Unravelling student challenges with quadratics:

A cognitive approach

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My secondary school mathematics students have often reported to me that quadratic relations are one of the most conceptually challenging aspects of the high school curriculum. From my own classroom experiences, there seemed to be several aspects to the students' challenges. Many students, even in their early secondary education, have difficulty with basic multiplication table fact retrieval. Difficulty retrieving multiplication facts directly influences students' ability to engage effectively in factorisation of quadratics, since factorisation is a process of finding products within the multiplication table. Finally, students also find it challenging to recognise and understand varied representations of the same quadratic relationship.

In my own classroom, I had explored various pedagogical strategies in order to mediate for the challenges that I have outlined — everything from rehearsal to real world applications. However, I felt that my pedagogical efforts lacked the necessary insight on how the brain creates memory and felt that my pedagogical directions might be enhanced with this knowledge. Therefore, in order to construct my own classroom solutions, I turned to cognitive science to assist me in better understanding the mechanisms of fact retrieval. I surmised that problems with quadratic relations might potentially be linked to the ways in which the brain constructs cognitive representations and this knowledge might in turn inform my pedagogical decision making as a classroom teacher. This article is a sharing of my investigation.

Linking cognitive science to pedagogy

To better understand the problems students experience with quadratic relations, I draw from cognitive science researchers, Phenix and Campbell (2001), who suggest that order matters in the brain's ability to retrieve numeric facts. Their research is useful in understanding students' problems with factorisation and with identifying varied representations of the same quadratic relationship. Before I detail their research, I begin with an overview of the kinds of memory capabilities our brains have in order to situate why, as teachers of mathematics, we need to pay attention to Phenix and Campbell's claims.

Making the right kind of mathematical memories

Butterworth (1999) suggests that there are three types of memory our brains can create. Long-term autobiographical memory stores events with generalised timelines of when and where the events occurred. A student's memory of grade eight graduation is an example of long-term autobiographical memory.

Long-term semantic memory stores general knowledge not identified by a timeline for when the event occurred. Multiplication facts, for example, are stored in long-term semantic memory. Semantic, as in long-term semantic memory, implies associations to specific memories or meanings. In mathematics, semantic implies the ability to access certain knowledge over other knowledge based upon context. I refer to this process as "linguistic discrimination" — the ability to access one meaning over another meaning from long-term semantic memory of mathematical text; i.e., symbolic, numeric, visual, graphic, etc. (Kotsopoulos, 2006).

Short-term memory stores information temporarily. This information, or knowledge, may be lost if not eventually stored in long-term semantic memory. Multiplication facts can, alternatively to long-term semantic memory, be relinquished to short-term memory and thus lost to students during, for example, factorisation. Our goal, as educators, is to structure learning opportunities to ensure that mathematical facts and/or multiple representations of mathematical objects are stored to long-term semantic memory and are, thus, potentially accessible to students in the form of prior learning.

Therefore, how do we, as teachers, create learning opportunities that facilitate long-term semantic mathematical memories? Furthermore, how do we teach students to discriminate linguistically between meanings and access the appropriate long-term semantic memory? I turn here to the distinction between two different types of mathematical long-term semantic memories — that of "procedural knowledge" or "conceptual knowledge." The distinction between the two might be useful understanding how to occasion certain long-term semantic memories over others.

Simply said, conceptual knowledge refers to deeper understanding of mathematical relationships beyond computations, while procedural knowledge is associated with calculations and algorithm use (Boaler, 1998; Hiebert & Carpenter, 1992). Procedural knowledge that is in long-term semantic memory allows one to perform mathematical computations effectively and efficiently. Conceptual knowledge is deeper and enables an individual to question whether findings make sense, and develop, if necessary, procedural strategies from other existing strategies in long-term semantic memory, when needed.

The debates concerning procedural versus conceptual knowledge have been extensive in mathematics education (Hiebert & Lefevre, 1987). Some researchers hypothesise that conceptual knowledge builds procedural fluency (Haapasalo & Kadijevich, 2000; 1987). While others propose that conceptual knowledge only follows from procedural competency — students need to know the basic facts before they can make sense of more complex mathematical problems (Davis, Gray, Simpson, Tall, & Thomas, 2000). Still others claim that there does not need to be a divide between procedural and conceptual knowledge, a one or the other approach to learning mathematics, in that one can and does support the other (Berger, 2004; Rittle-Johnson, Czarnocha & Baker, 2002).

The position I take most closely follows the last. I extend this position to say that, as teachers, we need to make intentional and informed decisions about the choices we make in teaching. This may imply that in some instances procedural knowledge may be emphasised more heavily (i.e., basic fact retrieval, multiplication, etc.) than conceptual knowledge, but not to the exclusion of conceptual knowledge. For example, in the case of multiplication, it is useful for students to have an understanding of repeated addition as the conceptual underpinnings of basic multiplication — particularly for when long-term semantic memory fails. However, as teachers, we understand how timely this can be when students resort to repeated addition in order to multiply. Therefore, in the case of number facts, procedural knowledge is important.

Long-term semantic memory: Order matters

Much insight can be gained in mathematics education from the cognitive sciences regarding our brains. One such example of this can be seen in Phenix and Campbell's (2001) research that considered the brain's function with respect to numeric fact retrieval. The intent of their study was to test whether numeric fact retrieval is order specific. For example, does 3 × 5 access the same cognitive representation as 5 × 3? Their results showed that order did matter! That is, 3 × 5 does not access the same cognitive representations as 5 × 3, despite the fact that the operands and the results are the same.

Phenix and Campbell's (2001) findings suggested to educators that when students learn the multiplication tables both 3 × 5 and 5 × 3 needed to be understood independently of the each other, rather than the latter as simply a commuted form of the former. Patterning activities that highlight the commutative property for multiplication are common in the early development of students' conceptual knowledge of multiplication tables and multiplicity. One example of this is the multiplication table itself, often used as a tool to demonstrate to students how half of the table is simply a replica of the other half (see Table 1). However, given that order does matter, patterning (and assumptions regarding a student's ability to recognise patterns) requires more careful contemplation on the part of educators if our goal is to reach students' long-term semantic memory. This is a prime case for ensuring both procedural and conceptual understanding. The emphasis on procedural understanding, however, must be organised around two separate multiplication facts rather than two related facts, given that order does matter. In other words, when students see these prob-

Table 1. Multiplication table.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

lems, 3 × 5 and 5 × 3, and engage in linguistic discrimination, different long-term semantic memory is being accessed.

We now move to quadratics; why then do students experience difficulty in factoring quadratics? What is it that makes factoring such a formidable task? Given that the factoring of quadratics is the writing of polynomials as a product of polynomials, students need to have both a strong conceptual understanding of multiplication of polynomials as well as the procedural knowledge to retrieve basic multiplication facts effectively. It is useful for students to have conceptual knowledge of how products of terms relate to one another (i.e., exponent laws, addition of like terms, etc.). With this understanding, students can do the necessary procedural steps in factorisation but also step back and ask themselves if the results make sense.

Astonishingly, many secondary mathematics students struggle with basic multiplication table fact retrieval, negative factors, and multiple factors. Worth visiting is Brownell's (1956) article discussing the underlying coherence that computational competency affords a student in mathematics. Although for some students multiplication facts are immediate and habitual, for others the facts are inaccessible without a calculator. This may make factoring simple quadratics ($ax^2 + bx + c$, $a = 1$) a considerable challenge while non-simple quadratics ($ax^2 + bx + c$, $a \neq 1$) become almost impossible. In both cases, students need to rely on procedural knowledge (e.g., multiplication facts) and conceptual understanding (e.g., the relationship between a, b and c). One reason, already already alluded to, that students experience challenges, could be in the ways in which the multiplication facts were initially learned (Ben-Yehuda, Lavy, Linchevski & Sfard, 2005).

Varied representations: Order still matters!

As I have already discussed, many grade 10 students are challenged when having to recall basic multiplication facts. Factoring of quadratics requires students to be able to quickly find factors of one number that also add to find another. This can be a drawn-out task if multiplication facts are not immediate. Drawing from cognitive science, I have suggested that one factor that might have influenced early storage of multiplication facts is the ways in which these facts were taught (i.e., false assumptions about patterning in that order does matter) and that these might have impacted whether or not the facts were stored in long-term semantic memory.

Order continues to matter in basic factorisation when it comes to quadratics. Order can be taken to mean the ordering of mathematical text in specific operand order, as was seen in Phenix and Campbell's work (2001). Order can also be taken to mean the general form in which mathematical information is presented. There are three forms of quadratic relations explored in many curriculums (e.g., National Council of Teachers of Mathematics (NCTM), 2000; Ontario Ministry of Education (OME), 2005). These are:

1. factored form, $y = a(x - r)(x - s)$
2. standard form, $y = ax^2 + bx + c$
3. vertex form, $y = a(x - h)^2 + k$.

When quadratic relations are presented as variations of the above forms, students can experience conceptual difficulties; that is, misunderstanding by the student may occur when the form or order of the above relations changes, thus limiting their ability to access the appropriate long-term semantic memory.

Take, for example, the following question: $x^2 + 3x + 1 = x + 4$. I have found in my teaching that when students are asked to factorise when solving this equation, they often experience challenges because the question is not in standard form, or, in other words, in an order that is familiar to them. In this other example, $y = (x - 1)(2 - x)$, students often experience challenges understanding this factored-form of the quadratic to be a non-simple quadratic. Often, the graphical representation is opening in the incorrect direction (up rather than down). The order of the x in the second bracket contributes to this dissonance.

The preceding examples suggest that commuted understandings cannot be left to chance in that order, as in form, potentially matters here as well. The examples from the previous paragraph should not be viewed as merely extensions of a particular form, but rather, as new forms unto themselves. As such, these new forms have to be made explicit to students. In this case, building conceptual understanding may support students in linking the new forms to prior cognitive representations. For example, students should be encouraged to make connections and broader multiple representations of quadratics to explicate equivalence of algebraic representations as part of building greater conceptual understanding — graphical representations here would be very useful (i.e., seeing that various algebraic forms yield the same graphical representation).

Classroom implications

In our teaching, we need to pay careful attention to ways in which the brain creates long-term semantic meaning. Conceptual knowledge alone of multiplication may not be sufficient to support students in quick fact retrieval. Students need to have procedural fluency and, in the case of multiplication, we see that this means that some of the patterning strategies used to facilitate conceptual understanding may be ineffective (Phenix & Campbell, 2001). That students experience difficulties with basic multiplication facts in secondary school suggests that there is a pressing need for a blending of procedural and conceptual knowledge through varied learning experiences for students, with careful attention to how our brains create long-term semantic memory. Assuming that conceptual knowledge will build procedural knowledge in multiplication fact retrieval, for example, may lead to some problems, particularly if the conceptual knowledge is structured around patterns, given that order does matter. I propose that order is also instrumental in the problems that students have with varied representations of the same quadratic relations.

Cognitive science gives us a unique lens from which to view our practices in the classroom. Understanding how the brain organises mathematical objects might help optimise students' chances for success. I intentionally chose to look more closely at how the brain works in coming to understand students' ability to retrieve facts. The cognitive model is one that could be more useful to education. It is indeed a fascination of mine. As educators, our work is to create learning opportunities that are mindful to how the brain organises material so that long-term semantic memories are occasioned.

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Investigation Ideas

- Crosses are placed on a 3 × 3 grid alternately by two players with the rule that no three crosses can be in a line. If the winner is the one that puts the last cross on the grid, investigate how both players can win. What are the options on a 4 × 4 grid?