Stephen: So you can do this two ways. You can throw out student one's number or you can keep it and say between student one and student two you have approximately .5 seconds. Then in between student two and student three there's .5 seconds. The distance the ball travelled is the same for each, so it's approximately...the numbers aren't exact, but it's approximately 4.02 metres per second. And that's what they [Linda and Lee] got the first time.
>
> Lee: 4.0184. Yeah, same thing.

Stephen and Veronica continued their presentation by outlining their second method of calculating the final answer and the viability of another approach.

Stephen: They're about the same thing. We went back and we saw if you took the initial time, which is .27, and the final time, 4.75, and then the distance in between those, you get the same exact thing. Well, it's a 4.0187. But the thing is...for each student getting a different time and a rate between each student is different....like, between student one and student two, we got a rate of 4 metres per second. Between student three and student four I've got a rate of 3.85 metres per second. So, that's a big difference. But the thing is we're talking students hitting stopwatches and we're talking about a bowling ball that has no internal, like, motor or anything. So, we're assuming acceleration is zero. We can assume that this velocity, or this speed, is the same. So 19 metres divided by 4.02 metres per second, we...gives you 4.72 seconds, but you have to also take into account the first student's time, .27 seconds. We got 4.99 seconds on a stopwatch, if you stopped it at the 19-metre mark.

Linda: We just didn't think that ours was very, like, accurate.

Stephen: But I think you gotta look at where your data's coming from. Ours is coming from a lot of uncertainty. So, if we were gonna do a prediction it's only gonna be an estimation.

Lee: Well, doing it either way is gets us within .03, .04, give or take, plus or minus.

Observing both methods, Lee wondered which procedure was most beneficial. Students showed differing opinions about this as exemplified in the following vignette.

Lee: You know how we did it both ways, like, taking the average between each second? Then taking like just from the end to the beginning? Well, we're wondering which way you think could be more accurate. Because, like taking from the beginning to the end just kind of auto corrects that average, y'know 'cause the students, as we've seen through the numbers...like one stops a little sooner, one a little later...

Stephen: I think with a lot more data points...I think by doing the average in-between would end up canceling out the error. If you think ...Okay, student one to student two to student three...student one stops his stopwatch early which means the rate between him or.... student one to student two to student three...student two stops his stopwatch early which means the rate between student one to student two is going to be small. But, the rate between student two and student three is already now larger. So, it's adding out. They're canceling each other out.

Dave: I like looking at each one of the intervals. I mean as long as we have these intervals, we might as well look at 'em. And not take the big leap from student one to student ten. If you got the data, look at it.

Adrian: Might as well use it.

Professor: So you look at it, and we've seen some variation.

Adrian: Slight.

Dave: That's what averages are for. Well, that's not all. You have to define your acceptable losses.

Neither Dave nor other members of the class explained what "acceptable losses" implied or how to resolve the issue of using an average over a finite number of points or an undetermined, larger number of points.

Episode 3. The third and final episode came from a class period in which the university professor reviewed what the class had decided upon as a procedure (or model) to describe and predict constant motion. Paul pointed out that calculating the rate or velocity should be done by interval (e.g., a final position minus an initial position divided by total time for each time period). The professor pointed out that there are two cases in her mind based on what the class did: where the change in positions between intervals is exact (as shown in some of the problems on the handout) and where the change is not exact, but includes error (as shown in the class experiments). Stephen believed that Paul's method of rate was still valid to use despite error and variation in the data. He believed it to be a good procedure that could come close to modelling a "perfect" experiment.

Stephen: If our equipment was perfect and our timing was perfect, and if our measurements were perfect, I think we would boil it down to something like that. Say the bowling ball goes ten metres. We can find the distance versus time from 2 to 4 metres and divide that by time. That should be the same rate if we did it from 6 to 8 metres and divided it by that time also.

Lee: Yeah, if it's truly constant you don't have to worry about the time in between, like, all that's important is the final and the initial.

Based on this argument, the question asked of the class was: How do you judge a motion to be "truly constant?"

Stephen: So this is like defining constant motion?

John: Is the question what do you do when you have constant motion or how do you know you have constant motion?

This brief exchange highlighted a tension between the mathematics and science realms. One may argue that someone may see a distinction between the model as descriptive (or as a representation) and the model as a calculational tool. Paul argued that the "average of a sum" is the same as "the sum of the averages." The professor disagreed because what happens in each of the intervals may not be the same. She brought up the bowling ball example where the roll is much faster at the beginning. Paul disagreed although the professor believed it depended on how you measure time intervals.

Dave: Adding up all the little averages is the same thing as taking one big average. You're adding up to the same thing.

Paul: Because if one has a larger velocity then the next time interval will have a smaller one.

John: But in the bowling ball example that we worked in class last time, what you were looking at was not the actual speed, but a lot of what seemed to be like a lot of error in the way the stopwatches were going and if you just looked at the last guy, and the last guy was very bad with the stopwatch, he would throw off the whole. So, in that example, you're better off looking at all the intervals.

Professor: But what Paul is saying is if you include that last interval, you've got that bad set of data for the last interval, too, and it's going to drag down your whole average.

John: Yeah, but it doesn't have the whole weight. It doesn't have the same weight as...

Professor: I think it depends upon how your data are sampled.

John: I have to think about that.

The professor asked: "When is it good to rely on Paul's method and when is it good to rely on Lee's method (averaging)?" Lee felt that if there was no pattern in the data, you must rule out constant motion. The professor reminded them that they did not rule out constant motion with the bowling ball experiment despite variation in the data. However, Lee said that's all they had to perform calculations and that it was extremely hard to find constant motion in any of the experiments because of other factors. However, the professor reminded them they had discussions of "good enough" regarding other constant motion experiments such as the ball rolling along the rim of a lid.

Dave: Is that change in there really significant? How do you look at your data and say, 'Well, this is a significant difference or not?' compared to those students who are each two metres apart (referring to the worksheet problem), but I guarantee you they're not two metres apart. They may be 2.004 or 2.02 metres apart, but the problem doesn't care about that small difference. So, if that's already your limiting factor, saying that you're exactly two metres apart...if that's a limiting factor, then your rate should also only be looked at to that limiting factor. Like, if it's 4.282 versus 4.284, well that little difference doesn't compare to the thing that is not looked at with the 2 metres apart. So, that's when we can tell, 'Is there a pattern or not?' 'Is this constant or not?'

Lee: But that's compounded by the fact that people are sitting there trying to stop it as close as they can.

Dave: But then there's always...you can't always do that either. You can't say, 'This is 2. This one's 6. I'm just gonna call that 4.' There's gotta be some way to limit that as well.

Stephen: If you knew a theoretical, what the speed should be, then you can compare it using statistical analysis.

Dave: If this is just a problem, you don't wanna say, 'I've got this data that already has what...I already have the answer.' That's not what physics ... we don't want to find an answer we already know.

Summary. In summary, all episodes helped support a conjecture with regard to average, scale, and, to a lesser degree, initial position. Figure 8 outlines the tensions more fully and provides a more concrete view that average and scale were more prominent issues than initial position. Therefore, selective coding proved fruitful in identifying similar tensions seen in the pilot study.

Participants working with the constant motion problem(s) encountered these tensions and attempted to resolve them. Examining all learners' modelling processes through the tensions lens resulted in the emergence of a critical theme: Construction of a model that is "good enough" based on human and experimental error and what the definition of "good enough" encompasses.

Furthermore, results also indicate that the participants possessed the capability of constructing and developing powerful mathematical models (e.g., $d = rt$, $p = rt + p_o$) using a constructivist approach based on inquiry rather than direct instruction. Within the process of constructing these models, learners attempted to reconcile conflicts or tensions among their personal experience with the phenomenon, learning standard mathematical concepts, and learning standard physics concepts.

Discussion

The results of the study reveal the complexity involved when constructing a mathematical model to describe and predict the motion of an object. When

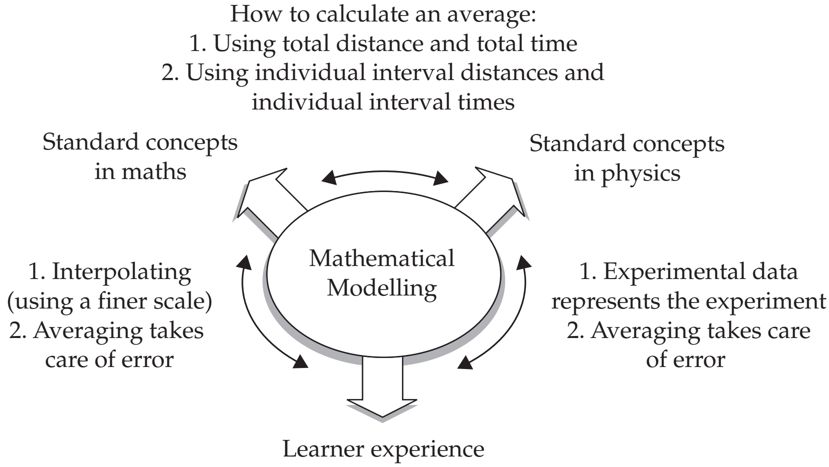


Figure 8. A summary of tensions related to average and scale.

immersed in a set of modelling activities that do not rely on direct instruction methods or procedures, learners become engaged in an authentic process that is both mathematical and scientific in nature. Such engagement aligns closely with expectations outlined by national standards and by national science and mathematics organisations and norms within the science and mathematics communities themselves. Within the process of constructing a mathematical model, learners attempt to reconcile conflicts or tensions among their personal experience with the phenomenon, learning standard mathematical concepts, and learning standard physics concepts. Analysis of efforts to link all three realms results in the emergence of critical themes highly relevant to both learners and teachers as they are engaged in the mathematical modelling process. The researcher examined the episodes identified as typical in the grounded theory coding using the tensions framework shown in Figure 4.

Constructing a Model that is "Good Enough"

Considerations of what makes a model "good enough" to use rest on deeply held convictions of how a mathematical model should or should not accurately and precisely describe and explain a real world phenomenon. While such a theme may seem obvious or trite, it is of profound significance for two reasons. First, the demand that learners make connections between mathematics and the real world has been, and will continue to be, at the forefront of most major reform efforts. Both teachers and students will often question the nature of mathematics and the reasons for learning mathematics as they try to meet educational goals. Such questions deserve to be answered and need to be addressed to support reform efforts. Secondly, learners' questions of what is "good enough" could rest on the development (or lack of development) of certain mathematical constructs. Exploring the nature of student thinking regarding these constructs could provide rich learning trajectories that could help students link the real world with more abstract, mathematical models in a far more conceptual way. It would also provide for them another facet of the nature of mathematical thinking and learning and provide mathematical empowerment they otherwise would not obtain through direct instruction (e.g., being told that the model is already accepted and they must learn it as such).

Limitations of the Study

Methodology. Given the use of the Grounded Theory approach to data collection, certain limitations of the method are evident. Taber (2000) warns of a researcher relying too formally on the "algorithm" for grounded theory. What appears to outline a procedure for making clear-cut decisions actually indicates that the development of a theory is never complete. According to Glaser (1978), "Grounded theory...makes [the analyst] humble to the fact that no matter how far he goes in generating theory, it appears as merely 'openers' to what he sees that could lie beyond" (p. 6). Given the implications of certain findings while conducting classroom research, there is always a concern that certain factors or thought processes could initially be overlooked or considered trivial. It is only through repeated implementations relying on sound conjectures that other factors can either be brought to the fore or determined irrelevant.

To discuss other limitations, the researchers rely on the framework presented by Cobb et al. (2001) and their work on analysing classroom mathematical practices using a modified Grounded Theory approach. Specifically, Cobb and his colleagues analyse their methodology in terms of trustworthines, replicability and commensurability, and usefulness.

Trustworthiness. The difficulty of presenting critical episodes in isolation cannot be overlooked. Episodes indicating certain mathematical threads of reasoning make sense only within the context of the entire study, and the reader must rely on the researcher's claim that presented inferences or themes span the entire data set. Furthermore, isolating certain episodes immediately leads to a tendency on the part of the reader to present alternative interpretations of reasoning exhibited in the vignette. This may be done without the reader realising the full scope of the analysis undertaken to choose the episodes as examples of an identified pattern of reasoning evident throughout the entire data set. The researchers hope that this issue can be addressed by selective coding and focusing on identifying core categories for future studies. The process of using more studies allows for development and refinement of initial conjectures as the researcher moves from one study to the next. The researchers also concede that conjectures developed from a grounded theory approach are always open to refutation and alternative interpretation. However, given the possibility of conducting further studies, the researcher feels confident that the validity of inferences and conjectures can become more firmly established.

Replicability and commensurability. Cobb et al. (2001) claim that mathematics education research is "replete with more than its share of disparate and irreconcilable findings" (p. 153). The researcher must answer the question of whether or not implementation of the same unit in a different classroom would yield the same findings and conclusions. The possibility of answering such a question stems from the importance of considering classroom context and setting not only when implementing the unit but also when analysing data. The advantage of the researcher's approach is that students' learning outcomes can be related to a learning situation, a desirable goal established by school reformers that the researcher feels would not be contested by professional teachers or mathematics and science education researchers.

Naturally, there are other limitations related to typical classroom practice that inhibit the implementation (though not the validity) of such an approach. One consideration is time constraints of the typical school schedule. A second limitation is the difficulty of analysing and documenting individual student learning. The interviews conducted in this study, for example, played a dual role in not only probing student thinking, but also evaluating how much the students had learned throughout the course of the unit. Analysis of these interviews is complex in that both roles may be used to develop conjectures and recommendations for further study and future implementations. Thirdly, while a grounded theory approach is context-based, the study presented does not account for either gender differences or other issues related to equity. For many schools, these are considerable factors for analysis and debate. Finally, one crucial factor is that the study was heavily concerned with mathematical meaning of critical concepts in kinematics. The researchers assume a certain level of content knowledge on the part of the teacher and his or her concern about whether or not such an approach will help teachers and students realise certain education goals.

During the pilot study and the present study, learners alluded to the significant role the teacher plays in inquiry-based learning and the importance of content knowledge:

> Joan: I just realised how much content knowledge that science and maths teachers have to have to get these kinds of discourse patterns in a classroom. To look at these relationships, the level of content knowledge that somebody needs is...well, that's my observation. But your strategy, though, to get people to see these relationships which has really moved away from just the procedural was to create these discourse patterns, and I look at the level of knowledge you have to have in order to really create that and to have people who are making sense of it all along the way. I was watching the pedagogy as much as trying to get the [ideas].

The researchers feel that the study makes a contribution by providing an indication, at least, of the type of content knowledge necessary or desirable for teachers to implement such an approach to studying kinematics (e.g., function and average) given calls for reform and guidelines presented by both national and state standards for mathematics and science.

Usefulness. The study presented provides a means to support discussions regarding professional development of teachers. Given that the study links classroom setting and learning, the opportunity for teachers to link context and instructional practice is evident. Through such studies, teachers can learn how to test, adapt, and modify certain approaches in the classroom based on student learning and desired outcomes. The complexity of such an approach, however, necessarily requires change to be a more time-consuming, and continuous, process of learning and implementing on the part of the classroom teacher.

Recommendations for Further Research

Critical themes embed epistemological considerations and cognitive conflict within an inquiry-based approach: "What is the meaning and role of abstraction, formalisation and generalisation in applications and modelling?" The following questions merit further study. If students are able to study, conceptualise, and understand possible approaches to resolving experimental error, will this allow them to accept and understand connections made between a formal model and their real-world experiences more readily? What makes a model "good enough" for students? Further implementations of the unit would provide more specific answers to this question.

Finally, teachers' beliefs and perceptions of models in science and science education have an effect on students' learning outcomes. The impact or implications on instruction or classroom practice requires further investigation to support implementation of modelling approaches. Such studies in this vein are rare (Justi & Gilbert, 2002; Van Driel & Verloop, 1999, 2002). Other specific issues, also raised by ICMI (2003, pp. 12-13) and related to epistemology, are also informed by our results: (1) How do teachers set up authentic applications and modelling tasks? and (2) What is essential in teacher education programs to prepare prospective teachers to teach applications and modelling?

Learners' involvement with the unit on kinematics highlights the need to bridge a gap between mathematics and physics concepts and the practices of experimentation, data gathering, and analysis of real world data. Woolnough (2000) emphasises that students must see "links between the mathematical processes they are using and the physics they are studying" (p. 259). In order to help students obtain learning goals, teachers must also be able to create and strengthen such links. A difficult, though notable, goal is to have teachers link not only the mathematics and physics worlds through critical concepts, but also link the mathematics and science realms to learners' experience. A model-based or inquiry-based approach appears to be the best means to reach this goal, though much work must be done in terms of teacher preparation and re-evaluating certain educational goals before substantial, worthwhile benefits are realised. Furthermore, establishing a uniform theory of modelling in mathematics and science classrooms can support efforts to reach the goal and maximise benefits for both teachers and students.

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Authors

David J. Carrejo, Department of Teacher Education, University of Texas at El Paso, 500 West University Avenue, EDUC 811, El Paso, Texas, 79968, USA. Email: <dcarrejo@utep.edu>

Jill Marshall, Department of Curriculum & Instruction, University of Texas at Austin, 1 University Station D5700, Austin, TX 78712, USA. Email: <marshall@mail.utexas.edu