DISCOVERY
with Neville de Mestre

Multigrades

Consider the magic square in Figure 1.

8	1	6
3	5	7
4	9	2

Figure 1

Clearly the first and third rows yield

$$8 + 1 + 6 = 4 + 9 + 2 = 15$$

but it is not obvious that

$$8^2 + 1^2 + 6^2 = 4^2 + 9^2 + 2^2 = 101.$$

Also from the first and third columns

$$8 + 3 + 4 = 6 + 7 + 2 = 15$$

and $8^2 + 3^2 + 4^2 = 6^2 + 7^2 + 2^2 = 89.$

So, is it true that for every 3×3 magic square the sums of the squares of the outside numbers in rows (or columns) are also equal? Your students could check this by algebra if you give them the general form of 3×3 magic squares. Here it is below.

$p + q + 2r$	p	$p + 2q + r$
$p + 2q$	$p + q + r$	$p + 2r$
$p + r$	$p + 2q + 2r$	$p + q$

So it remains for your students to show that

$$(p + q + 2r)^2 + p^2 + (p + 2q + r)^2$$
$$\equiv (p + r)^2 + (p + 2q + 2r)^2 + (p + q)^2$$

and

$$(p + q + 2r)^2 + (p + 2q)^2 + (p + r)^2$$
$$\equiv (p + 2q + r)^2 + (p + 2r)^2 + (p + q)^2$$

Sets of numbers where not only their sums are equal but the sums of other powers are also equal have been called multigrades (Bolt, 1982). The types considered so far are called second-order multigrades. Consider now the third-order multigrade:

$$1 + 5 + 8 + 12 = 2 + 3 + 10 + 11 = 26$$
$$1^2 + 5^2 + 8^2 + 12^2 = 2^2 + 3^2 + 10^2 + 11^2 = 234$$
$$1^3 + 5^3 + 8^3 + 12^3 = 2^3 + 3^3 + 10^3 + 11^3 = 2366$$

Although this seems to be a most unusual set of relationships between these numbers, others can be easily constructed from these by adding any constant to each of the original numbers. For example, if 3 is added to each then

$$4 + 8 + 11 + 15 = 5 + 6 + 13 + 14 = 38$$
$$4^2 + 8^2 + 11^2 + 15^2 = 5^2 + 6^2 + 13^2 + 14^2 = 426$$
$$4^3 + 8^3 + 11^3 + 15^3 = 5^3 + 6^3 + 13^3 + 14^3 = 5282$$

To see why this is true consider the general third-order multigrade

$$A1 + A2 + A3 + A4 = B1 + B2 + B3 + B4$$
$$A1^2 + A2^2 + A3^2 + A4^2 = B1^2 + B2^2 + B3^2 + B4^2$$
$$A1^3 + A2^3 + A3^3 + A4^3 = B1^3 + B2^3 + B3^3 + B4^3$$

Add k to each Ai, Bi (i = 1, 2, 3, 4). Clearly

$$(A1 + k) + (A2 + k) + (A3 + k) + (A4 + k)$$
$$= (B1 + k) + (B2 + k) + (B3 + k) + (B4 + k)$$

and

$$(A1+k)^2 + (A2 + k)^2 + (A3 + k)^2 + (A4 + k)^2$$
$$= (A1^2+ A2^2+A3^2+A4^2) + 2k(A1+A2+A3+A4) + 4k^2$$
$$= (B1^2+B2^2+B3^2+B4^2) + 2k(B1+B2+B3+B4) + 4k^2$$
$$= (B1 + k)^2 + (B2 + k)^2 + (B3 + k)^2 + (B4 + k)^2$$

A careful expansion by your students of the cubic expressions along similar lines would establish that

$$(A1 + k)^3 + (A2 + k)^3 + (A3 + k)^3 + (A4 + k)^3$$
$$= (B1 + k)^3 + (B2 + k)^3 + (B3 + k)^3 + (B4 + k)^3$$

How are multigrades obtained in the first place? Surprisingly, Bolt (1982) shows that this can be done in a very simple way. Start with a simple addition equation such as

$$1 + 5 = 2 + 4 \qquad (1)$$

Add a constant to each number in the equation so that not one of the original numbers is repeated. For example, the addition of 5 to each term yields

$$6 + 10 = 7 + 9 \qquad (2)$$

Swap the LHS and RHS of equation (2) around and add to equation (1) producing

$$1 + 5 + 7 + 9 = 2 + 4 + 6 + 10 = 22 \qquad (3)$$

and it is soon verified that

$$1^2 + 5^2 + 7^2 + 9^2 = 2^2 + 4^2 + 6^2 + 10^2 = 156$$

To form a third-order multigrade, use a similar process to the above, but perform it on a second-order multigrade. For example, add 10 to each term of equation (3) yielding

$$11 + 15 + 17 + 19 = 12 + 14 + 16 + 20 \qquad (4)$$

Now swap the LHS and RHS of equation (4) around and add them to equation (3) giving the result

$$1 + 5 + 7 + 9 + 12 + 14 + 16 + 20$$
$$= 2 + 4 + 6 + 10 + 11 + 15 + 17 + 19$$

with no repeated number. Verify that

$$1^N + 5^N + 7^N + 9^N + 12^N + 14^N + 16^N + 20^N$$
$$= 2^N + 4^N + 6^N + 10^N + 11^N + 15^N + 17^N + 19^N$$

for $N = 1, 2, 3$.

Your students may ask what happens to the process if a constant k is chosen to be added and a number is repeated? Again start with equation (1)

$$1 + 5 = 2 + 4$$

and now add 4 to each term generating

$$5 + 9 = 6 + 8 \qquad (5)$$

with the repeated number 5 occurring. Swap equation (5) around and add to equation (1) yielding

$$1 + 5 + 6 + 8 = 2 + 4 + 5 + 9$$

Since 5 is common to both sides eliminate it producing

$$1 + 6 + 8 = 2 + 4 + 9$$

which is one of the second-order multigrades from the original magic square. However, it is also instructive to note what happens when either 3 or 8 are added, as a multigrade can only be obtained if there is a repeated number on both sides of the added equation.

Your students should try to discover more multigrades and verify their properties using a calculator. By further extension of the process outlined in this article, they can even generate fourth-order, fifth-order, or higher-order multigrades.

Reference

Bolt, Brian (1982). *Mathematical Activities*. Cambridge University Press.