An investigation of the algebraic curve $y^3 - 3y + 2x = 0$

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Introduction

In this paper, I investigate the algebraic curve defined by the relation

$$y^3 - 3y + 2x = 0 \tag{1}$$

Treating this relation as a reduced cubic in the variable y, I use a procedure first discovered by the mathematician Scipione del Ferro (Nahin, 1998, pp. 8–10) to obtain an expression for y in terms of x, namely

$$y = \left(-x + \sqrt{x^2 - 1}\right)^{\frac{1}{3}} - \left(x + \sqrt{x^2 - 1}\right)^{\frac{1}{3}} \tag{2}$$

By applying de Moivre's theorem to each term on the right hand side of equation (2), I obtain three different branches of y and use them to show that the domain of the curve is the set of all real numbers. I find the derivative using implicit differentiation and use it to determine additional properties and features of these branches.

Although it might seem that a graphics calculator can be used to draw a graph of the curve from equation (2), there is a problem in that such graphs have a break over the interval $-1 < x < 1$. This suggests (incorrectly) that the interval $-1 < x < 1$ is not in the domain of the curve. I explain the reason for this problem in the last part of the paper.

Rationale for the paper

The investigation of an algebraic curve of the type exemplified in equation (1) has several potential uses:

- It exposes students to the procedure for solving a reduced cubic, that is, a cubic in which the second degree term is missing. While all senior secondary school mathematics students are familiar with the procedure for solving a general quadratic equation, few are familiar with or even

aware of the existence of the procedure for solving a general cubic equation. Using a suitable linear substitution rediscovered by the mathematician Girolamo Cardano (1501–1576), any cubic can be transformed into a reduced cubic (Nahin, 1998, pp. 16–17). The problem of solving a general cubic therefore becomes that of solving a reduced cubic.

- It illustrates how the complex number field can be used to find solutions in the real number field. The del Ferro solution to the reduced cubic was "central to the first progress made towards understanding the square root of minus one" (Nahin, 1998, p. 8) since square roots of negative numbers often arose in the del Ferro solution to cubic equations known to have only real solutions. The mathematician Rafael Bombelli (1526–1572) first proved that such "weird" solutions are real, but expressed in a very unfamiliar form, and gave an entirely original treatment of the imaginary quantities arising in such solutions (Nahin, 1998, pp. 18–22). He provided the first rules for addition, multiplication and division of imaginary numbers.
- Treating x as a parameter, the curve can be used to explore the roots of the cubic $y^3 - 3y + 2x = 0$ for different values of x.
- It applies a variety of mathematical knowledge and techniques found in typical senior secondary school advanced mathematics curricula to an unfamiliar problem.
- It provides an example (and perhaps a salutary warning) of how answers produced by a CAS or graphics calculator can be different (and in fact contextually incorrect) to those obtained from "by-hand" calculations.

The branches of $y^3 - 3y + 2x = 0$

Calculations

The algebraic curve defined by the relation $y^3 - 3y + 2x = 0$ has three different branches. First suppose one of these branches is given by y_1. Then the other two branches are given by

$$y_2 = \frac{1}{2}\left(-y_1 + \sqrt{3}\sqrt{4 - y_1^2}\right) \tag{3}$$

$$y_3 = \frac{1}{2}\left(-y_1 - \sqrt{3}\sqrt{4 - y_1^2}\right) \tag{4}$$

Equations (3) and (4) are derived as follows:

If y_1 is a branch of the curve, then $y = y_1$ is a solution to equation (1), and $y - y_1$ is a factor of $y^3 - 3y + 2x$. Dividing $y - y_1$ into $y^3 - 3y + 2x$ gives

$$y^3 - 3y + 2x = (y - y_1)(y^2 + y_1 y + [y_1^2 - 3])$$
$$\therefore\ y^3 - 3y + 2x = 0 \Leftrightarrow (y - y_1)(y^2 + y_1 y + [y_1^2 - 3]) = 0$$

Therefore the other two branches are solutions to the quadratic equation

$y^2 + y_1y + [y_1^2 - 3] = 0$.

Using the quadratic formula yields

$$y = \frac{1}{2}\left(-y_1 \pm \sqrt{3}\sqrt{4 - y_1^2}\right)$$

Equations (3) and (4) only define real values of y_2 and y_3 when $4 - y_1^2 \geq 0 \Rightarrow -2 \leq y_1 \leq 2$.

The next task is to find the Cartesian equation of each branch, which requires getting an expression for y in terms of x. This can be done by treating equation (1) as a reduced cubic in the variable y and using a procedure first discovered by the mathematician Scipione del Ferro (1465–1526) (Nahin, 1998, pp. 8–10). Let $y = u - v$ and substitute this expression into equation (1):

$$\begin{aligned}
(u - v)^3 - 3(u - v) + 2x &= 0\\
\therefore\ (u^3 - 3u^2v + 3uv^2 - v^3) - 3(u - v) + 2x &= 0\\
\therefore\ u^3 - v^3 + 3uv^2 - 3u^2v - 3(u - v) + 2x &= 0\\
\therefore\ u^3 - v^3 + 3uv(v - u) + 3(v - u) + 2x &= 0\\
\therefore\ u^3 - v^3 + 3(v - u)(uv + 1) + 2x &= 0 \qquad (5)
\end{aligned}$$

Equation (5) can be solved by solving the following pair of simultaneous equations:

$$u^3 - v^3 + 2x = 0 \qquad (6)$$

$$uv + 1 = 0 \qquad (7)$$

since equations (6) and (7) force the left hand side of equation (5) to equal zero. Therefore, substitute

$$v = -\frac{1}{u}$$

from equation (7) into equation (6):

$$u^3 + \frac{1}{u^3} + 2x = 0$$

$$\therefore u^6 + 2xu^3 + 1 = 0 \qquad (8)$$

Treat equation (8) as a quadratic equation in u^3 and use the quadratic formula to solve for u^3:

$$u^3 = \frac{-2x \pm \sqrt{4x^2 - 4}}{2} = -x \pm \sqrt{x^2 - 1} \qquad (9)$$

$$\therefore u = \left(-x \pm \sqrt{x^2 - 1}\right)^{\frac{1}{3}} \qquad (10)$$

Substitute equation (9) into equation (6):

$$-x \pm \sqrt{x^2 - 1} - v^3 + 2x = 0$$

$$\therefore v = \left(x \pm \sqrt{x^2 - 1}\right)^{\frac{1}{3}} \qquad (11)$$

Substitute

$$u=\left(-x+\sqrt{x^2-1}\right)^{\frac{1}{3}} \tag{12a}$$

$$v=\left(x+\sqrt{x^2-1}\right)^{\frac{1}{3}} \tag{12b}$$

into $y = u - v$:

$$y=\left(-x+\sqrt{x^2-1}\right)^{\frac{1}{3}}-\left(x+\sqrt{x^2-1}\right)^{\frac{1}{3}} \tag{13}$$

Note that equations (12a) and (12b) define three distinct complex expressions (branches) for u and v respectively. However, only those combinations of u and v which satisfy equation (7) are valid. Failure to recognise this has led to the erroneous conclusion that the del Ferro procedure sometimes yields incorrect solutions (see for example John, 2003). As will be seen below, for all values of x there is at least one combination of u and v satisfying equation (7) such that the imaginary parts of u and v are equal and hence $y = u - v$ is real. This shows that the domain of the curve is the set of all real numbers. Note that the solutions for u and v with the negative square root terms given in equations (10) and (11) also give equation (13) when substituted into $y = u - v$. These solutions are redundant and therefore are not used.

Case 1: $x^2 - 1 < 0 \Rightarrow -1 < x < 1$

When $-1 < x < 1$, the combinations of u and v which satisfy equation (7) always result in three distinct real expressions for y. This indicates that the curve defined by $y^3 - 3y + 2x = 0$ has three branches.

Example: $x = 0$

When $x = 0$ it follows from equations (12a) and (12b) that $u = i^{1/3}$ and $v = i^{1/3}$. The polar form of i is

$$i = cis\left(\frac{\pi}{2}+2n\pi\right),\ n \in J$$

Applying de Moivre's theorem then gives the following three distinct values of u and v respectively:

$$u_{-1}=-i,\ u_0=\frac{\sqrt{3}}{2}+\frac{i}{2},\ u_1=-\frac{\sqrt{3}}{2}+\frac{i}{2}$$
$$v_{-1}=-i,\ v_0=\frac{\sqrt{3}}{2}+\frac{i}{2},\ v_1=-\frac{\sqrt{3}}{2}+\frac{i}{2}$$

The combinations (u_n, v_m) of u and v which satisfy equation (7) are (u_{-1}, v_{-1}), (u_0, v_1) and (u_1, v_0). For each of these combinations, the imaginary parts of u and v are equal, resulting in three distinct and real values of y, namely

$$y_1=u_{-1}-v_{-1}=0,\ y_2=u_0-v_1=\sqrt{3},\ y_3=u_1-v_0=-\sqrt{3}$$

Note that:

- from equation (1), $x = 0$ gives three distinct real values of y:

$$y^3 - 3y = 0 \ \therefore\ y(y^2 - 3) = 0 \ \therefore\ y = 0, \pm\sqrt{3}$$

- from equations (3) and (4), $y_1 = 0$ gives

$$y_2 = \frac{1}{2}\left(0+\sqrt{3}\sqrt{4-0}\right) = \sqrt{3}$$
$$y_3 = \frac{1}{2}\left(0-\sqrt{3}\sqrt{4-0}\right) = -\sqrt{3}$$

General analysis for $-1 < x \le 0$

Let $A = -x$ and $B = \sqrt{x^2-1} = i\sqrt{1-x^2}$ so that $A = |A|$ and $B = i|B|$. It follows from equations (12a) and (12b) that

$$u = \left(A+B\right)^{\frac{1}{3}} = \left(|A| + i|B|\right)^{\frac{1}{3}}$$
$$v = \left(-A+B\right)^{\frac{1}{3}} = \left(-|A| + i|B|\right)^{\frac{1}{3}}$$

The polar forms of $|A| + i\,|B|$ and $-|A| + i\,|B|$ are

$$|A| + i\,|B| = cis(\alpha + 2n\pi),\ n \in J \tag{14a}$$
$$-|A| + i\,|B| = cis(\pi - \alpha + 2m\pi) = cis(-\alpha + (2m+1)\pi),\ m \in J \tag{14b}$$

where

$$\tan\alpha = \frac{|B|}{|A|} = \left|\frac{\sqrt{1-x^2}}{x}\right|,\ 0 < \alpha \le \frac{\pi}{2} \tag{15}$$

Note that $$r = \sqrt{|A|^2 + |B|^2} = \sqrt{x^2 + (1-x^2)} = 1$$

Applying de Moivre's theorem to equations (14a) and (14b) gives the following three distinct expressions (branches) for u and v respectively:

$$u_{-1} = cis\left(\frac{\alpha}{3} - \frac{2\pi}{3}\right),\ u_0 = cis\left(\frac{\alpha}{3}\right),\ u_1 = cis\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right)$$
$$v_{-1} = cis\left(-\frac{\alpha}{3} - \frac{\pi}{3}\right),\ v_0 = cis\left(-\frac{\alpha}{3} + \frac{\pi}{3}\right),\ v_1 = cis\left(-\frac{\alpha}{3} + \pi\right)$$

The combinations (u_m, v_n) of u and v which satisfy equation (7) are (u_{-1}, v_{-1}), (u_0, v_1) and (u_1, v_0). Application of the compound angle formulae shows that for each of these combinations, the imaginary parts of u and v are equal, resulting in three distinct and real expressions for y, namely

$$\begin{aligned} y_1 &= u_{-1} - v_{-1} \\ &= cis\left(\frac{\alpha}{3} - \frac{2\pi}{3}\right) - cis\left(-\frac{\pi}{3} - \frac{\alpha}{3}\right) \\ &= \cos\left(\frac{\alpha}{3} - \frac{2\pi}{3}\right) + i\sin\left(\frac{\alpha}{3} - \frac{2\pi}{3}\right) - \cos\left(-\frac{\pi}{3} - \frac{\alpha}{3}\right) - i\sin\left(-\frac{\pi}{3} - \frac{\alpha}{3}\right) \\ &= -\cos\left(\frac{\alpha}{3}\right) + \sqrt{3}\sin\left(\frac{\alpha}{3}\right) \end{aligned} \tag{16a}$$

$$\begin{aligned}
y_2 &= u_0 - v_1 \\
&= cis\left(\frac{\alpha}{3}\right) - cis\left(\pi - \frac{\alpha}{3}\right) \\
&= \cos\left(\frac{\alpha}{3}\right) + i\sin\left(\frac{\alpha}{3}\right) - \cos\left(\pi - \frac{\alpha}{3}\right) - i\sin\left(\pi - \frac{\alpha}{3}\right) \\
&= 2\cos\left(\frac{\alpha}{3}\right) \qquad (16b)
\end{aligned}$$

$$\begin{aligned}
y_3 &= u_1 - v_0 \\
&= cis\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right) - cis\left(\frac{\pi}{3} - \frac{\alpha}{3}\right) \\
&= \cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right) - \cos\left(\frac{\pi}{3} - \frac{\alpha}{3}\right) - i\sin\left(\frac{\pi}{3} - \frac{\alpha}{3}\right) \\
&= -\cos\left(\frac{\alpha}{3}\right) - \sqrt{3}\sin\left(\frac{\alpha}{3}\right) \qquad (16c)
\end{aligned}$$

Since $0 < \alpha \le \frac{\pi}{2}$, it follows that

$$-1 < y_1 \le 0,\ \sqrt{3} \le y_2 < 2,\ -\sqrt{3} \le y_3 < -1 \qquad (17)$$

General analysis for $0 \le x < 1$

Let $A = -x$ and $B = \sqrt{x^2 - 1} = i\sqrt{1 - x^2}$ so that $A = -|A|$ and $B = i|B|$. It follows from equations (12a) and (12b) that

$$u = (A + B)^{\frac{1}{3}} = (-|A| + i|B|)^{\frac{1}{3}}$$

$$v = (-A + B)^{\frac{1}{3}} = (|A| + i|B|)^{\frac{1}{3}}$$

Noting the symmetry between these expressions and those found in the analysis for $-1 < x \le 0$, it follows that there are three distinct and real expressions for y, namely

$$y_1 = \cos\left(\frac{\alpha}{3}\right) - \sqrt{3}\sin\left(\frac{\alpha}{3}\right) \qquad (18a)$$

$$y_2 = \cos\left(\frac{\alpha}{3}\right) + \sqrt{3}\sin\left(\frac{\alpha}{3}\right) \qquad (18b)$$

$$y_3 = -2\cos\left(\frac{\alpha}{3}\right) \qquad (18c)$$

where α is defined as in equation (15). It follows that

$$0 \le y_1 < 1,\ 1 < y_2 \le \sqrt{3},\ -2 < y_3 \le -\sqrt{3} \qquad (19)$$

Note that equations (16b), (16c), (18b) and (18c) follow from equations (3), (16a), (4) and (18a). For example, substitute equation (16a) into equation (3):

$$y_2 = \frac{1}{2}\left\{\cos\left(\frac{\alpha}{3}\right) - \sqrt{3}\sin\left(\frac{\alpha}{3}\right) + \sqrt{3}\sqrt{4 - \left[-\cos\left(\frac{\alpha}{3}\right) + \sqrt{3}\sin\left(\frac{\alpha}{3}\right)\right]^2}\right\} \qquad (20)$$

Now note that

$$\sqrt{4-\left[-\cos\left(\frac{\alpha}{3}\right)+\sqrt{3}\sin\left(\frac{\alpha}{3}\right)\right]^2}$$

$$=\sqrt{4-\cos^2\left(\frac{\alpha}{3}\right)+2\sqrt{3}\cos\left(\frac{\alpha}{3}\right)\sin\left(\frac{\alpha}{3}\right)-3\sin^2\left(\frac{\alpha}{3}\right)}$$

$$=\sqrt{4\cos^2\left(\frac{\alpha}{3}\right)+4\sin^2\left(\frac{\alpha}{3}\right)-\cos^2\left(\frac{\alpha}{3}\right)+2\sqrt{3}\cos\left(\frac{\alpha}{3}\right)\sin\left(\frac{\alpha}{3}\right)-3\sin^2\left(\frac{\alpha}{3}\right)}$$

$$=\sqrt{3\cos^2\left(\frac{\alpha}{3}\right)+\sin^2\left(\frac{\alpha}{3}\right)+2\sqrt{3}\cos\left(\frac{\alpha}{3}\right)\sin\left(\frac{\alpha}{3}\right)}$$

$$=\sqrt{\left[\sqrt{3}\cos\left(\frac{\alpha}{3}\right)+\sin\left(\frac{\alpha}{3}\right)\right]^2}$$

$$=\sqrt{3}\cos\left(\frac{\alpha}{3}\right)+\sin\left(\frac{\alpha}{3}\right) \qquad (21)$$

since $$3\cos\left(\frac{\alpha}{3}\right)+\sqrt{3}\sin\left(\frac{\alpha}{3}\right)>0 \text{ for } 0<\alpha\le\frac{\pi}{2}$$

Substituting equation (21) into equation (20) gives equation (16b).

Case 2: $x^2-1=0 \Rightarrow x=\pm 1$

When $x=\pm 1$, the combinations of u and v which satisfy equation (7) result in only two different real values of y (one of which is repeated).

Analysis for $x=1$

When $x=1$ it follows from equations (12a) and (12b) that $u=(-1)^{1/3}$ and $v=1^{1/3}$. Applying de Moivre's theorem gives the following three distinct values of u and v respectively:

$$u_{-1}=\frac{1}{2}-i\frac{\sqrt{3}}{2},\ u_0=\frac{1}{2}+i\frac{\sqrt{3}}{2},\ u_1=-1$$
$$v_{-1}=-\frac{1}{2}-i\frac{\sqrt{3}}{2},\ v_0=1,\ v_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

The combinations (u_n, v_m) of u and v which satisfy equation (7) are (u_{-1}, v_{-1}), (u_0, v_1) and (u_1, v_0). For each of these combinations, the imaginary parts of u and v are equal, resulting in two different real values of y, namely

$$y_1=u_{-1}-v_{-1}=1,\ y_2=u_0-v_1=1,\ y_3=u_1-v_0=-2 \qquad (22)$$

Note that:

- from equation (1), $x=1$ gives two different real values of y (one of which is repeated):

$$y^3-3y+2=0\ \therefore\ (y-1)^2(y+2)=0\ \therefore\ y=1,-2$$

- from equations (3) and (4), $y_1 = 1$ gives

$$y_2 = \frac{1}{2}\left(-1+\sqrt{3}\sqrt{4-1^2}\right)=1$$
$$y_3 = \frac{1}{2}\left(-1-\sqrt{3}\sqrt{4-1^2}\right)=-2$$

Analysis for $x = -1$
When $x = -1$ it follows from equations (12a) and (12b) that $u = 1^{1/3}$ and $v = (-1)^{1/3}$. Noting the symmetry between these values and those found in the analysis for $x = 1$, it follows that

$$y_1 = -1,\ y_2 = 2,\ y_3 = -1 \tag{23}$$

Note that these values are consistent with those derived from equation (1) and from equations (3) and (4).

It follows from inequations (17) and (19) and equations (22) and (23) that for $-1 \le x \le 1$,

$$-1 \le y_1 \le 1,\ 1 \le y_2 \le 2,\ -2 \le y_3 \le -1 \tag{24}$$

Note that these sets of values are consistent with those derived from equations (3) and (4) since

$$y_1 \le 1 \Rightarrow \begin{cases} y_2 \ge 1 \\ y_3 \ge -2 \end{cases} \text{ and } \ y_1 \ge -1 \Rightarrow \begin{cases} y_2 \le 2 \\ y_3 \le -1 \end{cases}$$

Case 3: $x^2 - 1 > 0 \Rightarrow x < -1$ or $x > 1$
When $x < -1$ or $x > 1$, combinations of u and v which satisfy equation (7) always result in only one real expression for y (and two non-real expressions).

General analysis for $\underline{x < -1}$
Let $A = -x$ and $B = \sqrt{x^2-1}$ so that $A = |\,A\,|$ and $B = |\,B\,|$. It follows from equations (12a) and (12b) that

$$u = \left(A+B\right)^{\frac{1}{3}} = \left(|\,A\,|+|\,B\,|\right)^{\frac{1}{3}}$$
$$v = \left(-A+B\right)^{\frac{1}{3}} = \left(-|\,A\,|+|\,B\,|\right)^{\frac{1}{3}}$$

Since $|\,A\,| + |\,B\,| > 0$ and $-|\,A\,| + |\,B\,| < 0$, the polar forms of $|\,A\,| + |\,B\,|$ and $-|\,A\,| + |\,B\,|$ are

$$|\,A\,|+|\,B\,| = \left(|\,A\,|+|\,B\,|\right) cis(2n\pi) = \left(-x+\sqrt{x^2-1}\right) cis(2n\pi),\ n \in J \tag{25a}$$

$$\begin{aligned} -|\,A\,|+|\,B\,| &= \left(|\,A\,|-|\,B\,|\right) cis(\pi+2m\pi) \\ &= \left(-x-\sqrt{x^2-1}\right) cis(\pi+2m\pi) \\ &= -\left(x+\sqrt{x^2-1}\right) cis([2m+1]\pi),\ m \in J \end{aligned} \tag{25b}$$

Applying de Moivre's theorem to equations (25a) and (25b) gives the following three distinct expressions (branches) for u and v respectively:

$$u_{-1} = \sqrt[3]{-x+\sqrt{x^2-1}}\ cis\left(-\frac{2\pi}{3}\right),\ u_0 = \sqrt[3]{-x+\sqrt{x^2-1}},\ u_1 = \sqrt[3]{-x+\sqrt{x^2-1}}\ cis\left(\frac{2\pi}{3}\right)$$

$$v_{-1} = -\sqrt[3]{x+\sqrt{x^2-1}}\ cis\left(-\frac{\pi}{3}\right),\ v_0 = -\sqrt[3]{x+\sqrt{x^2-1}}\ cis\left(\frac{\pi}{3}\right),\ v_1 = \sqrt[3]{x+\sqrt{x^2-1}}$$

The combinations (u_n, v_m) of u and v which satisfy equation (7) are (u_{-1}, v_{-1}), (u_0, v_1) and (u_1, v_0). However, only the combination (u_0, v_1) results in a real expression for y, namely

$$y_2 = u_0 - v_1 = \sqrt[3]{-x+\sqrt{x^2-1}} - \sqrt[3]{x+\sqrt{x^2-1}} \tag{26}$$

General analysis for $\underline{x > 1}$

Let $A = -x$ and $B = \sqrt{x^2-1}$ so that $A = -|\,A\,|$ and $B = |\,B\,|$. It follows from equations (12a) and (12b) that

$$u = \left(A+B\right)^{\frac{1}{3}} = \left(-|\,A\,|+|\,B\,|\right)^{\frac{1}{3}}$$

$$v = \left(-A+B\right)^{\frac{1}{3}} = \left(|\,A\,|+|\,B\,|\right)^{\frac{1}{3}}$$

Noting the symmetry between these expressions and those found in the analysis for $x < -1$, it follows that there is only one real expression for y, namely

$$y_3 = \sqrt[3]{-x+\sqrt{x^2-1}} - \sqrt[3]{x+\sqrt{x^2-1}} \tag{27}$$

Branches of $y^3 - 3y + 2x = 0$

The three branches y_1, y_2 and y_3 of $y^3 - 3y + 2x = 0$ are one-to-one functions defined by equations (16a), (18a), (15), (3), (26), (4) and (27):

$$y_1 = \begin{cases} -\cos\left(\frac{\alpha}{3}\right)+\sqrt{3}\sin\left(\frac{\alpha}{3}\right), & -1 \le x \le 0,\ -1 \le y \le 0 \\ \cos\left(\frac{\alpha}{3}\right)-\sqrt{3}\sin\left(\frac{\alpha}{3}\right), & 0 \le x \le 1,\ 0 \le y \le 1 \end{cases}$$

$$\text{where } \tan\alpha = \left|\frac{\sqrt{1-x^2}}{x}\right| \text{ and } 0 < \alpha \le \frac{\pi}{2} \tag{28}$$

$$y_2 = \begin{cases} \frac{1}{2}\left(-y_1+\sqrt{3}\sqrt{4-y_1^2}\right), & -1 \le x \le 1,\ 1 \le y \le 2 \\ \sqrt[3]{-x+\sqrt{x^2-1}} - \sqrt[3]{x+\sqrt{x^2-1}}, & -\infty < x \le -1,\ 2 \le y < +\infty \end{cases} \tag{29}$$

$$y_3 = \begin{cases} \frac{1}{2}\left(-y_1-\sqrt{3}\sqrt{4-y_1^2}\right), & -1 \le x \le 1,\ -2 \le y \le -1 \\ \sqrt[3]{-x+\sqrt{x^2-1}} - \sqrt[3]{x+\sqrt{x^2-1}}, & 1 \le x < +\infty,\ -\infty < y \le -2 \end{cases} \tag{30}$$

Note that since $-1 \le y_1 \le 1$ and the domain of y_1 is $-1 \le x \le 1$, equations (3) and (4) are only valid over the interval $-1 \le x \le 1$. Graphs of y_1, y_2 and y_3 are shown in Figure 1. From equations (29) and (30),

$$y \to \sqrt[3]{-x+x} - \sqrt[3]{x+x} = -\sqrt[3]{2x} \quad \text{as} \quad x \to \pm\infty$$

The curve $y = -\sqrt[3]{2x}$ is therefore an oblique asymptote of $y^3 - 3y + 2x = 0$ (see Figure 1(d)).

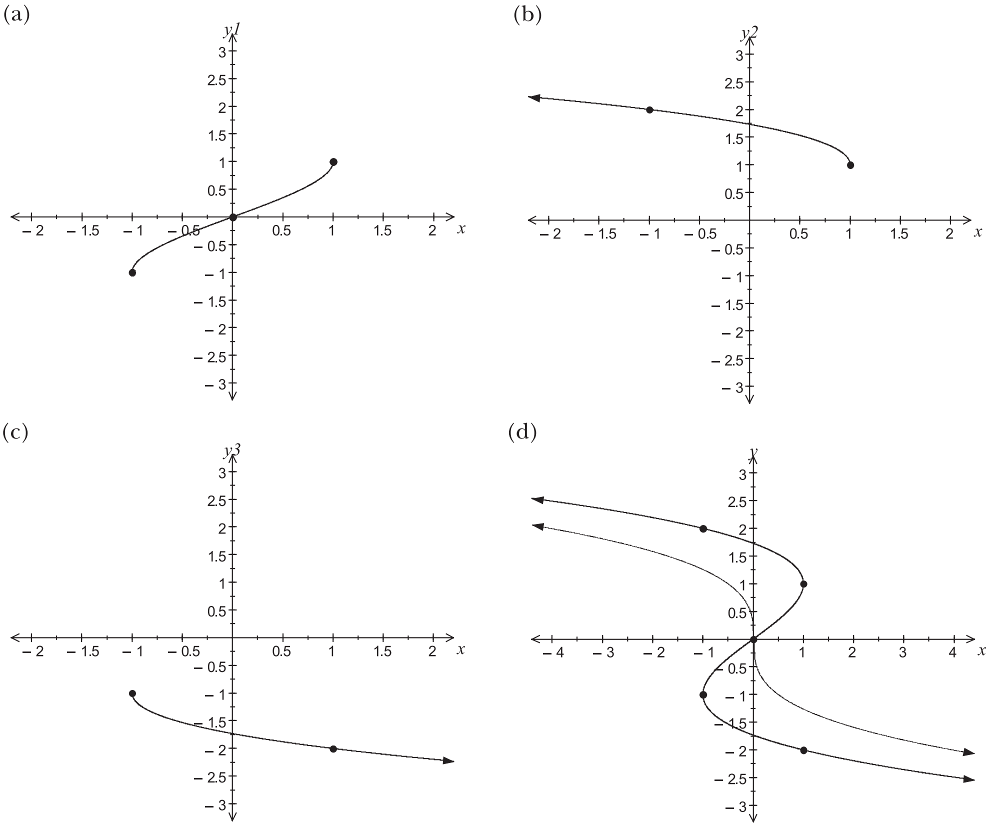

Figure 1. Graphs of the three branches y_1, y_2 and y_3 (defined in equations (28), (29) and (30) respectively) of $y^3 - 3y + 2x = 0$: (a) y_1, (b) y_2 and (c) y_3. (d) The curve $y^3 - 3y + 2x = 0$. The dotted line is the curve $y = -\sqrt[3]{2x}$.

Parametric representation of $y^3 - 3y + 2x = 0$

From equation (1),

$$x = \frac{3y - y^3}{2} \tag{31}$$

A possible parametric representation of $y^3 - 3y + 2x = 0$ is then

$$x = \frac{3t - t^3}{2} \tag{32}$$

$$y = t \tag{33}$$

where $-\infty < t < +\infty$. The three branches y_1, y_2 and y_3 are defined by $-1 \le t \le 1$, $1 \le t < +\infty$ and $-\infty < t \le -1$ respectively. Values found using equations (28), (29) and (30) are consistent with equations (32) and (33).

Inverse relation

Following from equation (31), the inverse relation of $y^3 - 3y + 2x = 0$ is defined by the equation

$$y = \frac{3x - x^3}{2} \tag{34}$$

It is readily confirmed that if the point (a, b) satisfies equation (34), then the

point (a, b) satisfies equations (28), (29) and (30). A graph of the inverse relation is shown in Figure 2. It is clearly the reflection of the curve $y^3 - 3y + 2x = 0$ around the line $y = x$.

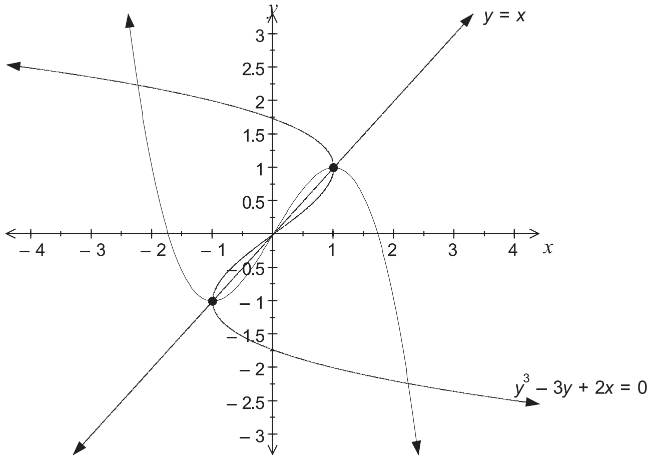

Figure 2. *Graph of* $y = \dfrac{3x - x^3}{2}$, *the inverse relation of* $y^3 - 3y + 2x = 0$.

The points of intersection of the relation and its inverse have coordinates $(1, 1)$, $(-1, -1)$, $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$. This can be verified using equations (32), (33) and (34). While the points $(1, 1)$ and $(-1, -1)$ obviously lie on the line $y = x$, the points $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$ do not. $y^3 - 3y + 2x = 0$ is therefore an example of a relation whose points of intersection with its inverse do not all lie on the line $y = x$.

In general, the points of intersection of a relation and its inverse always lie on one or both of the lines $y = x$ and $y = -x + c$, where c is a constant. Points of intersection that lie on the line $y = -x + c$ always occur in pairs of the form (a, b) and (b, a):

Consider a relation A with inverse A^{-1}. Let (a, b) be a point of intersection of A and A^{-1}. Then (a, b) satisfies both A and A^{-1}.

i. Since (a, b) satisfies A, the corresponding point satisfying A^{-1} is (b, a).

ii. Since (a, b) satisfies A^{-1}, the corresponding point satisfying A is (b, a).

Case 1: The points (a, b) and (b, a) are coincident.
$(a, b) \equiv (b, a) \Rightarrow a = b$. Therefore the intersection point lies on the line $y = x$.

Case 2: The points (a, b) and (b, a) are not coincident.
From (i) and (ii) above, it follows that (b, a) is a second point of intersection. The gradient of the line passing through the points and (b, a) is equal to -1. Then this pair of intersection points lies on the line with equation $y = -x + (a + b)$.

The inverse functions of the branches y_1, y_2 and y_3 are defined by equation (34) over the domains $-1 \le x \le 1$, $1 \le x < +\infty$ and $-\infty < x \le -1$ respectively.

Derivative

The derivative of the curve defined by $y^3 - 3y + 2x = 0$ can be found using implicit differentiation:

$$y^3 - 3y + 2x = 0 \Rightarrow 3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 2 = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2}{3}\frac{1}{y^2 - 1} \tag{35}$$

A graph of this derivative is shown in Figure 3.

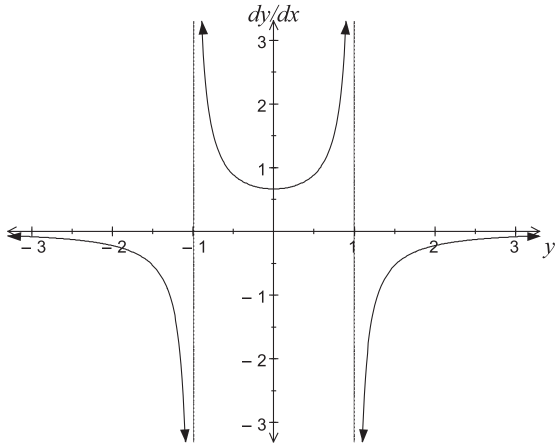

Figure 3. Graph of the derivative of $y^3 - 3y + 2x = 0$.

The following observations are made:

- The curve has a point of inflexion at (0, 0) since $\frac{dy}{dx}$ has a turning point at $y = 0$.
- The gradient of the tangent to the curve is undefined for $y = \pm 1$, that is, at the points (1, 1) and (–1, –1). These points are vertices of the curve.
- The branch y_1 is an increasing function since $-1 \le y_1 \le 1$ and $\frac{dy}{dx} > 0$ for $-1 \le y_1 \le 1$.
- The branch y_2 is a decreasing function since $1 \le y_2 < +\infty$ and $\frac{dy}{dx} < 0$ for $1 \le y_2 < +\infty$.
- The branch y_3 is a decreasing function since $-\infty < y_3 \le -1$ and $\frac{dy}{dx} < 0$ for $-\infty < y_3 \le -1$.

Problems with graphs drawn using a graphics calculator

Although it might seem that a graphics calculator can be used to draw a graph of the curve from equation (2), there is a problem in that such graphs have a break over the interval $-1 < x < 1$. This suggests (incorrectly) that the interval is not in the domain of the curve. In addition, the TI-89 calculator:

- returns a table of values showing undefined values of y for $-1 < x < 0$ and $0 < x < 1$ when in real mode (the TI-84 shows error for $-1 < x < 1$);
- draws no graph when in complex mode (the TI-84 draws the same graph as in real mode);
- returns a table of values showing non-real values of y for all real values

of x when in complex mode (the TI-84 shows error for $-1 < x < 1$).

The reason for these problems lies in the way in which the cube roots appearing in equation (2) are calculated. For example, when the TI-89 calculator is operating in real mode it uses the real branch (when it exists) for fractional powers that have a reduced exponent with odd denominator. When operating in complex mode, or when the real branch does not exist, the TI-89 uses the principle branch. Thus, for example, when operating in real and complex modes the TI-89 returns:

- the principle value $\frac{\sqrt{3}}{2}+\frac{i}{2}$ for $i^{1/3}$ (a real value does not exist);
- the real/principle value 1 for $1^{1/3}$ (the real and principle values are equal);
- the real and principle values -1 and $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ respectively for $(-1)^{1/3}$ (a real value exists and is different to the principle value). The TI-84 returns -1 in either mode, suggesting that in either mode it uses the real branch when it exists; otherwise the principle branch is used.

Note that when the parametric representation defined in equations (32) and (33) is used, a graphics calculator will draw an unbroken graph over $-\infty < x < +\infty$. This is expected since this representation does not require finding cube roots.

Case 1: $-1 < x < 1$

The values of u^3 and v^3 are always non-real for $-1 < x < 1$. When operating in either real or complex modes, the TI-89 calculator will therefore always use the principle values of u and v and therefore the combination (u_0, v_0) when calculating the value of $u - v$ for $-1 < x < 1$. With the exception of when $x = 0$, this leads to non-real values of y:

$$\begin{aligned} y &= u_0 - v_0 \\ &= cis\left(\frac{\alpha}{3}\right) - cis\left(\frac{\pi}{3}-\frac{\alpha}{3}\right) \\ &= \left[\cos\left(\frac{\alpha}{3}\right) - \sqrt{3}\sin\left(\frac{\alpha}{3}\right)\right]\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right] \\ &= -y_1\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right], \quad -1 < x \le 0 \end{aligned} \tag{36}$$

$$\begin{aligned} y &= \left[-\cos\left(\frac{\alpha}{3}\right) + \sqrt{3}\sin\left(\frac{\alpha}{3}\right)\right]\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right] \\ &= -y_1\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right], \quad 0 \le x < 1 \end{aligned} \tag{37}$$

Since non-real values do not get plotted, no graph is drawn over $-1 < x < 1$. Note that:

- the single point (0, 0) does not show on a graph but does appears in a table of values;

- equations (36) and (37) are not solutions to equation (1) since the combination (u_0, v_0) does not satisfy equation (7).

Although the TI-84 calculator also uses the principle values of u and v when $-1 < x < 1$, it returns graphs and tables showing an error for y when $x = 0$. A possible reason for this is that the method used by the calculator to find cube roots may lead to incomplete cancellation of imaginary terms in $u - v$ due to round-off error. This will lead to a value of y with a small but non-zero imaginary part, which the TI-84 cannot plot or tabulate.

Example: $x = -\frac{1}{2}$

When operating in complex mode, the TI-89 calculator returns $y \approx 0.1736 - 0.3008i$ for $x = -\frac{1}{2}$. This is consistent with equation (36):

From equation (16a),

$$y_1 = -\cos\left(\frac{\pi}{9}\right) + \sqrt{3}\sin\left(\frac{\pi}{9}\right)$$

since $$0 < \alpha \le \frac{\pi}{2}$$

and $$\tan\alpha = \left|\frac{\sqrt{1-\left(-\frac{1}{2}\right)^2}}{-\frac{1}{2}}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

Then $$y = -y_1\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$$

$$= -\left[-\cos\left(\frac{\pi}{9}\right) + \sqrt{3}\sin\left(\frac{\pi}{9}\right)\right]\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \approx 0.1736 - 0.3008\,i$$

Case 2: $x \le -1$ or $x \ge 1$

TI-89 calculator operating in real mode: $x \le -1$

For $x \le -1$, $u^3 > 0$ and $v^3 < 0$. When operating in real mode, the TI-89 calculator will therefore always use the positive value of u and the negative value of v when calculating the value of $u - v$ for $x \le -1$. This leads to values of y given by equation (29). A graph of the branch y_2 over $x \le -1$ is therefore drawn.

TI-89 calculator operating in real mode: $x \ge 1$

For $x \ge 1$, $u^3 < 0$ and $v^3 > 0$. When operating in real mode, the TI-89 calculator will therefore always use the negative value of u and the positive value of v when calculating the value of $u - v$ for $x \ge 1$. This leads to values of y given by equation (30). A graph of the branch y_3 over $x \ge 1$ is therefore drawn.

TI-89 calculator operating in complex mode

When operating in complex mode, the TI-89 calculator will always use the principle value of u and v and therefore the combination (u_0, v_0) when calculating the value of $u - v$. This leads to non-real values of y for all real values of x except $x = 0$ and hence no graph is drawn.

Example: $x = -2$

When $x = -2$, the TI-89 calculator returns $y \approx 2.1958$ and $y \approx 1.229 - 0.5583i$ when operating in real and complex modes respectively.

References

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